ON OSCILLATORY OSEENLETS: THE FORCES GENERATED BY THEM, THE FAR FIELD EXPANSION OBTAINED USING THEM, AND THEIR RELATION TO OSCILLATORY STOKESLETS

Edmund Chadwick & Rabea Elmazuzi
School of Computing, Science and Engineering, University of Salford, Salford M5 4WT, UK

ABSTRACT

Consider uniform flow past an oscillating body. Assume that the resulting far-field flow consists of both steady and time periodic components. The time periodic components can be decomposed into a Fourier expansion series of time harmonic components. The form of the steady component in terms of the steady oseenlet is well-known. However, the time-harmonic components in terms of the oscillatory oseenlet do not yet appear to be in the literature.

The oscillatory oseenlet solution is presented for the velocity and pressure, and the forces generated by them are calculated and shown to be oscillatory. The formulation is checked against known solutions in the following two limits: As the frequency of the oscillations tend to zero, it is shown that the steady oseenlet solution is recovered; Also, as the Reynolds number of the flow tends to zero, it is shown that the oscillatory stokeslet solution is recovered. Both the oscillatory stokeslet and oscillatory oseenlet solutions are represented by velocity potentials, and the oscillatory oseenlets are used to give a far-field velocity expansion.

1. INTRODUCTION

The problem of uniform flow past an oscillating body is a general one, examples being the flapping flight of birds and insects, and the swimming of mammals, fish and micro-organisms. Of particular interest to the authors is a feasibility study for the robotics centre of the university into the propulsion of miniaturised robot devices through fluid by means of an oscillatory swimming motion. The eventual goal is for devices at the milli- to nano-metre range that could be used within the blood stream for various medical purposes such as tissue or cell repair, and this is very much at the feasibility study stage. The steady forward velocity lends itself to the Oseen linearisation in the far-field, that the velocity perturbation is small compared to the uniform stream (or forward propulsive velocity).

The literature on time dependent Oseen and associated Stokes flows subdivides into transient analysis and oscillatory analysis, with the majority of work on transient rather than oscillatory analysis.

Price [16] uses transient oseenlets in order to model ship motions. Also, Chwang and co-workers [6] [13] describe the unsteady (transient) stokeslet and oseenlet and give applications related to acceleration and free surface waves. Childress [7] uses transient oseenlets to model the effect of flapping of a swimming mollusc as it thrusts forward. A numerical solution of the transient oseenlet analysis is employed. However, it is clear that if a steady oscillatory motion of the swimming mollusc is required instead, an oscillatory oseenlet would be preferable but is not currently available in the literature.

Riley's and Amin's work [17] [1] [18] does employ an oscillatory rather than transient analysis to model the flow generated by fixed oscillating bodies. Here, the focus is on matching the inner Stokes-type flow to an outer flow. Clarke et. al [8] consider the problem of a MEMS device vibrating in a fluid at rest. The device is treated as a slender body and the Stokes approximation is used. The oscillatory stokeslet given by Pozrikidis [15] is used. However, there is no uniform stream for these problems and so the outer flow is not an Oseen flow and very different from it. The problem of a forward moving body in the fluid additionally requires a far-field analysis where the velocity is linearized to a uniform stream. Within such a development, the oscillatory stokeslet is an inner near-field description to be matched to an outer far-field oscillatory oseenlet. In order to enable this, there is a requirement for the oscillatory oseenlet solution. Iima [9] considers a butterfly flapping and whether it can sustain a hovering motion. He formulates a two-dimensional far-field periodic Oseen representation for a small steady uniform flow motion and then lets that motion tend to zero. This representation is not expressed in terms of oseenlets, and instead uses an approach based upon that of Imai [10]. Yet the representation by singular (stokeslet, oseenlet) solutions has many advantages, one being that a body can be represented in a straightforward way by a distributed superposition of them [2], and another being the additional insight into the physical understanding of the flow such a model provides. For example, Iima's result is for two- rather than three-dimensional flow and it is difficult to see how the result can be extended by using Iima's formulation. By finding the oseenlets, it is anticipated that this will enable us to investigate the hovering paradox to three-dimensional flow, but this will be left for future work.

In the literature are currently available the steady, transient and oscillatory stokeslet as well as the steady and transient oseenlet. The omission of the oscillatory oseenlet representation within the literature is significant, and restricts the

armoury of techniques to be used on these important problems. In the present paper, we therefore give the oscillatory oseenlet solution which can then be applied to the important problems described above.
In the present paper, we shall give the time-harmonic oscillatory oseenlet representation and the force it generates. Furthermore, we shall show that it reduces to the steady oseenlet and oscillatory stokeslet solutions in appropriate limiting cases. It is noted that a steady streaming velocity perturbation is also expected in applications, but this shall not be detailed as this steady Oseen solution is well known [14]. Also, it is noted that the flow is time-periodic and not time-harmonic (for example in the formulation given by Lighthill [12]), but the time-periodic solution can be expressed as a Fourier series of time-harmonic terms (see for example Iima [9]).

2. THE GOVERNING EQUATIONS

The time-dependent incompressible Navier-Stokes equations are given by

$$\rho\frac{\partial u_j^\dagger}{\partial t}+\rho u_k^\dagger\frac{\partial u_j^\dagger}{\partial x_k}=-\frac{\partial p^\dagger}{\partial x_j}+\mu\frac{\partial^2 u_j^\dagger}{\partial x_k\partial x_k} \tag{1}$$

where: $u_j^\dagger$ is the velocity component in the j direction of a Cartesian coordinate system x_j, $1\le j\le 3$ and similarly $1\le k\le 3$; $p^\dagger$ is the fluid pressure; t denotes time; ρ is the fluid density; and μ is the fluid viscosity. A repeated suffix within a term denotes a summation, so for example $a_jb_j=a_1b_1+a_2b_2+a_3b_3$. For oscillatory flow, the fluid velocity and pressure can be represented by a steady component together with a time-periodic component. Also, in the regions where the Oseen representation is valid, such as the far-field region, the flow tends towards a uniform stream U in the x_1 direction. Applying a Fourier expansion to the time periodic motion then gives the velocity and pressure to be

$$u_j^\dagger=U\delta_{j1}+\sum_{n=-\infty}^{\infty}u_j^n e^{i\omega_n t}$$
$$p^\dagger=\sum_{n=-\infty}^{\infty}p^n e^{i\omega_n t} \tag{2}$$

where i is the imaginary number $\sqrt{-1}$, $\omega_n=2\pi n/T$, $\delta_{jl}=1$ when $j=l$ and zero otherwise, and T is the time period of the motion. Since the velocity and pressure represent real variables, then $u_j^n=\bar{u}_j^{(-n)}$ and $p^n=\bar{p}^{(-n)}$ where the bar denotes the complex conjugate. Equating terms in (1) with the same time-harmonic then gives

$$\rho U\frac{\partial u_j^0}{\partial x_1}=-\frac{\partial p^0}{\partial x_j}+\mu\frac{\partial^2 u_j^0}{\partial x_k\partial x_k} \tag{3}$$

producing the steady Oseen equations related to steady streaming, and a typical harmonic denoted with velocity u_j and pressure p as

$$\rho i\omega u_j+\rho U\frac{\partial u_j}{\partial x_1}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_k\partial x_k}. \tag{4}$$

Thus this equation holds for each n with frequency $\omega=\omega_n$. We now focus on the oscillatory oseenlet solution for this equation. Furthermore, we see that when $U\to 0$ in the above equation we obtain the oscillatory Stokes equation

$$\rho i\omega u_j=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_k\partial x_k}, \tag{5}$$

and when $\omega\to 0$ we obtain the steady Oseen equation

$$\rho U\frac{\partial u_j}{\partial x_1}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_k\partial x_k}. \tag{6}$$

Furthermore, when both $U \to 0$ and $\omega \to 0$, we obtain the steady Stokes equation

$$0 = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_k \partial x_k}. \tag{7}$$

We shall apply both these limits to the oscillatory oseenlets in order to demonstrate they tend to the oscillatory stokeslets and steady oseenlets respectively.

3. GREEN'S FUNCTIONS

We now proceed to determine the Green's functions $(u_j^{(m)}(z), p^{(m)}(z))$ for three solutions $m = 1,2,3$ which we call oscillatory oseenlets centred at the origin of a Cartesian coordinate system $z = (z_1, z_2, z_3)$. These functions satisfy (4), so the oscillatory factor $e^{i\omega t}$ in the Fourier representation has been removed. We know that near $z = 0$, the Green's functions must reduce to the form of the time-oscillatory Stokes' Greens functions, or stokeslets. (This is because the dimensionless length is zRe/a, where Re is the Reynolds number and a is a characteristic length. So, taking the limit $z \to 0$ for the oseenlet solution will give the Stokes flow limit.) This will ensure that the surface integral around the point z reduces to the appropriate velocity term. So we can determine the time-oscillatory oseenlet solution by imposing this limiting property, knowing the form of the oscillatory stokeslets are given by Pozrikidis [15]. The oscillatory stokeslets [15] as well as the steady oseenlets [14] have been given many times in the literature, and we can rewrite them using the potentials χ and ϕ given by Lamb [11], and also χ^* given by Chadwick [5] which is required to agree with the steady oseenlet representation of Oseen [14] and the oscillatory stokeslet representation of Pozrikidis [15]. We note, however, that although the Green's functions can be represented by the functions ϕ, χ and χ^*, the fluid velocity representation given by the Green's integral representation of these Green's functions in general cannot, as shown for example in the steady case by Chadwick [4]. So, the velocity cannot in general be represented by these potentials. The Green's function oseenlets are such that: (1) they are solutions to the oscillatory Oseen equations, (2) the Green's integral round a point in the flow gives the velocity at that point, and (3) the Green's integral in the far-field limit taken to infinity is zero. This is sufficient to then represent the velocity by a Green's integral representation.

3.1 Oscillatory stokeslets

The oscillatory Stokes equations are obtained from the oscillatory Oseen equations (4) by setting the stream velocity U to zero. The oscillatory stokeslets must therefore satisfy the equation

$$\rho i \omega u_j^{s(m)} = -\frac{\partial p^{s(m)}}{\partial z_j} + \mu \frac{\partial^2 u_j^{s(m)}}{\partial z_k \partial z_k}, \tag{8}$$

where the superscript s denotes the Stokes solution. So the three stokeslet solutions have velocity $u^{s(m)}(z)$ and pressure $p^{s(m)}(z)$ where $1 \le m \le 3$. We shall find that the stokeslets have the form

$$u_j^{s(m)} = \frac{\partial \phi^{s(m)}}{\partial z_j} + w_j^{s(m)}. \tag{9}$$

The potential $\phi^{s(m)}$ satisfies Laplace and is associated with flow outside the wake, and the velocity $w_j^{s(m)}$ is associated with the wake velocity. Further, we will try a solution with the form

$$w_j^{s(m)} = \frac{\partial \chi^{s(m)}}{\partial z_j} + \chi^{s*} \delta_{jm} \tag{10}$$

(We note that a term of the type $a^{(m)} b_m$ does not imply a summation; the brackets round the superscript integer are used to imply this difference from standard tensor notation.) Applying the divergence vector operator to the oscillatory Stokes equation (5) shows that the Stokes pressure satisfies the Laplace equation. Furthermore, in the steady case limit $\omega \to 0$, then the solution for the oscillatory stokeslet must tend to the solution for the steady stokeslet. The steady Stokes pressure also satisfies the Laplace equation. So, the solution for the pressure for both steady and oscillatory stokeslets must be the same. The steady stokeslet solution for pressure is given by Oseen [14], and so this is also the oscillatory stokeslet solution for pressure given by

$$p^{s(m)}(z) = \frac{1}{4\pi}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right) \tag{11}$$

where the radial distance from the stokeslet singularity is given by $R = |z|$. The pressure term can now be removed from the oscillatory Stokes equation for the stokeslet by letting

$$i\rho\omega\phi^{s(m)} = -p^{s(m)}, \tag{12}$$

which gives

$$\phi^{s(m)} = \frac{i}{4\pi\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right), \tag{13}$$

and

$$\frac{\partial^2 w_j^{s(m)}}{\partial z_l \partial z_l} - h^2 w_j^{s(m)} = 0, \tag{14}$$

where $h = \sqrt{i\rho\omega/\mu}$. We now use an approach for decomposing the Oseen equations, by introducing the potentials χ^{s*} and $\chi^{s(m)}$ defined above. In order to obtain the steady result, we must choose $\chi^{s(m)}$ such that in the steady limit $\omega \to 0$ the oscillatory dependence disappears and so this term cancels with the potential term $\phi^{s(m)}$. So, we must have

$$\chi^{s(m)} = -\frac{i}{4\pi\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right), \tag{15}$$

and consequently

$$\chi^{s*} = \frac{ih^2}{4\pi\rho\omega}\left(\frac{e^{-hR}}{R}\right). \tag{16}$$

This gives the oscillatory stokeslet solution as

$$u_j^{(m)} = \frac{i}{4\pi\rho\omega}\frac{\partial^2}{\partial z_j z_m}\left(\frac{1}{R}\right) - \frac{i}{4\pi\rho\omega}\frac{\partial^2}{\partial z_j z_m}\left(\frac{e^{-hR}}{R}\right) + \frac{ih^2}{4\pi\rho\omega}\frac{e^{-hR}}{R}\delta_{jm}. \tag{17}$$

Expanding the partial derivatives in (17) and collecting like terms produces the oscillatory stokeslet form given by Pozrikidis [15]. However, the form presented here is much more beneficial for us, as we seek to use (17) in order to infer the form of the oscillatory oseenlets given by the potentials $\phi^{(m)}$, χ^* and $\chi^{(m)}$. It is also noted that letting $h \to 0$ in (17), the steady stokeslet solution is recovered.

3.2 Oscillatory oseenlets

Let us look for oscillatory oseenlets velocity which have the form

$$u_j^{(m)} = \frac{\partial \phi^{(m)}}{\partial z_j} + w_j^{(m)}. \tag{18}$$

Then the pressure p and potential ϕ are related by the equation

$$\rho U\frac{\partial^2 \phi^{(m)}}{\partial z_1 \partial z_j} + \rho i\omega\frac{\partial \phi^{(m)}}{\partial z_j} = -\frac{\partial p^{(m)}}{\partial z_j}, \tag{19}$$

and integrating gives the pressure to be

$$p^{(m)} = -\rho U \frac{\partial \phi^{(m)}}{\partial z_1} - \rho i \omega \phi^{(m)}. \tag{20}$$

Applying the divergence vector to the oscillatory Oseen equation (4) shows that the Oseen pressure satisfies the Laplace equation. Furthermore, in the low Reynolds number limit the oscillatory oseenlet solution for the pressure must tend to the oscillatory stokeslet solution for the pressure. Since they satisfy the same governing (Laplace) equation, then we infer they must be the same. Therefore, the pressure for the oscillatory oseenlet has the same form as that for the oscillatory stokeslet which is given by Pozrikidis [15]

$$p^{(m)}(z) = \frac{1}{4\pi} \frac{\partial}{\partial z_m} \left(\frac{1}{R} \right). \tag{21}$$

From (20), the potential $\phi^{(m)}$ satisfies the equation

$$\rho U \frac{\partial}{\partial z_1} \left(\phi^{(m)} e^{i\omega z_1/U} \right) = -p^{(m)} e^{i\omega z_1/U}. \tag{22}$$

Integrating this equation gives

$$\phi^{(m)} = -\frac{e^{-i\omega z_1/U}}{4\pi\rho U} \int_{-\infty}^{z_1} e^{i\omega z_{1'}/U} \frac{\partial}{\partial z_{m'}} \left(\frac{1}{R'} \right) \mathrm{d}z_{1'}, \tag{23}$$

where $z_{2'} = z_2$, $z_{3'} = z_3$, $z_{1'}$ is the dummy integration variable, and $R' = |z'|$. However, we see that $\phi^{(m)}$ is indeterminate along the semi-infinite half line $z_1 \geq 0$, $z_2 = z_3 = 0$, or more succinctly $R = z_1$. This is because near $z = 0$ the integration of $p^{(m)}$ is unbounded. We may remove the term producing the singularity within the integrand in order to represent $\phi^{(m)}$ by a determinate integral and a singular term, such that

$$\phi^{(m)} = -\frac{e^{-i\omega z_1/U}}{4\pi\rho U} \left\{ \int_{-\infty}^{z_1} \left(e^{i\omega z_{1'}/U} - 1 \right) \frac{\partial}{\partial z_{m'}} \left(\frac{1}{R'} \right) \mathrm{d}z_{1'} - \frac{\partial}{\partial z_m} \ln(R - z_1) \right\}. \tag{24}$$

Substituting these oseenlet pressure and potential terms into the oscillatory Oseen equation (4) gives the wake velocity to be

$$\frac{\partial^2 w_j^{(m)}}{\partial z_l \partial z_l} = \frac{\rho i \omega}{\mu} w_j^{(m)} + 2k \frac{\partial w_j^{(m)}}{\partial z_1} \tag{25}$$

where $k = \rho U/(2\mu)$. Letting $w_j^{(m)} = w_j^{*(m)} e^{kz_1}$ then gives

$$\left(\frac{\partial^2}{\partial z_j \partial z_j} - k^{*2} \right) w_j^{*(m)} = 0 \tag{26}$$

where $k^{*2} = k^2 + \rho i \omega/\mu$. Solutions to this equation are given by [3]. We look for a solution that reduces to the oscillatory stokeslet in the limit as $z \to 0$, given by solutions of the type

$$w_j^{(m)} = \frac{\partial \chi^{(m)}}{\partial z_j} - \chi^* \delta_{jm} \tag{27}$$

where $\chi^* e^{-kz_1}$ satisfies the heat conduction equation (26). We choose a solution for χ^* which is given in [3] such that it reduces to the oscillatory stokeslet in the limit as $z \to 0$. This is given by

$$\chi^* = \frac{2k}{4\pi\rho U} \frac{e^{-k^* R} e^{kz_1}}{R}. \tag{28}$$

From the continuity equation $\frac{\partial w_j^{(m)}}{\partial z_j} = 0$, it follows that $\frac{\partial^2 \chi^{(m)}}{\partial z_j \partial z_j} - \frac{\partial \chi^*}{\partial z_m} = 0$. So from (25),

$$\frac{\rho i\omega}{\mu}\chi^{(m)}+2k\frac{\partial\chi^{(m)}}{\partial z_1}=\frac{\partial^2\chi^{(m)}}{\partial z_j\partial z_j}=\frac{\partial\chi^*}{\partial z_m}. \tag{29}$$

So $\chi^{(m)}$ is given by integrating the equation

$$2k\frac{\partial}{\partial z_1}\left(\chi^{(m)}e^{i\omega z_1/U}\right)=e^{i\omega z_1/U}\frac{\partial\chi^*}{\partial z_m}. \tag{30}$$

However, in a similar way as for $\phi^{(m)}$, integrating this equation directly produces an integral

$$\chi^{(m)}=\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1}e^{i\omega z_{1'}/U}\frac{\partial}{\partial z_{m'}}\left(\frac{e^{-k^*R'}e^{kz_{1'}}}{R'}\right)dz_{1'}, \tag{31}$$

which is indeterminate along $R=z_1$. Removing the singularity term gives a determinate integral together with a singular term

$$\chi^{(m)}=\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\left\{\int_{-\infty}^{z_1}\left[e^{i\omega z_{1'}/U}\frac{\partial}{\partial z_{m'}}\left(\frac{e^{-k^*R'}e^{kz_{1'}}}{R'}\right)-\frac{\partial}{\partial z_{m'}}\left(\frac{e^{-k(R'-z_{1'})}}{R'}\right)\right]dz_{1'}\right.$$
$$\left.-e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\right\}. \tag{32}$$

So the complete solution for the harmonic oscillatory oseenlets is given by

$$u_j^{(m)}=\frac{\partial\phi^{(m)}}{\partial z_j}+\frac{\partial\chi^{(m)}}{\partial z_j}-\chi^*\delta_{jm}$$
$$\phi^{(m)}=-\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1}e^{i\omega z_{1'}/U}\frac{\partial}{\partial z_{m'}}\left(\frac{1}{R'}\right)dz_{1'}$$
$$\chi^{(m)}=\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1}e^{i\omega z_{1'}/U}\frac{\partial}{\partial z_{m'}}\left(\frac{e^{-k^*R'}e^{kz_{1'}}}{R'}\right)dz_{1'}$$
$$\chi^*=\frac{2k}{4\pi\rho U}\frac{e^{-k^*R}e^{kz_1}}{R},\quad p^{(m)}=\frac{1}{4\pi}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right), \tag{33}$$

where $u_j^{(m)}=\frac{\partial\phi^{(m)}}{\partial z_j}+\frac{\partial\chi^{(m)}}{\partial z_j}-\chi^*\delta_{jm}$, $k=\rho U/(2\mu)$, and $k^*=\sqrt{k^2+\rho i\omega/\mu}$. Furthermore, this result checks with two limiting cases: as $\omega\to 0$ it reduces to the steady oseenlets, and as $k\to 0$ it reduces to the oscillatory stokeslets, shown next.

Case $\omega\to 0$

When $\omega\to 0$, then $k^*\to k$, $\phi^{(m)}$ reduces to

$$\phi^{(m)}\Big|_{\omega\to 0}=-\frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_{m'}}\left(\frac{1}{R'}\right)dz_{1'}$$
$$=\frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}\ln(R-z_1) \tag{34}$$

since $1/R=-(\partial/\partial z_1)\ln(R-z_1)$, which is the steady oseenlet solution for $\phi^{(m)}$ [14]. Similarly, $\chi^{(m)}$ reduces to

$$\chi^{(m)}\Big|_{\omega\to 0} = \frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_{m'}}\left(\frac{e^{-k(R'-z_{1'})}}{R'}\right)dz_{1'}$$

$$= -\frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_{m'}}\left(e^{-k(R'-z_{1'})}\frac{\partial}{\partial z_{1'}}\ln(R'-z_{1'})\right)dz_{1'}$$

$$= -\frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_{1'}}\left(e^{-k(R'-z_{1'})}\frac{\partial}{\partial z_{m'}}\ln(R'-z_{1'})\right)dz_{1'}$$

$$= -\frac{1}{4\pi\rho U}e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1) \qquad (35)$$

which is the steady oseenlet solution for $\chi^{(m)}$ [14]. Finally, χ^* reduces to

$$\chi^*\Big|_{\omega\to 0} = \frac{2k}{4\pi\rho U}\frac{e^{-k(R-z_1)}}{R} \qquad (36)$$

which is the steady oseenlet solution for χ^*. Therefore, the oscillatory oseenlet solution reduces to the steady oseenlet solution in the limit as $\omega \to 0$.

Case $k \to 0$

When $k \to 0$, then $k^* \to h$, $\phi^{(m)}$ reduces to

$$\phi^{(m)}\Big|_{k\to 0} == -\lim_{k\to 0}\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1}e^{i\omega z_{1'}/U}\frac{\partial}{\partial z_{m'}}\left(\frac{1}{R'}\right)dz_{1'}$$

$$= -\lim_{k\to 0}\frac{e^{-h^2 z_1/2k}}{4\pi\rho U}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right)\frac{e^{h^2 z_1/2k}}{h^2/2k}$$

$$= \frac{2k}{4\pi\rho U h^2}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right) \qquad (37)$$

which is the oscillatory stokeslet solution for $\phi^{(m)}$. Similarly, $\chi^{(m)}$ reduces to

$$\chi^{(m)}\Big|_{k\to 0} == -\lim_{k\to 0}\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1}e^{i\omega z_{1'}/U}\frac{\partial}{\partial z_{m'}}\left(\frac{e^{-hR'}}{R'}\right)dz_{1'}$$

$$= -\lim_{k\to 0}\frac{e^{-h^2 z_1/2k}}{4\pi\rho U}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right)\frac{e^{h^2 z_1/2k}}{h^2/2k}$$

$$= \frac{2k}{4\pi\rho U h^2}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right) \qquad (38)$$

which is the oscillatory stokeslet solution for $\chi^{(m)}$. Finally, χ^* reduces to

$$\chi^*\Big|_{k\to 0} = \frac{2k}{4\pi\rho U}\frac{e^{-hR}}{R} \qquad (39)$$

which is the oscillatory stokeslet solution for χ^*. Therefore, the oscillatory oseenlet solution reduces to the oscillatory stokeslet solution in the limit as $k \to 0$.

4. GREEN'S INTEGRAL REPRESENTATION

We follow the Green's integral formulation as given by Oseen [14], except apply it to the oscillatory rather than steady or transient case. Consider four solutions for the velocity and pressure field given by $u_j(y)$, $p(y)$ and $u_j^{(m)}(z)$, $p^{(m)}(z)$ where $1 \le m \le 3$. The first solution refers to a general velocity and pressure field, and the subsequent three solutions refer to the specific Green's functions satisfying a Green's integral which we shall construct. We consider distinct Cartesian coordinates y_j and $z_j = x_j - y_j$. The coordinate y parameterises a point on or within a fixed closed surface and the coordinate x shall refer to a general fluid point. The four fluid solutions then satisfy the equations

$$\rho i\omega u_j(y) + \rho U \frac{\partial u_j(y)}{\partial y_1} = -\frac{\partial p(y)}{\partial y_j} + \mu \frac{\partial^2 u_j(y)}{\partial y_l \partial y_l}, \tag{40}$$

and

$$\rho i\omega u_j^{(m)}(z) + \rho U \frac{\partial u_j^{(m)}(z)}{\partial z_1} = -\frac{\partial p^{(m)}(z)}{\partial z_j} + \mu \frac{\partial^2 u_j^{(m)}(z)}{\partial z_l \partial z_l}. \tag{41}$$

Since $z = x - y$, then the adjoint equation in y is satisfied as $\frac{\partial}{\partial z_j} = -\frac{\partial}{\partial y_j}$, which gives

$$\rho i\omega u_j^{(m)}(z) - \rho U \frac{\partial u_j^{(m)}(z)}{\partial y_1} = \frac{\partial p^{(m)}(z)}{\partial y_j} + \mu \frac{\partial^2 u_j^{(m)}(z)}{\partial y_l \partial y_l}. \tag{42}$$

Following the method of Oseen [14] to get Oseen's Greens function representation, we multiply (40) by $u_j^{(m)}$ and take it from the multiplication of (41) by u_j. Applying the continuity equation $\frac{\partial u_j}{\partial y_j} = 0$ then gives

$$\rho U \frac{\partial}{\partial y_1}\left(u_j(y) u_j^{(m)}(z)\right) = -\frac{\partial}{\partial y_j}\left(u_j^{(m)}(z) p(y) + u_j(y) p^{(m)}(z)\right)$$
$$+ \mu \frac{\partial}{\partial y_l}\left(u_j^{(m)}(z) \frac{\partial u_j(y)}{\partial y_l} - u_j(y) \frac{\partial u_j^{(m)}(z)}{\partial y_l}\right). \tag{43}$$

This holds within a volume V of fluid bounded by the surface S where the Oseen approximation is valid, and parameterised by the coordinate system y. We consider the surface S consisting of a surface S_δ of a sphere radius $\delta \to 0$ around the point $z = 0$, a surface S_B enclosing the oscillating body, and a surface S_R of a sphere radius $R \to \infty$ centred at the point $z = 0$, see figure 1.

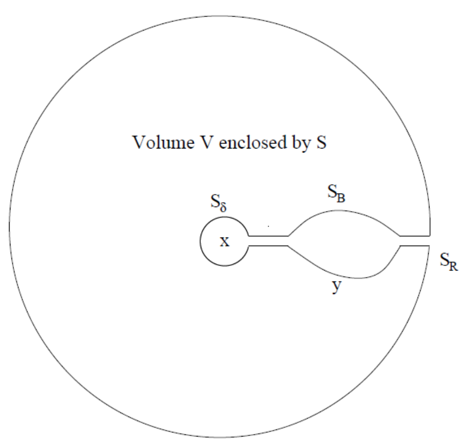

Figure 1: The surface S

Then, applying the divergence theorem and noting that the oscillatory oseenlet approximates to the steady oseenlet close to the point $y = x$, and also noting that the integrand of the Green's integral obtained for the oscillatory Oseen flow case (44) is identical to the integrand of the Green's integral obtained for the steady Oseen flow case, then an identical analysis can be followed for the oscillatory case as for the steady case [14] to obtain the Green's integral representation for the oscillatory Oseen velocity

$$u_m(x) = -\iint_{S_B}\{\rho U u_j(y)u_j^{(m)}(z)n_1 + u_j^{(m)}(z)p(y)n_j + u_j(y)p^{(m)}(z)n_j$$

$$-\mu u_j^{(m)}(z)\frac{\partial u_j(y)}{\partial y_l}n_l + \mu u_j(y)\frac{\partial u_j^{(m)}(z)}{\partial y_l}n_l\}\mathrm{d}S. \tag{44}$$

5. FAR-FIELD OSEEN VELOCITY EXPANSION

We expand the velocity and pressure for the Green's functions by a Taylor series in the integral representation (44). The oseenlet representations for the velocity and pressure expanded by a Taylor series are

$$u_j^{(m)}(z) = u_j^{(m)}(x-y) = u_j^{(m)}(x) - y_k\frac{\partial}{\partial x_k}u_j^{(m)}(x) + (1/2)y_k y_l\frac{\partial^2}{\partial x_k \partial x_l}u_j^{(m)}(x) - ....$$

$$p^{(m)}((z) = p^{(m)}(x-y) = p^{(m)}(x) - y_k\frac{\partial}{\partial x_k}p^{(m)}(x) + (1/2)y_k y_l\frac{\partial^2}{\partial x_k \partial x_l}p^{(m)}(x) - .... \tag{45}$$

Substituting this into the integral representation (44) then gives the velocity expansion

$$u_m = A_j u_j^{(m)} + B_{jk}\frac{\partial}{\partial x_k}u_j^{(m)} + ... \tag{46}$$

where

$$A_j = -\iint_{S_B}\left\{\rho U u_j n_1 + p n_j + i\rho\omega y_j u_k n_k - \mu\frac{\partial u_j}{\partial y_l}\right\}n_l \mathrm{d}S \tag{47}$$

and

$$B_{jk} = -\iint_{S_B}\{-\rho U y_k u_j n_1 - y_k p n_j + \delta_{k1} y_j u_l n_l$$

$$+(1/2)i\rho\omega y_j y_k u_l n_l + \mu y_k\frac{\partial u_j}{\partial y_l}n_l\}\mathrm{d}S. \tag{48}$$

Differentiating the factor $e^{-i\omega x_1/U}$ in the oseenlet velocity terms (33) with respect to x_1 brings down a coefficient factor in ω. So, the expansion (45) will only be convergent when ω is small. In particular, as $\omega \to 0$, we see that the expansion (45) reduces to the far-field Oseen velocity expansion for steady flow [4].

6. FORCES GENERATED BY THE OSEENLETS

Denote the surface of the oscillating body given by $B_t(x,t)$. The force on the body due to the action of the fluid is then

$$F_j = \iint_{B_t}\tau_{jl}^{\dagger}n_l \mathrm{d}S \tag{49}$$

where

$$\tau_{jl}^{\dagger} = -p^{\dagger}\delta_{jl} + \mu\left(\frac{\partial u_j^{\dagger}}{\partial x_l} + \frac{\partial u_l^{\dagger}}{\partial x_j}\right)$$

is the symmetric Navier-Stokes stress tensor. So, the Navier-Stokes equations (1) can be re-written as

$$\rho\frac{\partial u_j^{\dagger}}{\partial t} = \frac{\partial}{\partial x_l}\left(\tau_{jl}^{\dagger} - \rho u_j^{\dagger}u_l^{\dagger}\right), \tag{50}$$

since the continuity equation $\dfrac{\partial u_l^{\dagger}}{\partial x_l} = 0$ holds. Denoting the velocity on the body surface to be u_j^B, then the total force F_j is

$$
\begin{aligned}
F_j &= \iint_{B_t} \{\tau_{jl}^{\dagger} - \rho u_j^{\dagger} u_l^{\dagger} + \rho u_j^B u_l^B\} n_l \mathrm{d}S \\
&= \int_{B_\infty} \{\tau_{jl}^{\dagger} - \rho u_j^{\dagger} u_l^{\dagger}\} n_l \mathrm{d}S - \iiint_V \frac{\partial}{\partial y_l}(\tau_{jl}^{\dagger} - \rho u_j^{\dagger} u_l^{\dagger}) \mathrm{d}V \\
&+ \iint_{B_t} \rho u_j^B u_l^B n_l \mathrm{d}S \\
&= \iint_{B_\infty} \{\tau_{jl}^{\dagger} - \rho u_j^{\dagger} u_l^{\dagger}\} n_l \mathrm{d}S \\
&- \iiint_V \rho \frac{\partial u_j^{\dagger}}{\partial t} \mathrm{d}V + \iint_{B_t} \rho u_j^B u_l^B n_l \mathrm{d}S
\end{aligned} \tag{51}
$$

where B_∞ is an enclosing surface a sufficiently large distance away from the body, and V is the volume of fluid exterior to B_t. On B_∞, assume that the surface is sufficiently far from the disturbance that the Oseen approximation (2)

$$
u_j^{\dagger} = U\delta_{j1} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t}, \quad |u_j^n| < U, \tag{52}
$$

holds. The force then becomes

$$
\begin{aligned}
F_j &= \sum_{n=-\infty}^{\infty} \left[\iint_{B_\infty} \{\tau_{jl}^n n_l - \rho U \delta_{j1} u_l^n n_l - \rho U \delta_{l1} u_j^n n_l\} \mathrm{d}S \right. \\
&\left. - \iiint_V \rho i \omega_n u_j^n \mathrm{d}V \right] e^{i\omega_n t} + \iint_{B_t} \rho u_j^B u_l^B n_l \mathrm{d}S,
\end{aligned} \tag{53}
$$

where we define $\tau_{jl}^n = -p^n \delta_{jl} + \mu\left(\dfrac{\partial u_j^n}{\partial x_l} + \dfrac{\partial u_l^n}{\partial x_j}\right)$, and p^n is the pressure associated with the velocity field u_j^n.

However,

$$
\begin{aligned}
\iiint_V u_j^n \mathrm{d}V &= \iiint_V \frac{\partial}{\partial y_l}(y_j u_l^n) \mathrm{d}V \\
&= \iint_{B_\infty} y_j u_l^n n_l \mathrm{d}S - \iint_{B_t} y_j u_l^n n_l \mathrm{d}S.
\end{aligned} \tag{54}
$$

Also,

$$
\sum_{-\infty}^{\infty} \left(\iint_{B_t} \rho i \omega_n u_l^n y_j n_l \mathrm{d}S \right) e^{i\omega_n t} = \iint_{B_t} \rho y_j \frac{\partial u_l^B}{\partial t} n_l \mathrm{d}S \tag{55}
$$

and $\iint_{B_\infty} u_l n_l \mathrm{d}S = 0$ from the continuity equation requiring not fluid outflow. So we can write

$$
F_j = \sum_{n=-\infty}^{\infty} f_j^n e^{i\omega_n t} + f_j^B \tag{56}
$$

where

$$
f_j^n = \iint_{B_\infty} \{\tau_{jl}^n n_l - \rho U \delta_{l1} u_j^n n_l - \rho i \omega_n y_j u_k^n n_l\} \mathrm{d}S \tag{57}
$$

and

$$f_j^B = \iint_{B_t} \{\rho u_j^B u_l^B n_l + \rho y_j \frac{\partial u_l^B}{\partial t} n_l\} \mathrm{d}S. \tag{58}$$

So let the the integral representation for the force with Fourier component of frequency ω be f_j where

$$f_j = \iint_{B_\infty} \{(-p\delta_{jl} + \mu\left(\frac{\partial u_l}{\partial y_j} + \frac{\partial u_j}{\partial y_l}\right))n_l - \rho U \delta_{l1} u_j n_l - \rho i \omega y_j u_l n_l\} \mathrm{d}S. \tag{59}$$

Comparing with A_j in (46), we see that all the terms are the same except for an additional term $\iint_{B_\infty} \mu \frac{\partial u_l}{\partial y_j} n_l \mathrm{d}S$ in f_j. However, in the limit as $\omega \to 0$, $\iint_{B_\infty} \mu \frac{\partial u_l}{\partial y_j} n_l \mathrm{d}S = 0$, and so for the steady case $A_j = f_j$ as expected from [?]. In order to obtain the force generated by the oscillatory oseenlet we consider an oseenlet $u_j^{(m)}$ inside the body. A sphere S_δ of radius δ, is central at the force point $u_j^{(m)}$ and the volume V_t is bounded by the body surface B_t and S_δ. Substituting an oscillatory oseenlet $u_j^{(m)}$ into the force (54), the volume integral V_t obtained from using the divergence theorem is identically zero, giving an integral over the surface S_δ rather than B_t which is

$$F_j^{(m)} = \sum_{n=-\infty}^{\infty} \left[\iint_{S_\delta} \{\tau_{jl}^{n(m)} n_l - \rho U \delta_{l1} u_j^{n(m)} n_l\} dS \right.$$
$$\left. + \iint_{S_\delta} i\rho\omega_n z_j u_l^{n(m)} \; n_l \; ds \right] e^{i\omega_n t} + \iint_{B_t} \rho y_j \frac{\partial u_l^B}{\partial t} n_l dS + \iint_{B_t} \rho u_j^B u_l^B n_l \;\; dS, \tag{60}$$

where $F_j^{(m)}$ is the force generated by the oscillatory oseenlet, $\tau_{jl}^{n(m)}$ is the oscillatory Oseen stress tensor. Around zero the oscillatory oseenlets approximate to the steady stokeslet;

$$u_j^{(m)} \approx \frac{-1}{8\pi\mu} \{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}, \tag{61}$$

so the second and the third integrals in the force representation (??) can be approximated. Therfore, the force generated by an oscillatory oseenlet is

$$F_j^{(m)} = \sum_{n=-\infty}^{\infty} \delta_{jm} e^{i\omega_n t} + \iint_{B_t} \{\rho y_j \frac{\partial u_l^B}{\partial t} n_l + \rho u_j^B u_l^B n_l\} \;\; dS. \tag{62}$$

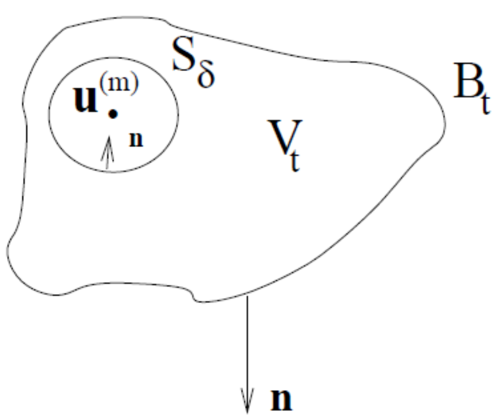

Figure 2: oseenlet inside the oscillatory body

As the limit $\omega \to 0$, $u_j^B = 0$ hence $F_j^{(m)} = \delta_{jm}$ which is the force generated by a steady oseenlet [14]. Also, the equation (62) shows that the force is oscillatory.

7. DISCUSSION

The oscillatory oseenlet and corresponding Green's integral representation have been presented which enables time-periodic Oseen flow to be modelled. A far-field velocity expansion is given, but it is shown that this will only be convergent for small values of ω. In the limit as $\omega \to 0$ it is shown that the far-field steady Oseen velocity expansion of Chadwick [4] is recovered. The force generated by the oscillatory oseenlet is shown itself to be oscillatory, and so any net propulsive force is related to the steady oseenlet only. This completes all the cases for oseenlets and stokeslets in steady, transient and oscillatory flows, the oscillatory oseenlet being the final case not yet present in the literature. This representation can now be used to apply to the variety of important practical problems discussed in the introduction.

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