ON SOME PROPERTIES OF DIVISORS OF ORDER k

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ABSTRACT

The purpose of this paper is to present some properties about the arithmetic functions which use divisors of order k. We also study the mean value of some of them.

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1. INTRODUCTION

A series of results in the Number Theory are based on the study of multiplicative arithmetic functions. Many important relations involving arithmetic functions can be developed by investigating some classes of divisors. It is necessary to present several types of divisors found in some books on the Number Theory. First we compile a short history of evolution of some classes of divisors.

Let n be a positive integer number. Starting from the set of divisors of n were defined two main multiplicative arithmetic functions, namely, $\sigma(n)$ is the sum of the divisors of n and $\tau(n)$ is the number of divisors of n.

E. Cohen [2] introduced the notion of *unitary divisors* in the following way: a divisor d of n is a unitary divisor when $\left(d,\frac{n}{d}\right)=1$. But, a few years before, R. Vaidyanathaswamy in [14] used the notion of block-factor for the same thing.

In 1966, M. V. Subbarao and L. J. Warren [10] introduced the *unitary perfect numbers* satisfying $\sigma^*(n)=2n$, where $\sigma^*(n)$ denoted the sum of the unitary divisors of n. Let $\tau^*(n)$ denote the number of unitary divisors of n, which is, in fact, the number of the squarefree divisors of n. Several characterization of these arithmetical function are given in [8,10].

The class of *exponential divisor* was introduced by M. V. Subbarao in [9] in the following way: d is said to be an *exponential divisor* (or *e-divisor*) of $n=p_1^{a_1}...p_r^{a_r}>1$, if $d=p_1^{b_1}...p_r^{b_r}$, where $b_i\,|\,a_i$ for any $1\le i\le r$. A series of results related to the exponential divisors are given in many sources, such as: [8,9,10,11].

L. Tóth and N. Minculete in [13] presented some properties of the arithmetical functions which use *exponential unitary divisors* or *e-unitary divisors* of $n=p_1^{a_1}...p_r^{a_r}>1$, if $d=p_1^{b_1}...p_r^{b_r}$, where b_i is an unitary divisor of a_i, so $\left(b_i,\frac{a_i}{b_i}\right)=1$, for any $1\le i\le r$.

In [6], we have introduced a class of exponential semiproper divisors, thus a divisor d of n, so that $\gamma(d)=\gamma(n)$ and $\left(\frac{d}{\gamma(n)},\frac{n}{d}\right)=1$, was called an *exponential semiproper divisor* or an *e-semiproper divisor* of n, where $\gamma(n)=p_1p_2...p_r$, for $n=p_1^{a_1}...p_r^{a_r}>1$ and $\gamma(1)=1$.

We also generalized in [5] the class of unitary divisors and the class of exponential semiproper divisor based on the arithmetic function $\gamma_k:\mathbb{N}^*\to\mathbb{C}$ such that $\gamma_k(1)=1$ and $\gamma_k(n)=p_1^{a_1}p_2^{a_2}...p_u^{a_u}(p_{u+1}...p_r)^k$, when $n=p_1^{a_1}...p_u^{a_u}p_{u+1}^{a_{u+1}}...p_r^{a_r}>1$ where $a_1,a_2,...,a_u<k+1$ and $a_{u+1},...,a_r\ge k+1$, and $k\ge 0$ $(\mathbb{N}^*=\{1,2,3...\})$ and $\mathbb{C}$ is the set of complex numbers.

Therefore, a divisor d of n, so that $\gamma_k(d)=\gamma_k(n)$ and $\left(\frac{d}{\gamma_k(n)},\frac{n}{d}\right)=1$, was called the *divisor of order* k of n. In this case we write $d\,|_{(k)}\,n$. Note that for $k=0$ the notion of the divisor of order 0 is identical with the notion of the unitary divisors and for $k=1$ the notion of the divisor of order 1 is identical with the notion of the exponential semiproper divisor.

Let $\tau^{(k)}(n)$ denote the number of the divisors of order k of n, and $\sigma^{(k)}(n)$ denote the sum of the divisors of order k of n, so that $\sigma^{(k)}(1)=\tau^{(k)}(1)=1$.

According to the things mentioned in [5], we have

$$\tau^{(k)}(p^a)=\begin{cases}1, \text{ for } a<k+1\\ 2, \text{ for } a\ge k+1,\end{cases} \tag{1}$$

and

$$\sigma^{(k)}(p^a)=\begin{cases}p^a, \text{ for } a<k+1\\ p^a+p^k, \text{ for } a\ge k+1.\end{cases} \tag{2}$$

Similar to the unitary analogue of Euler's totient (see e.g. [8]), was defined in [5] the multiplicative function $\varphi^{(k)}:\mathbb{N}^*\to\mathbb{C}$, so that $\varphi^{(k)}(1)=1$ and

$$\varphi^{(k)}(p^a)=\begin{cases}p^a, \text{ for } a<k+1\\ p^a-p^k, \text{ for } a\ge k+1.\end{cases} \tag{3}$$

By particularize the values of k we observe that $\varphi^{(0)}(n)=\varphi^*(n)$, where φ^* is the unitary analogue of Euler's arithmetic function, and $\varphi^{(1)}(n)=\varphi^{(e)s}(n)$, where the multiplicative functions $\varphi^{(e)s}:\mathbb{N}^*\to\mathbb{C}$, is defined as $\varphi^{(e)s}(1)=1$ and

$$\varphi^{(e)s}(p^a)=\begin{cases}p, \text{ for } a=1\\ p^a-p, \text{ for } a\ge 2.\end{cases} \tag{4}$$

In [5], we presented another function of Möbius type, namely, the arithmetic function given by $\mu^{(k)}(1)=1$ and

$$\mu^{(k)}(p^a)=\begin{cases}1, \text{ for } a<k+1\\ -1, \text{ for } a\ge k+1.\end{cases} \tag{5}$$

The arithmetic functions $\tau^{(k)},\sigma^{(k)},\varphi^{(k)}$ and $\mu^{(k)}$ are multiplicative and we have

$$\begin{gathered}\tau^k(n)=2^t,\sigma^{(k)}(n)=p_1^{a_1}\dots p_u^{a_u}\prod_{i=u+1}^{r}(p_i^{a_i}+p_i^k),\\ \varphi^{(k)}(n)=p_1^{a_1}\dots p_u^{a_u}\prod_{i=u+1}^{r}(p_i^{a_i}-p_i^k),\ \mu^{(k)}(n)=(-1)^t,\end{gathered} \tag{6}$$

where $n=p_1^{a_1}\dots p_u^{a_u}p_{u+1}^{a_{u+1}}\dots p_r^{a_r}$, with $a_i\le k$ for any $i\in\{1,\dots,u\}$ and $a_i\ge k+1$ for any $i\in\{u+1,\dots,r\}$ and $t=r-u$, so t is the number of the exponents in the prime factorization of n which are $\ge k+1$.

It is known [4] that the number-theoretical function $f(n)$ has a *mean value* if the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(n)=m(f) \tag{7}$$

exists. The questions of the existence of the mean value $m(f)$ can be found in most text on the theory of multiplicative arithmetic functions.

A result is due to H. Delange [3] and can be expressed as:

Theorem 1.1 *Suppose* $f(n)$ *is a multiplicative function satisfying the following conditions:*
1) $|f(n)| \leq 1$, *for all* $n \geq 1$;
2) $\sum_{p\ prime} \frac{f(p)-1}{p}$ *converges.*
Then the limit

$$m(f) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n)$$

exists, and

$$m(f) = \prod_{p\ prime} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots\right). \tag{8}$$

We mention another general result related to $m(f)$ which is due to L. Tóth [11].
Theorem 1.2 *Let* f *be a complex valued multiplicative function such that* $|f(n)| \leq 1$, *for every* $n \geq 1$, *and* $f(p) = 1$, *for every prime* p. *Then*

$$\sum_{n\leq x} f(n) = m(f)x + O(x^{\frac{1}{2}} \log x). \tag{9}$$

where

$$m(f) = \prod_{p} \left(1 + \sum_{a=2}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right) \tag{10}$$

is the mean value of f.

2. MAIN RESULTS

Motivated by the above results, the purpose of this paper is to discuss further properties of these arithmetic functions.
Theorem 2.1 *For* $k \geq 1$, *we have*

$$\sum_{n\leq x} \mu^{(k)}(n) = Ax + O\left(x^{\frac{1}{2}} \log x\right), \tag{11}$$

where

$$A = \prod_{p\ prime} \left(1 - \frac{2}{p^{k+1}}\right).$$

Proof. We apply Theorem 1.2 for the function $\mu^{(k)}$ and we obtain the statement.

Theorem 2.2 *For* $k \geq 1$, *we have*

$$\sum_{n\leq x} \varphi^{(k)}(n) = Bx + O(x^{3/2} \log x), \tag{12}$$

where

$$B = \frac{1}{2} \prod_{p\ prime} \left(1 - \frac{1}{p^{k+1}(p+1)}\right).$$

Proof. We consider the multiplicative function $f(n) = \frac{\varphi^{(k)}(n)}{n}$, which has the properties: $|f(n)| \leq 1$ for all $n \geq 1$ and $f(p) = 1$ for every p prime number. Applying Theorem 1.2 or a corollary of Erdös-Renyi [4] for this function we deduce the relation

$$\sum_{n\le x}\frac{\varphi^{(k)}(n)}{n}=2Bx+O(x^{1/2}\log x), \tag{13}$$

where

$$B=\frac{1}{2}\prod_{p\ prime}\left(1-\frac{1}{p^{k+1}(p+1)}\right).$$

We use the theorem of partial summation [7], which is given thus

$$\sum_{n\le x}f(n)g(n)=F(x)g(x)-\int_1^x F(t)g^{'}(t)dt, \tag{14}$$

where $f(n), g(n)$ are two arithmetic functions, $x\ge 2$, $g(t)$ is continuously differentiable on [1,x] and $F(x)=\sum_{n\le x}f(n)$.

Therefore, for $f(n)=\frac{\varphi^{(k)}(n)}{n}$, $g(n)=n$ and $F(x)=2Bx+O(x^{1/2}\log x)$ (from relation (13)), we have, from relation (14), the following relation

$$\sum_{n\le x}\varphi^{(k)}(n)=\sum_{n\le x}\frac{\varphi^{(k)}(n)}{n}\cdot n=$$

$$[2Bx+O(x^{1/2}\log x)]x-\int_1^x[2Bt+O(t^{1/2}\log t)]dt=$$

$$=2Bx^2+O(x^{3/2}\log x)-Bx^2-O\left(\int_1^x t^{1/2}\log t dt\right)$$

$$=Bx^2+O(x^{3/2}\log x).$$

Consequently, we obtain the average order of $\varphi^{(k)}$.

Remark 2.1 *The mean value of the multiplicative function* $\frac{\varphi^{(k)}(n)}{n}$ *is given by* $\prod_{p\ prime}\left(1-\frac{1}{p^{k+1}(p+1)}\right)$.

Next, we consider the function $h^{(k)}=\sigma^{(k)}*\mu$ given by the Dirichlet product, where μ is the Möbius's function. Since the functions $\sigma^{(k)}$ and μ are multiplicative and taking into account that the Dirichlet product keeps multiplicativity [1], it follows that the function $h^{(k)}$ is multiplicative and implies the relation

$$h^{(k)}(p^a)=\begin{cases}\varphi(p^a),\ a\ne k+1\\ p^a,\ a=k+1\end{cases} \tag{15}$$

where φ is the Euler's totient.

Theorem 2.3 *For* $k\ge 1$, *we have*

$$\sum_{n\le x}h^{(k)}(n)=Cx^2+O(x), \tag{16}$$

where

$$C=\prod_{p\ prime}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\frac{1}{p^{k+2}}\right).$$

Proof. We consider the function $f(n)=\frac{h^{(k)}(n)}{n}$, which has the properties $0<f(n)\le 1$, for all $n\ge 1$, and the series $\sum_p\frac{f(p)-1}{p}=-\sum_p\frac{1}{p^2}$ converges. Applying Theorem 1.1, we deduce the equality

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{h^{(k)}(n)}{n}=\prod_{p}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\frac{1}{p^{k+2}}\right)=2C,$$

which means that

$$F(x)=\sum_{n\le x}\frac{h^{(k)}(n)}{n}=2Cx+O(1). \tag{17}$$

Therefore, for $f(n)=\frac{h^{(k)}(n)}{n}$, $g(n)=n$ and $F(x)$ given in relation (17), we have, from relation (14), the relation

$$\sum_{n\le x}h^{(k)}(n)=\sum_{n\le x}\frac{h^{(k)}(n)}{n}\cdot n=$$
$$F(x)\cdot x-\int_1^x F(t)dt=Cx^2+O(x).$$

Thus, the proof is complete.

Remark 2.2 *From relation (17) we deduce that the mean value of the multiplicative function* $\frac{h^{(k)}(n)}{n}$ *is given by*

$$\prod_{p\ prime}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\frac{1}{p^{k+2}}\right).$$

3. CONCLUSIONS

Similar to the function introduced by S.S. Pillai (see e. g. [8,12]) $P(n)=\sum_{d|n}d\varphi(d)$, we can introduce the following arithmetic function:

$$P^{(k)}(n)=\sum_{d|_{(k)}n}d\varphi^{(k)}(d).$$

This arithmetic function may generate a number of properties, which can be study by reader.

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