THE NULLSPACE OF NESTED MAGIC SQUARES

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ABSTRACT
In this paper we formulate the general structure of the nullspace and subspaces of the nullspace for nested magic squares, where we consider two different types of centre. Further, we study the properties of these spaces.

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1. INTRODUCTION

We consider magic squares here as matrices and study the algebraic properties for them. Hence, a semi magic square is a n by n matrix such that the sum of the entries in each row and columns is the same. The common value is called the magic constant. If, in addition, the sum of all entries in each left-broken diagonal and each right-broken diagonal is the magic constant, then we call the matrix a pandiagonal magic square. It is well-known that the following structure

A	B	C	2s–A–B–C
E	2s– A–B–E	A+E – C	B+C – E
s – C	A+B+C – s	s – A	s – B
s–A–E+C	s–B–C+E	s – E	A+B+E–s

is a general structure of the pandiagonal magic square 4 by 4. Here, the magic constant is 2s.

When we want to obtain pandiagonal magic squares 6 by 6, having a similar structure to that of the pandiagonal magic squares 4 by 4, we consider the matrix

a_1	a_2	a_3	a_4	a_5	a_6
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
$s - a_4$	$s - a_5$	$s - a_6$	$s - a_1$	$s - a_2$	$s - a_3$
$s - a_{10}$	$s - a_{11}$	$s - a_{12}$	$s - a_7$	$s - a_8$	$s - a_9$
$s - a_{16}$	$s - a_{17}$	$s - a_{18}$	$s - a_{13}$	$s - a_{14}$	$s - a_{15}$

We then require the following conditions:

$$\begin{aligned}
&a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 3s \\
&a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} = 3s \\
&a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} = 3s \\
&a_1 - a_4 + a_7 - a_{10} + a_{13} - a_{16} = 0 \\
&a_2 - a_5 + a_8 - a_{11} + a_{14} - a_{17} = 0 \\
&a_3 - a_6 + a_9 - a_{12} + a_{15} - a_{18} = 0
\end{aligned}$$

These conditions suffice to ensure that the resulting square is pandiagonal. We have as a possible form of these squares:

J–L –K+J+I+H+E+D+C – 3/2 s	L–J+G–E+B	K–I+F–D+A	–L–K–H–G–F–C–B–A+ 9/2 s	L	K
–J–I–H–G–F+3s	J	I	H	G	F
–E–D–C–B–A+3s	E	D	C	B	A
L+K+H+G+F+C+B+A–7/2 s	s–L	s–K	L+K–J–I–H–E–D–C+ 5/2s	s–L+J–G+E–B	s–K+I–F+D–A
s–H	s–G	s–F	J+I+H+G+F–2s	s–J	s–I
s– C	s–B	s–A	E+D+C+B+A –2s	s–E	s–D

We call this kind of squares the pandiagonal magic square 6 by 6 of the special structure. Note that the magic constant is now 3s.

The nullspace of a matrix A is the solution set of the homogenous system Ax = 0. The pandiagonal magic square 4 by 4 possesses a nontrivial null space. Using the previous notation for a pandiagonal magic square 4 by 4 we can easily prove (see [1]) that this nullspace can be written in the following form

$$\{z\begin{pmatrix} A+2B+C-2s \\ C-A \\ 2s-A-2B-C \\ A-C \end{pmatrix} : z \in R\}$$

Note that the sum of the entries is zero. Furthermore, the multiplication of the square as a matrix with the vector

$$z\begin{pmatrix} A+2B+C-2s \\ C-A \\ 2s-A-2B-C \\ A-C \end{pmatrix} + \frac{q}{2s}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad , \forall q, z \in R$$

yields the vector

$$\begin{pmatrix} q \\ q \\ q \\ q \end{pmatrix}$$

This is due to definition of the magic square. In [1] we find that the pandiagonal magic square 6 by 6 of the special structure possesses also a nontrivial nullspace, which can be written in the following form:

$$\{z\begin{pmatrix} -(DJ - DG + AG + BI - EI + EF - AJ - BF) \\ -(3Is - BI - 2AI - 2IC - EI - 2AH + 2DH + 3As - 3Ds - 3Fs + EF + BF + DJ + 2DF + 2CF - AG - AJ + DG) \\ -(BI - EI - 2EH + 2BH - 3Bs - 3Js + 3Gs + 3Es - AG - EF - 2CG + BF - 2EG + 2BJ + 2JC + DJ - DG + AJ) \\ (DJ - DG + AG + BI - EI + EF - AJ - BF) \\ (3Is - BI - 2AI - 2IC - EI - 2AH + 2DH + 3As - 3Ds - 3Fs + EF + BF + DJ + 2DF + 2CF - AG - AJ + DG) \\ (BI - EI - 2EH + 2BH - 3Bs - 3Js + 3Gs + 3Es - AG - EF - 2CG + BF - 2EG + 2BJ + 2JC + DJ - DG + AJ) \end{pmatrix} : z \in R\}$$

We will therefore denote the vectors in this nullspace as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}$$

In this case the multiplication of the square with the vector

$$z\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} + \frac{q}{3s}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \qquad \forall q, z \in R$$

yields the vector

$$\begin{pmatrix} q \\ q \\ q \\ q \\ q \\ q \end{pmatrix}$$

Nested magic squares

By nested squares we mean matrices having certain properties, which make them and some of their submatrices semi magic squares or pandiagonal magic squares. By a nested semi magic square 6 by 6 with a pandiagonal magic square 4 by 4 we mean a matrix of the form

r_{11}	b_{11}	b_{12}	b_{13}	b_{14}	d_{11}
p_{11}	A	B	C	2s–A–B–C	s – p_{11}
P_{21}	E	2s–A–B–E	A+E–C	B +C – E	s – p_{21}
P_{31}	s– C	A+B+C–s	s–A	s – B	s – p_{31}
P_{41}	s–A–E+C	S–B–C+E	s–E	A+B+E–s	s – p_{41}
e_{11}	s – b_{11}	s – b_{12}	s – b_{13}	s – b_{14}	f_{11}

where we require

$$d_{11} = 3s - r_{11} - b_{11} - b_{12} - b_{13} - b_{14}$$
$$e_{11} = 3s - r_{11} - p_{11} - p_{21} - p_{31} - p_{41} \text{..........} \mathbf{(1)}$$
$$f_{11} = b_{11} + b_{12} + b_{13} + b_{14} - e_{11} - s$$

This ensures that the matrix as a whole is a semi magic square with 3s as a magic constant. In the centre we have a pandiagonal magic square. We introduce also the concept of the multi-nested semi magic square 2n by 2n with a pandiagonal magic square 4 by 4. This will be the following matrix

r_{mm}	$r_{m(m-1)}$	.	.	.	r_{m1}	b_{m1}	b_{m2}	b_{m3}	b_{m4}	d_{m1}	.	.	.	$d_{m(m-1)}$	d_{mm}
$r_{(m-1)m}$	.				.	.	.	.	.	.				.	$s - r_{(m-1)m}$
.		.			.	.	.	.	.	.			.		.
.			.		.	.	.	.	.	.		.			.
r_{2m}	.	.	.	r_{22}	r_{21}	B_{21}	b_{22}	b_{23}	b_{24}	d_{21}	d_{22}	.	.	.	$s - r_{2m}$
r_{1m}	.	.	.	r_{12}	r_{11}	B_{11}	b_{12}	b_{13}	b_{14}	d_{11}	$s - r_{12}$	.	.	.	$s - r_{1m}$
p_{1m}	.	.	.	p_{12}	p_{11}	A	B	C	2s–A–B–C	$s - p_{11}$	$s - p_{12}$	.	.	.	$s - p_{1m}$
P_{2m}	.	.	.	P_{22}	P_{21}	E	2s–A–B–E	A+E – C	B +C – E	$s - P_{21}$	$s - P_{22}$	.	.	.	$s - P_{2m}$
P_{3m}	.	.	.	P_{32}	P_{31}	s – C	–s + A +B+C	s – A	s – B	$s - P_{31}$	$s - P_{32}$	.	.	.	$s - P_{3m}$
P_{4m}	.	.	.	P_{42}	P_{41}	s–A–E+C	s–B–C+E	s – E	A+B+E–s	$s - P_{41}$	$s - P_{42}$	.	.	.	$s - P_{4m}$
e_{1m}	.	.	.	e_{12}	e_{11}	$s - b_{11}$	$s - b_{12}$	$s - b_{13}$	$s - b_{14}$	f_{11}	$s - e_{12}$	.	.	.	$s - e_{1m}$
e_{2m}	.	.	.	e_{22}	$s - r_{21}$	$s - b_{21}$	$s - b_{22}$	$s - b_{23}$	$s - b_{24}$	$s - d_{21}$	f_{22}	.	.	.	$s - e_{2m}$
.			.		.	.	.	.	.	.		.			.
.		.			.	.	.	.	.	.			.		.
$e_{(m-1)m}$	.				.	.	.	.	.	.				.	$s - e_{(m-1)m}$
e_{mm}	$s - r_{m(m-1)}$	.	.	.	$s - r_{m1}$	$s - b_{m1}$	$s - b_{m2}$	$s - b_{m3}$	$s - b_{m4}$	$s - d_{m1}$	.	.	.	$s - d_{m(m-1)}$	f_{mm}

where m= n – 2 and

$$d_{mm} = ns - r_{mm} - r_{m(m-1)} - ... - r_{m1} - b_{m1} - b_{m2} - b_{m3} - b_{m4} - d_{m1} - ... - d_{m(m-1)}$$

$$e_{mm} = ns - r_{mm} - r_{(m-1)m} - ... - r_{1m} - p_{1m} - p_{2m} - p_{3m} - p_{4m} - e_{1m} - ... - e_{(m-1)m}$$

$$f_{mm} = r_{m(m-1)} + ... + r_{m1} + b_{m1} + b_{m2} + b_{m3} + b_{m4} + d_{m1} + ... + d_{m(m-1)} - e_{mm} - (n-2)s$$

Note that we obtain semi magic square each time we remove the outer frame of the square.

By a multi-nested semi magic square 2n by 2n with a pandiagonal magic square 6 by 6 we mean the following matrix:

p_{mm}	$p_{m(m-1)}$	…	P_{m1}	V_{m1}	V_{m2}	V_{m3}	V_{m4}	V_{m5}	V_{m6}	r_{m1}	…	$r_{m(m-1)}$	r_{mm}
$P_{(m-1)m}$	.		.	.	.	.	.	.	.	.		.	$s - p_{(m-1)m}$
.	.		.	.	.	.	.	.	.	.			.
.			.	.	.	.	.	.	.	.			.
.		p_{22}	p_{21}	v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	r_{21}	r_{22}		.
P_{1m}	…	p_{12}	p_{11}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	r_{11}	$s - p_{12}$	…	$s - p_{1m}$
u_{1m}	…	u_{12}	u_{11}	$-L-K+J+I+H+E+D+C-3/2s$	$L-J+G-E+B$	$K-I+F-D+A$	$-L-K-H-G-F-C-B-A+9/2s$	L	K	$s-u_{11}$	$s-u_{12}$	…	$s-u_{1m}$
u_{2m}	…	u_{22}	u_{21}	$-J-I-H-G-F+3s$	J	I	H	G	F	$s-u_{21}$	$s-u_{22}$	…	$s-u_{2m}$
u_{3m}	…	u_{32}	u_{31}	$-E-D-C-B-A+3s$	E	D	C	B	A	$s-u_{31}$	$s-u_{32}$	…	$s-u_{3m}$
u_{4m}	…	u_{42}	u_{41}	$L+K+H+G+F+C+B+A-7/2\,s$	$s-L$	$s-K$	$L+K-J-I-H-E-D-C+5/2s$	$s-L-J-G+E-B$	$s-K+I-F+D-A$	$s-u_{41}$	$s-u_{42}$	…	$s-u_{4m}$
u_{5m}	…	u_{52}	u_{51}	$s-H$	$s-G$	$s-F$	$J+I+H+G+F-2s$	$s-J$	$s-I$	$s-u_{51}$	$s-u_{52}$	…	$s-u_{5m}$
u_{6m}	…	u_{62}	u_{61}	$s-C$	$s-B$	$s-A$	$E+D+C+B+A-2s$	$s-E$	$s-D$	$s-u_{61}$	$s-u_{62}$	…	$s-u_{6m}$
w_{1m}	...	w_{12}	w_{11}	$s-v_{11}$	$s-v_{12}$	$s-v_{13}$	$s-v_{14}$	$s-v_{15}$	$s-v_{16}$	t_{11}	$s-w_{12}$	…	$s-w_{1m}$
.		w_{22}	$s-p_{21}$	$s-v_{21}$	$s-v_{22}$	$s-v_{23}$	$s-v_{24}$	$s-v_{25}$	$s-v_{26}$	$s-r_{21}$	t_{22}		.
.			.	.	.	.	.	.	.	.			.
.			.	.	.	.	.	.	.	.			.
$w_{(m-1)m}$	.		.	.	.	.	.	.	.	.		.	$s-w_{(m-1)m}$
w_{mm}	$s-p_{m(m-1)}$	.	$s-p_{m1}$	$s-v_{m1}$	$s-v_{m2}$	$s-v_{m3}$	$s-v_{m4}$	$s-v_{m5}$	$s-v_{m6}$	$s-r_{m1}$	.	$s-r_{m(m-1)}$	t_{mm}

where m = n – 2,

$$r_{mm} = ns - p_{mm} - p_{m(m-1)} - ... - p_{m1} - v_{m1} - v_{m2} - v_{m3} - v_{m4} - v_{m5} - v_{m6} - r_{m1} - ... - r_{m(m-1)},$$

$$w_{mm} = ns - p_{mm} - p_{(m-1)m} - ... - p_{1m} - u_{1m} - u_{2m} - u_{3m} - u_{4m} - u_{5m} - u_{6m} - w_{1m} - ... - w_{(m-1)m},$$

$$t_{mm} = p_{m(m-1)} + ... + p_{m1} + v_{m1} + v_{m2} + v_{m3} + v_{m4} + v_{m5} + v_{m6} + r_{m1} + ... + r_{m(m-1)} - w_{mm} - (n-2)s.$$

For example, we define the nested semi magic square 8 by 8 with a 6 by 6 pandiagonal square as the following square:

p_{11}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	r_{11}
u_{11}	– L –K + J +I +H + E + D + C – 3/2 s	L – J + G – E + B	K–I–D+F+A	–L –K –H –G –F –C –B –A + 9/2 s	L	K	s – u_{11}
u_{21}	–J –I –H –G–F+ 3s	J	I	H	G	F	s – u_{21}
u_{31}	–E –D –C –B–A+ 3s	E	D	C	B	A	s – u_{31}
u_{41}	L + K + H + G + F + C + B + A – 7/2 s	s – L	s –K	L + K – J – I –H –E –D – C + 5/2s	s – L +J – G+ E –B	s – K – F +D – A +I	s – u_{41}
u_{51}	s – H	s – G	s – F	J + I + H + G +F – 2s	s – J	s – I	s – u_{51}
u_{61}	s –C	s – B	s – A	E+D+C + B+A –2s	s – E	s – D	s – u_{61}
w_{11}	s – v_{11}	s – v_{12}	s – v_{13}	s–v_{14}	s–v_{15}	s – v_{16}	t_{11}

where

$$r_{11} = 4s - p_{11} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16}$$
$$w_{11} = 4s - p_{11} - u_{11} - u_{21} - u_{31} - u_{41} - u_{51} - u_{61}$$
$$t_{11} = v_{11} + v_{12} + v_{13} + v_{14} + v_{15} + v_{16} - w_{11} - 2s$$

2. MAIN RESULTS

We will prove several statements about the nullspace of the mentioned nested magic squares. By doing this we will use mathematical induction. Hence, some results will be used for proving others. We start with

Proposition 1: The nested semi magic square 6 by 6 with a pandiagonal magic square 4 by 4 has the following space

$$\{z\begin{pmatrix} \alpha_1 \\ A+2B+C-2s-\frac{\alpha_1}{2} \\ C-A-\frac{\alpha_1}{2} \\ 2s-A-2B-C-\frac{\alpha_1}{2} \\ A-C-\frac{\alpha_1}{2} \\ \alpha_1 \end{pmatrix} : z \in R\}, \alpha_1 \in R$$

as a subspace of the nullspace of this square.

Proof: We search for real numbers $a_1, ..., a_6$ such that

$$a_1 \times C_1 + ... + a_6 \times C_6 = 0$$

where the capital letters denote the columns of the matrix. We set

$a_1 = a_6$ and $q = -sa_1$. We set

$$\begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = z \begin{pmatrix} A+2B+C-2s \\ C-A \\ 2s-A-2B-C \\ A-C \end{pmatrix} + \frac{q}{2s} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots \ (2)$$

where z is a unknown real number. Now, the expression

$$a_1 \times C_1 + \ldots + a_6 \times C_6$$

is the result of matrix multiplication of the square with the vector

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

The result of this operation will be according to our choice a vector which has the following middle entries

$$\begin{pmatrix} p_{11}a_1 \\ p_{21}a_1 \\ p_{31}a_1 \\ p_{41}a_1 \end{pmatrix} + \begin{pmatrix} -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \end{pmatrix} + \begin{pmatrix} sa_1 - p_{11}a_1 \\ sa_1 - p_{21}a_1 \\ sa_1 - p_{31}a_1 \\ sa_1 - p_{41}a_1 \end{pmatrix}$$

Hence, the middle entries are zero and we have to equate the first and last entry of the vector to zero. We will use this requirement to determine the values of a_1 and z. Thus, we solve the following two equations:

$$r_{11}a_1 + b_{11}a_2 + b_{12}a_3 + b_{13}a_4 + b_{14}a_5 + d_{11}a_1 = 0$$

$$e_{11}a_1 + (s-b_{11})a_2 + (s-b_{12})a_3 + (s-b_{13})a_4 + (s-b_{14})a_5 + f_{11}a_1 = 0$$

This system can be simplified to

$$(r_{11}+d_{11})a_1 + b_{11}a_2 + b_{12}a_3 + b_{13}a_4 + b_{14}a_5 = 0 \quad \ldots\ldots\ldots\ldots \ (3)$$

$$(e_{11}+f_{11})a_1 + s(a_2+a_3+a_4+a_5) - b_{11}a_2 - b_{12}a_3 - b_{13}a_4 - b_{14}a_5 = 0$$

According to the definition of the variables (see (1) and (2)) we have

$$r_{11} + d_{11} = 3s - b_{11} - b_{12} - b_{13} - b_{14}$$

$$e_{11} + f_{11} = b_{11} + b_{12} + b_{13} + b_{14} - s$$

$$a_2 + a_3 + a_4 + a_5 = \frac{2q}{s} = -2a_1$$

Note that we used the sign properties of the vectors belonging to the nullspace of the 4 by 4 square. Upon using the last relations we convert the system (3) into

$$(3s - b_{11} - b_{12} - b_{13} - b_{14})a_1 + b_{11}a_2 + b_{12}a_3 + b_{13}a_4 + b_{14}a_5 = 0$$

$$(b_{11} + b_{12} + b_{13} + b_{14} - 3s)a_1 - b_{11}a_2 - b_{12}a_3 - b_{13}a_4 - b_{14}a_5 = 0$$

We recognize that one equation is redundant. We substitute the values of a_2, a_3, a_4 and a_5. Then, we obtain from the last equation

$$\frac{3}{2}(2s - b_{11} - b_{12} - b_{13} - b_{14})a_1 + (b_{11}(A+2B+C-2s) + b_{12}(C-A) + b_{13}(2s-A-2B-C) + b_{14}(A-C))\,z = 0$$

If the coefficient of a_1 is not zero, then we conclude that

$$a_1 = \alpha_1 z$$

Upon substituting this value for a_1 we are done with the proof. If the coefficient of a_1 is zero, then we conclude that a_1 can be any number, while z might be zero. In this case we can say

$$a_1 = \alpha_1 \mathrm{u}, \quad \alpha_1 = 1, \quad \mathrm{u} \in R$$

and the proof is done with u instead of z.

Remark: 1) We note that the sum of all entries of any vector in the subspace is zero.

2) When we consider the following numerical example for a nested semi magic square 6 by 6

9	7	4	1	8	–26
5	2	3	0	–3	– 4
2	6	–9	8	–3	–1
0	1	4	–1	–2	1
3	–7	4	–5	10	–2
–16	–6	–3	0	–7	35

We obtain the following reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -\frac{35}{11} \\ 0 & 0 & 1 & 0 & 0 & \frac{19}{11} \\ 0 & 0 & 0 & 1 & 0 & \frac{46}{11} \\ 0 & 0 & 0 & 0 & 1 & -\frac{8}{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the nullity of the square is one. Therefore, the described subspace in proposition 1 is actually the nullspace of the square.

Proposition 2: The multi-nested semi magic square 2n by 2n with a pandiagonal magic square 4 by 4 has the following space

$$\{z\begin{pmatrix} \alpha n-2 \\ \alpha n-3-\frac{\alpha n-2}{n-1} \\ \alpha n-4-\frac{\alpha n-3}{n-2}-\frac{\alpha n-2}{n-1} \\ . \\ . \\ . \\ \alpha 1-\frac{\alpha 2}{3}-\frac{\alpha 3}{4}-...-\frac{\alpha n-2}{n-1} \\ A+2B+C-2s-\frac{\alpha 1}{2}-...-\frac{\alpha n-2}{n-1} \\ C-A-\frac{\alpha 1}{2}-...-\frac{\alpha n-2}{n-1} \\ 2s-A-2B-C-\frac{\alpha 1}{2}-...-\frac{\alpha n-2}{n-1} \\ A-C-\frac{\alpha 1}{2}-...-\frac{\alpha n-2}{n-1} \\ \alpha 1-\frac{\alpha 2}{3}-\frac{\alpha 3}{4}-...-\frac{\alpha n-2}{n-1} \\ . \\ . \\ . \\ \alpha n-4-\frac{\alpha n-3}{n-2}-\frac{\alpha n-2}{n-1} \\ \alpha n-3-\frac{\alpha n-2}{n-1} \\ \alpha n-2 \end{pmatrix} : z \in R\}$$

for all $n \geq 3$ as a subspace of the nullspace of this square.

Proof: We will use mathematical induction in our proof. We start with

Basis step: when $n = 3$ we obtain the semi magic square 6 by 6, which has according to proposition 1 a subspace as claimed.
We continue now with

Induction step: we suppose that the given form of the nullspace is true for $n = k$, i. e. there exists a subspace of the nullspace of any multi-nested square $2k$ by $2k$ with a pandiagonal magic square 4 by 4, which has the following structure

$$\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_k \\ x_{k+1} \\ . \\ . \\ . \\ x_{2k} \end{pmatrix} = z \begin{pmatrix} \alpha_{k-2} \\ \alpha_{k-3} - \frac{\alpha_{k-2}}{k-1} \\ \alpha_{k-4} - \frac{\alpha_{k-3}}{k-2} - \frac{\alpha_{k-2}}{k-1} \\ . \\ . \\ . \\ \alpha_1 - \frac{\alpha_2}{3} - \frac{\alpha_3}{4} - ... - \frac{\alpha_{k-2}}{k-1} \\ A + 2B + C - 2s - \frac{\alpha_1}{2} - ... - \frac{\alpha_{k-2}}{k-1} \\ C - A - \frac{\alpha_1}{2} - ... - \frac{\alpha_{k-2}}{k-1} \\ 2s - A - 2B - C - \frac{\alpha_1}{2} - ... - \frac{\alpha_{k-2}}{k-1} \\ A - C - \frac{\alpha_1}{2} - ... - \frac{\alpha_{k-2}}{k-1} \\ \alpha_1 - \frac{\alpha_2}{3} - \frac{\alpha_3}{4} - ... - \frac{\alpha_{k-2}}{k-1} \\ . \\ . \\ . \\ \alpha_{k-4} - \frac{\alpha_{k-3}}{k-2} - \frac{\alpha_{k-2}}{k-1} \\ \alpha_{k-3} - \frac{\alpha_{k-2}}{k-1} \\ \alpha_{k-2} \end{pmatrix}, \quad z \in R$$

We construct now a subspace of the nullspace of the multi-nested square $2k+2$ by $2k+2$. We search for $a_1, ..., a_{2k+2}$ such that the following relation holds

$$a_1 \times C_1 + ... + a_{2k+2} \times C_{2k+2} = 0$$

where the capital letters denote the columns of the square. We choose $a_1 = a_{2k+2}$, $q = -sa_1$. We choose further

$$\begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ . \\ . \\ . \\ a_{2k+1} \end{pmatrix} = z \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_k \\ x_{k+1} \\ . \\ . \\ . \\ x_{2k} \end{pmatrix} + \frac{q}{ks} \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad ......(4)$$

Since the statement of the proposition holds for $n = k$, we conclude that the multiplication of the multi-nested $2k$ by $2k$ square inside the multi-nested square $2k+2$ by $2k+2$ by this vector yields a vector, where each entry is equal to q. Hence, the multiplication of the multi-nested square $2k+2$ by $2k+2$ by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_{2k+2} \end{pmatrix}$$

yields

$$\begin{pmatrix} r_{m(m+1)}a_1 \\ r_{(m-1)(m+1)}a_1 \\ . \\ . \\ . \\ r_{2(m+1)}a_1 \\ r_{1(m+1)}a_1 \\ p_{1(m+1)}a_1 \\ p_{2(m+1)}a_1 \\ p_{3(m+1)}a_1 \\ p_{4(m+1)}a_1 \\ e_{1(m+1)}a_1 \\ e_{2(m+1)}a_1 \\ . \\ . \\ . \\ e_{m(m+1)}a_1 \end{pmatrix} + \begin{pmatrix} -sa_1 \\ -sa_1 \\ . \\ . \\ . \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ . \\ . \\ . \\ -sa_1 \end{pmatrix} + \begin{pmatrix} sa_1 - r_{m(m+1)}a_1 \\ sa_1 - r_{(m-1)(m+1)}a_1 \\ . \\ . \\ . \\ sa_1 - r_{2(m+1)}a_1 \\ sa_1 - r_{1(m+1)}a_1 \\ sa_1 - p_{1(m+1)}a_1 \\ sa_1 - p_{2(m+1)}a_1 \\ sa_1 - p_{3(m+1)}a_1 \\ sa_1 - p_{4(m+1)}a_1 \\ sa_1 - e_{1(m+1)}a_1 \\ sa_1 - e_{2(m+1)}a_1 \\ . \\ . \\ . \\ sa_1 - e_{m(m+1)}a_1 \end{pmatrix}$$

Therefore, all the entries are zero except the first and last entry. We will use this requirement to determine z and a_1. We obtain therefore the following equations

$$r_{(m+1)(m+1)}a_1 + r_{(m+1)m}a_2 + \ldots + r_{(m+1)1}a_{k-1} + b_{(m+1)1}a_k + b_{(m+1)2}a_{k+1} + b_{(m+1)3}a_{k+2}$$
$$+ b_{(m+1)4}a_{k+3} + d_{(m+1)1}a_{k+4} + \ldots + d_{(m+1)(m+1)}a_1 = 0$$
$$e_{(m+1)(m+1)}a_1 + (s - r_{(m+1)m})a_2 + \ldots + (s - r_{(m+1)1})a_{k-1} + (s - b_{(m+1)1})a_k + (s - b_{(m+1)2})a_{k+1}$$
$$+ (s - b_{(m+1)3})a_{k+2} + (s - b_{(m+1)4})a_{k+3} + (s - d_{(m+1)1})a_{k+4} + \ldots + f_{(m+1)(m+1)}a_1 = 0$$

This linear system can be simplified to

$$(r_{(m+1)(m+1)} + d_{(m+1)(m+1)})a_1 + r_{(m+1)m}a_2 + \ldots + r_{(m+1)1}a_{k-1} + b_{(m+1)1}a_k + b_{(m+1)2}a_{k+1} + b_{(m+1)3}a_{k+2}$$
$$+ b_{(m+1)4}a_{k+3} + d_{(m+1)1}a_{k+4} + \ldots + d_{(m+1)m}a_{2k+1} = 0$$

$$(e_{(m+1)(m+1)} + f_{(m+1)(m+1)})a_1 + s(a_2 + a_3 + ... + a_{2k+1}) - r_{(m+1)m}a_2 - ... - r_{(m+1)1}a_{k-1} - b_{(m+1)1}a_k$$
$$- b_{(m+1)2}a_{k+1} - b_{(m+1)3}a_{k+2} - b_{(m+1)4}a_{k+3} - d_{(m+1)1}a_{k+4} - ... - d_{(m+1)m}a_{2k+1} = 0$$

According to the definition of our variables (see (4)) we have

$$r_{(m+1)(m+1)} + d_{(m+1)(m+1)} = (k+1)s - r_{(m+1)m} - r_{(m+1)(m-1)} - ... - r_{(m+1)1} - b_{(m+1)1}$$
$$- b_{(m+1)2} - b_{(m+1)3} - b_{(m+1)4} - d_{(m+1)1} - ... - d_{(m+1)m}$$
$$e_{(m+1)(m+1)} + f_{(m+1)(m+1)} = r_{(m+1)m} + ... + r_{(m+1)1} + b_{(m+1)1} + b_{(m+1)2} + b_{(m+1)3} + b_{(m+1)4}$$
$$+ d_{(m+1)1} + ... + d_{(m+1)m} - (k-1)s$$
$$a_2 + a_3 + ... + a_{2k+1} = \frac{2q}{s} = -2a_1$$

According to the definition of $f_{(m+1)(m+1)}$ and $d_{(m+1)(m+1)}$ we obtain the following linear system

$$((k+1)s - r_{(m+1)m} - ... - r_{(m+1)1} - b_{(m+1)1} - b_{(m+1)2} - b_{(m+1)3} - b_{(m+1)4} - d_{(m+1)1} - ... - d_{(m+1)m})a_1$$
$$+ r_{(m+1)m}a_2 + ... + r_{(m+1)1}a_{k-1} + b_{(m+1)1}a_k + b_{(m+1)2}a_{k+1} + b_{(m+1)3}a_{k+2} + b_{(m+1)4}a_{k+3}$$
$$+ d_{(m+1)1}a_{k+4} + ... + d_{(m+1)m}a_{2k+1} = 0$$
$$(r_{(m+1)m} + ... + r_{(m+1)1} + b_{(m+1)1} + b_{(m+1)2} + b_{(m+1)3} + b_{(m+1)4} + d_{(m+1)1} + ... + d_{(m+1)m} - (k+1)s)a_1$$
$$- r_{(m+1)m}a_2 - ... - r_{(m+1)1}a_{k-1} - b_{(m+1)1}a_k - b_{(m+1)2}a_{k+1} - b_{(m+1)3}a_{k+2} - b_{(m+1)4}a_{k+3}$$
$$- d_{(m+1)1}a_{k+4} - ... - d_{(m+1)m}a_{2k+1} = 0$$

We recognize that one equation is redundant. We substitute the values of a_2, …, a_{2k+1} and obtain the following equation:

$$\frac{k+1}{k}(ks - r_{(m+1)m} - ... - r_{(m+1)1} - b_{(m+1)1} - b_{(m+1)2} - b_{(m+1)3} - b_{(m+1)4} - d_{(m+1)1} - ... - d_{(m+1)m})a_1$$
$$+ (r_{(m+1)m}x_1 + ... + r_{(m+1)1}x_{k-2} + b_{(m+1)1}x_{k-1} + b_{(m+1)2}x_k + b_{(m+1)3}x_{k+1} + b_{(m+1)4}x_{k+2} +$$
$$d_{(m+1)1}x_{k+3} + ... + d_{(m+1)m}x_{2k})z = 0$$

We can say using a similar argument to the one used in the proof of proposition 1

$$a_1 = \alpha_{k-1}z$$

In analogy to that proof we are done.

Remark: We note that the sum of all entries of any vector in the subspace is zero.

We turn our attention now to the squares, which have a pandiagonal magic square 6 by 6 of the special structure in the centre. We can prove very similar results for this type of squares like the results for multi-nested squares with 4 by 4 square in the centre.

Proposition 3: The nested semi magic square 8 by 8 with a 6 by 6 pandiagonal square has the following space

$$\{z\begin{pmatrix}\beta_1\\ x_1-\frac{\beta_1}{3}\\ x_2-\frac{\beta_1}{3}\\ x_3-\frac{\beta_1}{3}\\ -x_1-\frac{\beta_1}{3}\\ -x_2-\frac{\beta_1}{3}\\ -x_3-\frac{\beta_1}{3}\\ \beta_1\end{pmatrix} : z\in R\}$$

as a subspace of the nullspace of the square.

Proof: We search for real numbers $a_1,\ldots,a_8$ such that

$$a_1\times C_1+\ldots+a_8\times C_8=0$$

where the capital letters denote the columns of the matrix. We set $a_1=a_8$ and $q=-sa_1$. We set

$$\begin{pmatrix}a_2\\a_3\\a_4\\a_5\\a_6\\a_7\end{pmatrix}=z\begin{pmatrix}x_1\\x_2\\x_3\\-x_1\\-x_2\\-x_3\end{pmatrix}+\frac{q}{3s}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix},z\in R$$

where z is an unknown real number. Now, the expression

$$a_1\times C_1+\ldots+a_8\times C_8$$

is the result of matrix multiplication of the square with the vector

$$\begin{pmatrix}a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\end{pmatrix}$$

The result of this operation will be according to our choice a vector which has the following middle entries

$$\begin{pmatrix}u_{11}a_1\\u_{21}a_1\\u_{31}a_1\\u_{41}a_1\\u_{51}a_1\\u_{61}a_1\end{pmatrix}+\begin{pmatrix}-sa_1\\-sa_1\\-sa_1\\-sa_1\\-sa_1\\-sa_1\end{pmatrix}+\begin{pmatrix}sa_1-u_{11}a_1\\sa_1-u_{21}a_1\\sa_1-u_{31}a_1\\sa_1-u_{41}a_1\\sa_1-u_{51}a_1\\sa_1-u_{61}a_1\end{pmatrix}$$

Hence, the middle entries are zero and we have to equate the first and last entry of the vector to zero. We will use this requirement to determine the values of a_1 and z. Thus, we solve the following two equations:

$$p_{11}a_1+v_{11}a_2+v_{12}a_3+v_{13}a_4+v_{14}a_5+v_{15}a_6+v_{16}a_7+r_{11}a_1=0$$

$$w_{11}a_1 + (s - v_{11})a_2 + (s - v_{12})a_3 + (s - v_{13})a_4 + (s - v_{14})a_5 + (s - v_{15})a_6 + (s - v_{16})a_7 + t_{11}a_1 = 0$$

This system can be simplified to

$$(p_{11}+r_{11})a_1 + v_{11}a_2 + v_{12}a_3 + v_{13}a_4 + v_{14}a_5 + v_{15}a_6 + v_{16}a_7 = 0$$
$$(w_{11} + t_{11})a_1 + s(a_2 + a_3 + a_4 + a_5 + a_6 + a_7) - v_{11}a_2 - v_{12}a_3 - v_{13}a_4 - v_{14}a_5 - v_{15}a_6 - v_{16}a_7 = 0$$

According to the definition of the variables we have

$$p_{11} + r_{11} = 4s - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16}$$
$$w_{11} + t_{11} = v_{11} + v_{12} + v_{13} + v_{14} + v_{15} + v_{16} - 2s$$
$$a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = \frac{2q}{s} = -2a_1$$

Note that we used the sign properties of the vectors belonging to the nullspace of the 6 by 6 square. Upon using the last relations we convert the system into

$$(4s - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16})a_1 + v_{11}a_2 + v_{12}a_3 + v_{13}a_4 + v_{14}a_5 + v_{15}a_6 + v_{16}a_7 = 0$$
$$(v_{11} + v_{12} + v_{13} + v_{14} + v_{15} + v_{16} - 4s)a_1 - v_{11}a_2 - v_{12}a_3 - v_{13}a_4 - v_{14}a_5 - v_{15}a_6 - v_{16}a_7 = 0$$

We recognize that one equation is redundant. We substitute the values of a_2, a_3, a_4, a_5, a_6 and a_7. Then, we obtain from the last equation

$$\frac{4}{3}(3s - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16})a_1 + (v_{11}x_1 + v_{12}x_2 + v_{13}x_3 - v_{14}x_1 - v_{15}x_2 - v_{16}x_3)z = 0$$

If the coefficient of a_1 is not zero, then we conclude that

$$a_1 = \beta_1 z$$

Upon substituting this value for a_1 we are done with the proof. If the coefficient of a_1 is zero, then we conclude that a_1 can be any number, while z might be zero. In this case we can say

$$a_1 = \beta_1 \mathrm{u}, \quad \alpha_1 = 1, \quad \mathrm{u} \in R$$

and we have the proof done with u instead of z.

Remark: 1) We note that the sum of all entries of any vector in the subspace is zero.
2) When we consider the following numerical example for a nested semi magic square 8 by 8

8	2	4	0	1	3	7	–17
6	1	2	0	4	1	–2	– 4
2	0	0	1	1	0	4	0
5	1	3	1	–3	4	0	–3
3	–2	1	4	1	0	2	–1
0	1	2	–2	2	2	1	2
1	5	–2	2	1	–1	1	1
–17	0	–2	2	1	–1	–5	30

We obtain the following reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{37}{21} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{463}{63} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{65}{63} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{17}{7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{505}{63} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{23}{63} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the nullity of the square is one. Therefore, the described subspace in proposition 3 is actually the nullspace of the square.

<u>Proposition 4</u>: The nested semi magic square 2n by 2n with a 6 by 6 pandiagonal square has the following space

$$\{z\begin{pmatrix} \beta_{n-3} \\ \beta_{n-4} - \frac{\beta_{n-3}}{n-1} \\ \beta_{n-5} - \frac{\beta_{n-4}}{n-2} - \frac{\beta_{n-3}}{n-1} \\ . \\ . \\ . \\ \beta_1 - \frac{\beta_2}{4} - \frac{\beta_3}{5} - \ldots - \frac{\beta_{n-3}}{n-1} \\ x_1 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - \ldots - \frac{\beta_{n-3}}{n-1} \\ x_2 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - \ldots - \frac{\beta_{n-3}}{n-1} \\ x_3 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - \ldots - \frac{\beta_{n-3}}{n-1} \\ -x_1 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - \ldots - \frac{\beta_{n-3}}{n-1} \\ -x_2 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - \ldots - \frac{\beta_{n-3}}{n-1} \\ .-x_3 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - \ldots - \frac{\beta_{n-3}}{n-1} \\ \beta_1 - \frac{\beta_2}{4} - \frac{\beta_3}{5} - \ldots - \frac{\beta_{n-3}}{n-1} \\ . \\ . \\ . \\ \beta_{n-5} - \frac{\beta_{n-4}}{n-2} - \frac{\beta_{n-3}}{n-1} \\ \beta_{n-4} - \frac{\beta_{n-3}}{n-1} \\ \beta_{n-3} \end{pmatrix} : z \in R\}$$

as a subspace of the nullspace of the square for all $n \geq 4.$

Proof: We will use mathematical induction in our proof. We start with

Basis step: when $n = 4$ we obtain the semi magic square 8 by 8, which has according to proposition 3 a subspace as claimed.

We continue now with

Induction step: we suppose that the given form of the nullspace is true for $n = k$, i.e. there exists a subspace of the nullspace of the multi-nested square $2k$ by $2k$, which has the following structure

$$\begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_k \\ y_{k+1} \\ . \\ . \\ . \\ y_{2k} \end{pmatrix} = z \begin{pmatrix} \beta_{k-3} \\ \beta_{k-4} - \frac{\beta_{k-3}}{k-1} \\ \beta_{k-5} - \frac{\beta_{k-4}}{k-2} - \frac{\beta_{k-3}}{k-1} \\ . \\ . \\ . \\ \beta_1 - \frac{\beta_2}{4} - \frac{\beta_3}{5} - ... - \frac{\beta_{k-3}}{k-1} \\ x_1 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - ... - \frac{\beta_{k-3}}{k-1} \\ x_2 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - ... - \frac{\beta_{k-3}}{k-1} \\ x_3 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - ... - \frac{\beta_{k-3}}{k-1} \\ -x_1 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - ... - \frac{\beta_{k-3}}{k-1} \\ -x_2 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - ... - \frac{\beta_{k-3}}{k-1} \\ .-x_3 - \frac{\beta_1}{3} - \frac{\beta_2}{4} - ... - \frac{\beta_{k-3}}{k-1} \\ \beta_1 - \frac{\beta_2}{4} - \frac{\beta_3}{5} - ... - \frac{\beta_{k-3}}{k-1} \\ . \\ . \\ . \\ \beta_{k-5} - \frac{\beta_{k-4}}{k-2} - \frac{\beta_{k-3}}{k-1} \\ \beta_{k-4} - \frac{\beta_{k-3}}{k-1} \\ \beta_{k-3} \end{pmatrix} , z \in R$$

We construct now a subspace of the nullspace of the multi-nested square $2k+2$ by $2k+2$. We search for $a_1,...,a_{2k+2}$ such that the following relation holds

$$a_1 \times C_1 + ... + a_{2k+2} \times C_{2k+2} = 0$$

where the capital letters denote the columns of the square. We choose $a_1 = a_{2k+2}$, $q = -sa_1$. We choose further

$$\begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ . \\ . \\ . \\ a_{2k+1} \end{pmatrix} = z \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_k \\ y_{k+1} \\ . \\ . \\ . \\ y_{2k} \end{pmatrix} + \frac{q}{ks} \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Since the statement of the proposition holds for $n = k$ we conclude that the multiplication of the multi-nested $2k$ by $2k$ square inside the multi-nested square $2k+2$ by $2k+2$ by this vector yields a vector, where each entry is equal to q. Hence, the multiplication of the multi-nested square $2k+2$ by $2k+2$ by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_{2k+2} \end{pmatrix}$$

yields

$$\begin{pmatrix} p_{m(m+1)}a_1 \\ p_{(m-1)(m+1)}a_1 \\ . \\ . \\ . \\ p_{1(m+1)}a_1 \\ u_{1(m+1)}a_1 \\ u_{2(m+1)}a_1 \\ u_{3(m+1)}a_1 \\ u_{4(m+1)}a_1 \\ u_{5(m+1)}a_1 \\ u_{6(m+1)}a_1 \\ w_{1(m+1)}a_1 \\ . \\ . \\ . \\ w_{(m-1)(m+1)}a_1 \\ w_{m(m+1)}a_1 \end{pmatrix} + \begin{pmatrix} -sa_1 \\ -sa_1 \\ . \\ . \\ . \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ -sa_1 \\ . \\ . \\ . \\ -sa_1 \\ -sa_1 \end{pmatrix} + \begin{pmatrix} sa_1 - p_{m(m+1)}a_1 \\ sa_1 - p_{(m-1)(m+1)}a_1 \\ . \\ . \\ . \\ sa_1 - p_{1(m+1)}a_1 \\ sa_1 - u_{1(m+1)}a_1 \\ sa_1 - u_{2(m+1)}a_1 \\ sa_1 - u_{3(m+1)}a_1 \\ sa_1 - u_{4(m+1)}a_1 \\ sa_1 - u_{5(m+1)}a_1 \\ sa_1 - u_{6(m+1)}a_1 \\ sa_1 - w_{1(m+1)}a_1 \\ . \\ . \\ . \\ sa_1 - w_{(m-1)(m+1)}a_1 \\ sa_1 - w_{m(m+1)}a_1 \end{pmatrix}$$

All the entries are zero except the first and last entry. We will use this requirement to determine z and a_1. We obtain therefore the following equations

$$\begin{aligned} & p_{(m+1)(m+1)}a_1 + p_{(m+1)m}a_2 + \ldots + p_{(m+1)1}a_{k-2} + v_{(m+1)1}a_{k-1} + v_{(m+1)2}a_k + v_{(m+1)3}a_{k+1} \\ & + v_{(m+1)4}a_{k+2} + v_{(m+1)5}a_{k+3} + v_{(m+1)6}a_{k+4} + r_{(m+1)1}a_{k+5} + \ldots + r_{(m+1)(m+1)}a_1 = 0 \\ & w_{(m+1)(m+1)}a_1 + (s - p_{(m+1)m})a_2 + \ldots + (s - p_{(m+1)1})a_{k-2} + (s - v_{(m+1)1})a_{k-1} + (s - v_{(m+1)2})a_k \\ & + (s - v_{(m+1)3})a_{k+1} + (s - v_{(m+1)4})a_{k+2} + (s - v_{(m+1)5})a_{k+3} + (s - v_{(m+1)6})a_{k+4} + \\ & (s - r_{(m+1)1})a_{k+5} + \ldots + t_{(m+1)(m+1)}a_1 = 0 \end{aligned}$$

This linear system can be simplified to

$$\begin{aligned} & (p_{(m+1)(m+1)} + r_{(m+1)(m+1)})a_1 + p_{(m+1)m}a_2 + \ldots + p_{(m+1)1}a_{k-2} + v_{(m+1)1}a_{k-1} + v_{(m+1)2}a_k + v_{(m+1)3}a_{k+1} \\ & + v_{(m+1)4}a_{k+2} + v_{(m+1)5}a_{k+3} + v_{(m+1)6}a_{k+4} + r_{(m+1)1}a_{k+5} + \ldots + r_{(m+1)m}a_{2k+1} = 0 \end{aligned}$$

$$(w_{(m+1)(m+1)}+t_{(m+1)(m+1)})a_1+s(a_2+a_3+...+a_{2k+1})-p_{(m+1)m}a_2-...-p_{(m+1)1}a_{k-2}-v_{(m+1)1}a_{k-1}$$
$$-v_{(m+1)2}a_k-v_{(m+1)3}a_{k+1}-v_{(m+1)4}a_{k+2}-v_{(m+1)5}a_{k+3}-v_{(m+1)6}a_{k+4}-r_{(m+1)1}a_{k+5}$$
$$-...-r_{(m+1)m}a_{2k+1}=0$$

According to the definition of our variables we have

$$p_{(m+1)(m+1)}+r_{(m+1)(m+1)}=(k+1)s-p_{(m+1)m}-p_{(m+1)(m-1)}-...-p_{(m+1)1}-v_{(m+1)1}$$
$$-v_{(m+1)2}-v_{(m+1)3}-v_{(m+1)4}-v_{(m+1)5}-v_{(m+1)6}-r_{(m+1)1}-...-r_{(m+1)m}$$
$$w_{(m+1)(m+1)}+t_{(m+1)(m+1)}=p_{(m+1)m}+...+p_{(m+1)1}+v_{(m+1)1}+v_{(m+1)2}+v_{(m+1)3}+v_{(m+1)4}$$
$$+v_{(m+1)5}+v_{(m+1)6}+r_{(m+1)1}+...+r_{(m+1)m}-(k-1)s$$
$$a_2+a_3+...+a_{2k+1}=\frac{2q}{s}=-2a_1$$

According to the definition of $r_{(m+1)(m+1)}$ and $t_{(m+1)(m+1)}$ we obtain the following linear system

$$((k+1)s-p_{(m+1)m}-...-p_{(m+1)1}-v_{(m+1)1}-v_{(m+1)2}-v_{(m+1)3}-v_{(m+1)4}-v_{(m+1)5}-v_{(m+1)6}$$
$$-r_{(m+1)1}-...-r_{(m+1)m})a_1+p_{(m+1)m}a_2+...+p_{(m+1)1}a_{k-2}+v_{(m+1)1}a_{k-1}+v_{(m+1)2}a_k+v_{(m+1)3}a_{k+1}$$
$$+v_{(m+1)4}a_{k+2}+v_{(m+1)5}a_{k+3}+v_{(m+1)6}a_{k+4}+r_{(m+1)1}a_{k+5}+...+r_{(m+1)m}a_{2k+1}=0$$
$$(p_{(m+1)m}+...+p_{(m+1)1}+v_{(m+1)1}+v_{(m+1)2}+v_{(m+1)3}+v_{(m+1)4}+v_{(m+1)5}+v_{(m+1)6}+r_{(m+1)1}$$
$$+...+r_{(m+1)m}-(k+1)s)a_1-p_{(m+1)m}a_2-...-p_{(m+1)1}a_{k-2}-v_{(m+1)1}a_{k-1}-v_{(m+1)2}a_k$$
$$-v_{(m+1)3}a_{k+1}-v_{(m+1)4}a_{k+2}-v_{(m+1)5}a_{k+3}-v_{(m+1)6}a_{k+4}-r_{(m+1)1}a_{k+5}-...-r_{(m+1)m}a_{2k+1}=0$$

We recognize that one equation is redundant. We substitute the values of a_2 , …, a_{2k+1} and obtain the following equation

$$\frac{k+1}{k}(ks-p_{(m+1)m}-...-p_{(m+1)1}-v_{(m+1)1}-v_{(m+1)2}-v_{(m+1)3}-v_{(m+1)4}-v_{(m+1)5}-v_{(m+1)6}$$
$$-r_{(m+1)1}-...-r_{(m+1)m})a_1+(p_{(m+1)m}y_1+...+p_{(m+1)1}y_{k-3}+v_{(m+1)1}y_{k-2}+v_{(m+1)2}y_{k-1}+v_{(m+1)3}y_k$$
$$+v_{(m+1)4}y_{k+1}+v_{(m+1)5}y_{k+2}+v_{(m+1)6}y_{k+3}+r_{(m+1)1}y_{k+4}+...+r_{(m+1)m}y_{2k})z=0$$

We can say using a similar argument to the one in the proof of proposition 3

$$a_1=\alpha_{k-1}z$$

In analogy to that proof we are done.

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