THE DIVISOR PROBLEM ON SQUARE-FREE INTEGERS

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ABSTRACT
For any real number $x > 1$, let $D(x) = \{d \le x \mid d \text{ is a square} - \text{free divisor integer}\}$, we study the properties of $\sum_{n \le x} \sum_{\substack{d|n \\ d \in D(x)}} 1$, and get a sharp asymptotic formula about it.

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1. INTRODUCTION

A square-free integer is one divisible by no perfect square, except 1. For example, 10 is a square-free integer, but 18 is not. For $x \ge 1$, let $D(x)$ denotes the set of square-free integers between 1 and x, then (see Ref. [1])

$$|D(x)| = \frac{6}{\pi^2} x + O\left(x^{\frac{1}{2}}\right). \tag{1}$$

Under the Riemann hypothesis, the error term can be reduced (see Ref. [2])

$$|D(x)| = \frac{6}{\pi^2} x + O\left(x^{\frac{17}{54}+\epsilon}\right). \tag{2}$$

For $x \ge 1$, $d(n)$ is the divisor function, Dirichlet (see Ref. [3]) showed that the average order of the divisor function satisfies the following inequality

$$\sum_{n < x} d(n) = x \log x + (2\gamma - 1)x + O\left(x^{\frac{1}{2}}\right), \tag{3}$$

where γ is Euler's constant. Improving the bound in this formula is known as Dirichlet's divisor problem. Precisely stated, the Dirichlet divisor problem is to find the infimum of all values θ for which

$$\sum_{n \le x} d(n) - (x \log x + (2\gamma - 1)x) = O\left(x^{\theta+\epsilon}\right), \tag{4}$$

holds true, for any $\epsilon > 0$. At present the best bound is $O(x^{\frac{131}{416}+\epsilon})$. So, the true value of $\inf \theta$ lies somewhere between 1/4 and 131/416; it is widely conjectured to be exactly 1/4.

In this paper, we want to know the distributive properties about the divisor function involving square-free integers as the following

$$\sum_{n \le x} \sum_{\substack{d|n \\ d \in D(x)}} 1$$

It is very interesting because it can show us some internal evidence on the divisor problem which is very complicated and has been unsolved completely so far. However it is very easy for us to get a weak asymptotic formula from (1) by the elementary method, that is

$$\sum_{n \le x} \sum_{\substack{d|n \\ d \in D(x)}} 1 = \frac{6}{\pi^2} x \log x + O(x). \tag{5}$$

Could the bound be reduced? The answer is definite. Hence we have studied further more the asymptotic properties of $\sum_{n \le x} \sum_{\substack{d|n \\ d \in D(y)}} 1$ and have got some non-trivial properties about it. About this problem, we know very little at present. At least we have not found it in any reference that we could find. Therefore, in this paper, we have obtained the sharp asymptotic formula. That is, we will prove the following theorem.

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Theorem For any real number $x \geq 1$, let $D(x) = \{d \leq x \mid d \text{ is a square} - \text{free divisor integer}\}$, we have the following asymptotic formula

$$\sum_{n\leq x}\sum_{\substack{d|n\\ d\in D(x)}} 1 = \frac{6}{\pi^2}x\log x - \frac{6}{\pi^2}(2\log 2\pi - 24\log A)x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where A is the Gtaisher-Kinkelin constant (see Ref. [4]),ϵ is any fixed positive real number .

Remark. Under the Riemann hypothesis, by the elementary method, from (2) we can not change the error term of $\sum_{n\leq x}\sum_{\substack{d|n\\ d\in D(x)}} 1$. But to our surprise, following the proof of Theorem we can reduce the error term to $x^{\frac{1}{4}+\epsilon}$. It is strange for us to get such a sharp bound. So it is still an open problem.

2. SOME LEMMAS

In this section, we shall give several lemmas which are necessary in the proof of the theorem.

Lemma 1. (Perron's formula) Let $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converge absolutely for $\sigma = \text{Re}s > 1$ and let $|a_n| < C\Phi(n)$, where $C > 0$ and for $x \geq x_0 \Phi(x)$ is monotonically increasing. Let further

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma-1)^{-\alpha}$$

as $\sigma \to 1+0$ for some $\alpha > 0$. If $w = u + iv$ (u, v real) is arbitrary, $b > 0, T > 0, u + b > 1$, then

$$\sum_{n\leq x} a_n n^{-w} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} A(s+w)x^s s^{-1} ds + O(x^b T^{-1}(u+b-1)^{-\alpha}) + \\ +O(T^{-1}\Phi(2x)x^{1-u}\log 2x) + O(\Phi(2x)x^{-u}), \tag{6}$$

and the estimate is uniform in x, T, b and u provided that b and u are bounded.

Proof. See A.10 in Ref. [6].

Lemma 2. Let $s = \sigma + it$, there is an absolute constant $C > 0$, such that

$$\frac{1}{\zeta(s)} = O\left(\log^{\frac{2}{3}} T(\log\log T)^{\frac{1}{3}}\right).$$

in the region $\sigma \geq 1 - \dfrac{C}{\log^{\frac{2}{3}} T(\log\log T)^{\frac{1}{3}}}$, $T_0 < t \leq T$.

Proof. See Lemma 12.3 in Ref. [6] .

Lemma 3. Let complex number $s = \sigma + it$, then we have the following estimate

$$\zeta(\sigma+it) = \begin{cases} 1, & \sigma \geq 2; \\ \log t, & 1 \leq \sigma \leq 2; \\ t^{\frac{1-\sigma}{2}}\log t, & 0 \leq \sigma \leq 1; \\ t^{\frac{1}{2}-\sigma}\log t, & \sigma \leq 0. \end{cases}$$

Proof. See Ref. [6].

Lemma 4. Let complex number $s = \sigma + it$, for any $\frac{1}{2} < \sigma < 1$ fixed we define $m(\sigma)$ as the supremum of all numbers $m\ (\geq 4)$ such that

$$\int_1^T |\zeta(\sigma+it)|^m dt \ll T^{1+\epsilon}$$

for any $\epsilon > 0$. Then for $\frac{1}{2} < \sigma < \frac{5}{8}$, we have

$$m(\sigma) = \frac{4}{3-4\sigma}.$$

Proof. See Ref. [6].

3. PROOF OF THEOREMS

In this section, we shall complete the proof of the theorem. Let $b = 1 + \frac{1}{\log x}, s = \sigma + it, (\sigma > 1)$, $A(s) = \sum_{nm=1}^{\infty} \frac{\mu^2(n)}{(nm)^s} = \frac{\zeta^2(s)}{\zeta(2s)}$ and $w = 0, \alpha = 2$, according to Lemma 1, we have

$$\sum_{n\leq x} \sum_{\substack{d\mid n \\ d\in D(x)}} 1 = \sum_{dl\leq x} \mu^2(d)$$

$$= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta^2(s)}{\zeta(2s)} \frac{x^s}{s} ds + O(x \log x \cdot T^{-1}). \tag{7}$$

Moving the line of integration in (7) to Re$s = \frac{1}{2} - \delta \left(\delta = \frac{C}{2} \log^{-\frac{2}{5}} T (\log\log T)^{-\frac{1}{5}}\right)$ in view of Lemma 2, we encounter poles at $s = 1$ of order 2 with residues $Res_{s=1}\left(\frac{\zeta^2(s)}{\zeta(2s)}\frac{x^s}{s}\right)$, hence from Residues Theorem, we get

$$Res_{s=1}\left(\frac{\zeta^2(s)}{\zeta(2s)}\frac{x^s}{s}\right) = \lim_{s\to 1}\left(\frac{\zeta^2(s)(s-1)^2}{\zeta(2s)}\frac{x^s}{s}\right)'$$

$$= \lim_{s\to 1}\left(\frac{\zeta^2(s)(s-1)^2}{\zeta(2s)}\right)'\frac{x^s}{s} + \lim_{s\to 1}\frac{\zeta^2(s)(s-1)^2}{\zeta(2s)}\left(\frac{x^s}{s}\right)'$$

$$= x\,\frac{2\gamma\zeta(2) - 2\zeta'(2)}{\zeta^2(2)} + \frac{1}{\zeta(2)}(x\log x - x)$$

$$= \frac{6}{\pi^2} x \log x - \frac{6}{\pi^2}(2\log 2\pi - 24\log A)x, \tag{8}$$

where we have used $\zeta(2) = \frac{6}{\pi^2}, \zeta'(2) = \frac{\pi^2}{6}(\gamma + \log 2\pi - 12\log A)$ (A is the Gtaisher-Kinkelin constant (see Ref. [4])), and γ is the Euler's constant. Therefore from (7), (8) and Lemma 2, 3, we have

$$\sum_{n\leq x} \sum_{\substack{d\mid n \\ d\in D(x)}} 1$$

$$= \frac{6}{\pi^2} x\log x - \frac{6}{\pi^2}(2\log 2\pi - 24\log A)x +$$

$$+ \frac{1}{2\pi i}\left(\int_{b-iT}^{\frac{1}{2}-\delta-iT} + \int_{\frac{1}{2}-\delta+iT}^{b+iT} + \int_{\frac{1}{2}-\delta-iT}^{\frac{1}{2}-\delta+iT}\right)\frac{\zeta^2(s)}{\zeta(2s)}\frac{x^s}{s} ds + O\left(x\log x \cdot T^{-1}\right). \tag{9}$$

Denoting the integrals in (9) by I_1, I_2 and I_3 respectively, we will estimate each of them in the following. Firstly, we estimate I_2, from Lemma 2, we have

$$|I_2| = \int_{\frac{1}{2}-\delta}^{b} \frac{\zeta^2(\sigma+iT)}{\zeta(2\sigma+2iT)} \frac{x^{\sigma+iT}}{\sigma+iT} d\sigma \ll x^b T^{-1} \int_{\frac{1}{2}-\delta}^{b} \log^C T\, d\sigma$$

$$\ll xT^{-1+\epsilon}, \tag{10}$$

where we have used $b = 1 + \frac{1}{\log x}$, and a corresponding bound for I_1. Next we estimate I_3. From

$$|I_3| = \int_{-T}^{T} \frac{\zeta^2\left(\frac{1}{2} - \delta + it\right)}{\zeta(1 - 2\delta + 2it)} \frac{x^{\frac{1}{2}-\delta+it}}{\frac{1}{2} - \delta + it} dt$$

$$\ll \int_0^{T_0} \frac{x^{\frac{1}{2}-\delta}}{\frac{1}{2} - \delta} dt + \int_{T_0}^{T} \frac{x^{\frac{1}{2}-\delta}}{\delta t} \left|\zeta^2\left(\frac{1}{2} - \delta + it\right)\right| dt$$

$$\ll x^{\frac{1}{2}-\delta} + \frac{x^{\frac{1}{2}-\delta}}{\delta} \int_{T_0}^{T} \left|\zeta^2\left(\frac{1}{2} - \delta + it\right)\right| t^{-1}\, dt, \tag{11}$$

using the functional equation

$$\zeta(s) = \left(\frac{2\pi}{t}\right)^{s-\frac{1}{2}} e^{i\left(t+\frac{\pi}{4}\right)}(1 + O(t^{-1}))\zeta(1 - s) \quad (t > t_0 > 0),$$

we have

$$\int_{T_0}^{T} \left|\zeta^2\left(\frac{1}{2} - \delta + it\right)\right| t^{-1}\, dt \ll \int_{T_0}^{T} t^{-1+\delta} \left|\zeta^2\left(\frac{1}{2} + \delta - it\right)\right| dt, \tag{12}$$

so according to the integration by parts, if we will prove the following, we can immediately get the estimate of I_3.

$$\int_{T_0}^{T} \left|\zeta^2\left(\frac{1}{2} + \delta - it\right)\right| dt \ll T^{1+\epsilon}.$$

In fact by Hölder's inequality and Lemma 4, we have

$$\int_{T_0}^{T} \left|\zeta^2\left(\frac{1}{2} + \delta - it\right)\right| dt \le \left(\int_{T_0}^{T} \left|\zeta\left(\frac{1}{2} + \delta - it\right)\right|^{2p} dt\right)^{\frac{1}{p}} \left(\int_{T_0}^{T} dt\right)^{\frac{1}{q}} \ll T^{1+\epsilon}, \tag{13}$$

where $0 < \delta < \frac{1}{8}$ and the integers p and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

Therefore, from (12) and (13), and taking $T = x$, we have

$$\sum_{n \le x} \sum_{\substack{d|n \\ d \in D(x)}} 1$$

$$= \frac{6}{\pi^2} x \log x - \frac{6}{\pi^2}(2\log 2\pi - 24\log A)x + O\left(x^{\frac{1}{2}+\varepsilon}\right). \tag{14}$$

This proves the Theorem.

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