COMMON FIXED POINT THEOREM FOR SUB COMPATIBLE AND SUB SEQUENTIALLY CONTINUOUS MAPS IN FUZZY METRIC SPACE USING IMPLICT RELATION

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ABSTRACT

The present paper introduces the new concepts of sub compatibility and subsequential continuity in fuzzy metric spaces using Implict Relation which are weaker than occasionally weak compatibility and reciprocal continuity. In general all known results on commuting, weakly commuting, compatible, weak compatible, semi compatible and occasionally weak compatible maps in fuzzy metric spaces are generalized in this note.

Keywords: *Common Fixed point, Fuzzy Metric space, Continuous t-norm, subcompatibility and subsequential continuity.*

1. INTRODUCTION

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by L.A.Zadeh [16] which laid the foundation of fuzzy mathematics. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineer sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc.

Kramosil and Michalek [13] introduced the notion of a fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [4] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [13]. There are many view points of the notion of the metric space in fuzzy topology for instance one can refer to Kaleva and Seikkala [15], Kramosil and Michalek [13], George and Veeramani [4].

Regan and Abbas[1] obtained some necessary and sufficient conditions for the existence of common fixed point in fuzzy metric spaces .

Popa ([18]-[19]) introduced the idea of implicit function to prove a common fixed point theorem in metric spaces . Singhand Jain [14] further extended the result of Popa ([18]-[19]) in fuzzy metric spaces. For the reader convenience, we recall some terminology from the theory of fuzzy metric space.

Using the concept of R-weak commutativity of mappings, R.Vasuki [21] proved the fixed point theorems for Fuzzy metric space. Recently in 2009, using the concept of subcompatible maps, H.Bouhadjera et. al. [12] proved common fixed point theorems. Using the concept of compatible maps of type (A), Jain et. al. [5] proved a fixed point theorem for six self maps in a fuzzy metric space. Using the concept of compatible maps of type (β), Jain et. al. [6] proved a fixed point theorem in Fuzzy metric space. In 2010 and 2011, B.Singh et. al. [8,9,10] proved fixed point theorems in Fuzzy metric space and Menger space using the concept of semi-compatibility, weak compatibility and compatibility of type (β) respectively.

In this paper, we introduce the new concepts of subcompatibility and subsequential continuity which are respectively weaker than occasionally weak compatibility and reciprocal continuity in Fuzzy metric space using Implict Relation and estabilish a common fixed point theorem.

For the sake of completeness, we recall some definitions and known result in Fuzzy metric space.

2. PRELIMINARIES

Definition 2.1. [16] Let X be any non empty set. A fuzzy set M in X is a function with domain X and values in [0,1].

Definition 2.2. [24] A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norm if it satisfy the following condition:

(i) $*$ is associative and commutative .
(ii) $*$ is continous function.
(iii) $a*1=a$ for all $a\in[0,1]$
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]$
Examples of t-norms are $a * b = a b$ and $a * b = \min\{a, b\}$.

Definition 2.3. [22] The 3 − tuple (X,M, $*$) is called a fuzzy metric space i if X is an arbitrary set, is a continuous t − norm and M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions:
(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0,\infty) \to [0, 1]$ is a left continuous function, for all $x, y, z \in X$ and $t, s > 0$.
(FM-6) $\lim_{t\to\infty} M(x, y, t) =1$.
Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$.The following example shows that every metric space induces a Fuzzy metric space

Example 2.4. [22] Let (X,d) be a metric space. Define $a* b = \min \{a, b\}$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y\in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.5. [22] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \ge n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.
A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchysequence in it converges to a point in it.

Definition 2.6. [23] Self mappings A and S of a Fuzzy metric space(X, M, *) are said to be weakly commuting if $M(ASz, SAz, t) \ge M(Az, Sz, t)$ for all $z \in X$ and $t > 0$.

Definition 2.7. [7] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.8. [21] Two mappings A and S of a Fuzzy metric space (X, M, *) are called reciprocally continuous if $ASx_n \to Az$ and $SAx_n \to Sz$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to z$ for some z in X.

If A and S are both continuous, then they are obviously reciprocally continuous but converse is not true. Moreover, in the setting of common fixed point theorems for compatible pair of mappings satisfying contractive conditions, continuity of one of the mappings A and S implies their reciprocal continuity but not conversely.

Definition 2.9. Suppose A and S be self mappings of a Fuzzy metric space (X, M, *). A point x in X is called a coincidence point of A and S if and only if $Ax = Sx$. In this case, $w = Ax = Sx$ is called a point of coincidence of A and S.

Definition 2.10. [6] Self maps A and S of a Fuzzy metric space (X, M, *) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e if $Ap = Sp$ for some $p\in X$ then $ASp = SAp$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11. Self maps A and S of a Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

In this paper, we weaken the above notion by introducing a new concept called subcompatibility just as defined by H.Bouhadjera et. al. [12] in metric space as follows :

Definition 2.12. Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be subcompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $z \in X$ and satisfy $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$

Obviously, two owc maps are subcompatible, however the converse is not true in general.
Now, our second objective is to introduce subsequential continuity in Fuzzy metric space which weaken the concept of reciprocal continuity which was introduced by Pant et. al. [20] just as introduced by H.Bouhadjera et. al. [12] in metric space, as follows :

Definition 2.13. Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be subsequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $z \in X$ and satisfy $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$

Clearly, if A and S are continuous or reciprocally continuous then they are obviously subsequentially continuous. However, the converse is not true in general.

3. IMPLICIT RELATIONS [2]:

(a) Let **(Φ)** be the set of all real continuous functions $\emptyset : (R^+)^6 \to R^+$ satisfying the condition
$\emptyset$: (u, u, v, v, u, u) ≥ 0 imply $u \geq v$, for all u, v $\in [0,1]$.

(b) Let **(Φ)** be the set of all real continuous functions $\emptyset : (R^+)^5 \to R^+$ satisfying the condition
$\emptyset$: (u, u, v, u, u) ≥ 0 imply $u \geq v$, for all u, v $\in [0,1]$.

4. MAIN RESULTS

Theorem 4.1 Let A, B, S and T be four self maps of Fuzzy metric space (X, M, *) with continuous t norm * defined by $t * t \geq t$ for all $t \in [0,1]$. If the pairs (A, S) and (B, T) are subcompatible and subsequentially continuous, then

(a) A and S have a coincidence point.

(b) B and T have a coincidence point

(c) For some $\emptyset \in \mathbf{\Phi}$ and for all $x,y \in X$ and every $t > 0$,
$$\emptyset\{M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, t), M(Ty, Ax, t)\} \geq 0$$

Then A, B, S and T have a unique common fixed point.

Proof : Since the pairs (A, S) and (B, T) are subcompatible and subsequentially continuous, then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $z \in X$ and satisfy $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = M(Az, Sz, t) = 1$

$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z'$, $z' \in X$ and satisfy $\lim_{n\to\infty} M(BTy_n, TBy_n, t) = M(Bz', Sz', t) = 1$

Therefore, Az = Sz and Bz' = Tz'; that is, z is a coincidence point of A and S and z' is a coincidence point of B and T.

Now, we prove z = z'.

Put $x = x_n$ and $y = y_n$ in inequality (c), we get

$$\emptyset\{M(Ax_n, By_n, t), M(Sx_n, Ty_n, t), M(Sx_n, Ax_n, t), M(Ty_n, By_n, t), M(Sx_n, By_n, t), M(Ty_n, Ax_n, t)\} \geq 0$$

Taking the limit as $n \to \infty$, we get

$$\emptyset\{M(z, z', t), M(z, z', t), M(z, z, t), M(z', z', t), M(z, z', t), M(z', z, t)\} \geq 0$$

$$\emptyset\{M(z, z', t), M(z, z', t), 1, 1, M(z, z', t), M(z, z', t)\} \geq 0$$

In view of **Φ** (3-a) we get z = z'

Again, we claim that Az = z.

Substitute x = z and y = y_n in inequality (c), we get

$\emptyset\{M(Az,By_n, t), M(Sz, Ty_n, t), M(Sz, Az, t), M(Ty_n, By_n, t), M(Sz, By_n, t), M(Ty_n, Az, t)\} \geq 0$

Taking the limit as $n \to \infty$, we get

$\emptyset\{M(Az, z', t), M(Az, z', t), M(Az, Az, t), M(z',z', t), M(Az, z', t), M(z',Az, t)\} \geq 0$

$\emptyset\{M(Az, z', t), M(Az, z', t), 1, 1, M(Az, z', t), M(Az, z', t)\} \geq 0$

In view of **Φ** (3-a) we get Az = z' = z

Again, we claim that Bz = z.

Substitute x = z and y = z in inequality (c), we get

$\emptyset\{M(Az, Bz, t), M(Sz, Tz, t), M(Sz, Az, t), M(Tz, Bz, t), M(Sz, Bz, t), M(Tz, Az, t)\} \geq 0$

$\emptyset\{M(z, Bz, t), M(z, Bz, t), M(z, z, t), M(Bz, Bz, t), M(z, Bz, t), M(Bz, z, t)\} \geq 0$

$\emptyset\{M(z, Bz, t), M(z, Bz, t), 1, 1, M(z, Bz, t), M(z, Bz, t)\} \geq 0$

In view of **Φ** (3-a) we get z = Bz = Tz

Therefore, z = Az = Bz = Sz = Tz; that is z is common fixed point of A, B, S and T.

Uniqueness : Let w be another common fixed point of A, B, S and T. Then
Aw = Bw = Sw = Tw = w.
Put x = z and y = w in inequality (c), we get

$\emptyset\{M(Az, Bw, t), M(Sz, Tw, t), M(Sz, Az, t), M(Tw, Bw, t), M(Sz, Bw, t), M(Tw, Az, t)\} \geq 0$

$\emptyset\{M(z, w, t), M(z, w, t), M(z, z, t), M(w, w, t), M(z, w, t), M(w, z, t)\} \geq 0$

$\emptyset\{M(z, w, t), M(z, w, t), 1, 1, M(z, w, t), M(z, w, t)\} \geq 0$

In view of **Φ** (3-a) we get z = w.

Therefore, uniqueness follows.

If we take S = T in theorem 3.1, we get the following result:

Corollary 4.2 Let A, B,and S be three self maps of Fuzzy metric space (X, M, *) with continuous t norm * defined by $t * t \geq t$ for all $t \in [0, 1]$. If the pairs (A, S) and (B, S) are subcompatible and subsequentially continuous, then

(a) A and S have a coincidence point.

(b) B and S have a coincidence point

(c) For some $\emptyset \in \boldsymbol{\Phi}$ and for all $x,y \in X$ and every $t > 0$,
$\emptyset\{M(Ax, By, t), M(Sx, Sy, t), M(Sx, Ax, t), M(Sy, By, t), M(Sx, By, t), M(Sy, Ax, t)\} \geq 0$

Then A, B, and S have a unique common fixed point.

Theorem 4.3 Let A, B, S and T be four self maps of Fuzzy metric space (X, M, *) with continuous t norm * defined by $t * t \geq t$ for all $t \in [0, 1]$. If the pairs (A, S) and (B, T) are subcompatible and subsequentially continuous, then
(a) A and S have a coincidence point.
(b) B and T have a coincidence point
(c) For some $\emptyset \in \mathbf{\Phi}$ and for all $x,y \in X$ and every $t > 0$,

$$\emptyset\left\{M(Ax,By,t), M(Sx,Ty,t), \frac{M(Sx,Ax,t)+M(Ty,By,t)}{2}, M(Sx,By,t), M(Ty,Ax,t)\right\} \geq 0$$

Then A, B, S and T have a unique common fixed point.

Proof : Since the pairs (A, S) and (B, T) are subcompatible and subsequentially continuous, then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $z \in X$ and satisfy $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = M(Az,Sz,t) = 1$

$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z'$, $z' \in X$ and satisfy $\lim_{n\to\infty} M(BTy_n, TBy_n, t) = M(Bz',Sz',t) = 1$

Therefore, Az = Sz and Bz' = Tz'; that is, z is a coincidence point of A and S and z' is a coincidence point of B and T.
Now, we prove z = z'.
Put $x = x_n$ and $y = y_n$ in inequality (c), we get

$$\emptyset\left\{M(Ax_n,By_n,t)M(Sx_nTy_n,t), \frac{M(Sx_n,Ax_n,t)+M(Ty_n,By_n,t)}{2}, M(Sx_n,By_n,t), M(Ty_n,Ax_n,t)\right\} \geq 0$$

Taking the limit as $n \to \infty$, we get

$$\emptyset\left\{M(z,z',t), M(z,z',t), \frac{M(z,z,t)+M(z',z',t)}{2}, M(z,z',t), M(z',z,t)\right\} \geq 0$$

$$\emptyset\{M(z,z',t), M(z,z',t), 1, M(z,z',t), M(z,z',t)\} \geq 0$$

In view of $\mathbf{\Phi}$ (3-b) we get z = z'

Again, we claim that Az = z.
Substitute x = z and $y = y_n$ in inequality (c), we get

$$\emptyset\left\{M(Az,By_n,t), M(Sz,Ty_n,t),, \frac{M(Sz,Az,t)+M(Ty_n,By_n,t)}{2}, M(Sz,By_n,t), M(Ty_n,Az,t)\right\} \geq 0$$

Taking the limit as $n \to \infty$, we get

$$\emptyset\left\{M(Az,z',t), M(Az,z',t), \frac{M(Az,Az,t)+M(z',z',t)}{2}, M(Az,z',t), M(z',Az,t)\right\} \geq 0$$

$$\emptyset\{M(Az,z',t), M(Az,z',t), 1, M(Az,z',t), M(Az,z',t)\} \geq 0$$

In view of $\mathbf{\Phi}$ (3-b) we get Az = z' = z

Again, we claim that Bz = z.
Substitute x = z and y = z in inequality (c), we get

$$\emptyset\left\{M(Az,Bz,t), M(Sz,Tz,t), \frac{M(Sz,Az,t)+M(Tz,Bz,t)}{2}, M(Sz,Bz,t), M(Tz,Az,t)\right\} \geq 0$$

$$\emptyset\left\{M(z,Bz,t), M(z,Bz,t), \frac{M(z,z,t)+M(Bz,Bz,t)}{2}, M(z,Bz,t), M(Bz,z,t)\right\} \geq 0$$

$$\emptyset\{M(z,Bz,t), M(z,Bz,t), 1, M(z,Bz,t), M(z,Bz,t)\} \geq 0$$

In view of $\mathbf{\Phi}$ (3-b) we get z = Bz = Tz

Therefore, z = Az = Bz = Sz = Tz; that is z is common fixed point of A, B, S and T.

Uniqueness : Let w be another common fixed point of A, B, S and T.
Then Aw = Bw = Sw = Tw = w.

Put x = z and y = w in inequality (c), we get

$$\emptyset\left\{M(Az,Bw,t),M(Sz,Tw,t),\frac{M(Sz,Az,t)+M(Tw,Bw,t)}{2},M(Sz,Bw,t),M(Tw,Az,t)\right\}\geq 0$$

$$\emptyset\left\{M(z,w,t),M(z,w,t),\frac{M(z,z,t)+M(w,w,t)}{2},M(z,w,t),M(w,z,t)\right\}\geq 0$$

$$\emptyset\{M(z,w,t),M(z,w,t),1,M(z,w,t),M(z,w,t)\}\geq 0$$

In view of **Φ** (3-b) we get z = w. Therefore, uniqueness follows.

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