

A BILINEAR TRANSFORMATION METHOD OF ELLIPTIC IIR LOWPASS FILTER (LPF) DESIGN

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ABSTRACT

An Elliptic lowpass digital filter was designed using the MATLAB tool box. The programme code was based on the bilinear transformation method through computation and simulation based on the necessary equations underlying filter design. At input sampling frequency, $f_s = 400\text{Hz}$, peak ripple value at passband, $A_{\text{pass}} = 1\text{Hz}$ and peak ripple value at bandstop, $A_{\text{stop}} = 0.45\text{Hz}$, the Elliptic LPF has its lower 3dB point at 0Hz and upper 3dB point at 85Hz, giving a BW of 85Hz. It was observed that the ripples of the Elliptic LPF is independently adjustable which makes the filter maximally insensitive to component variations due to a faster transition gain between the passband and the stopband. This makes it suitable for use to process baseband signals in radio transmitting stations that employ pulse coded modulation (PCM). Its steep roll off is highly efficient in blocking off harmonic distortions arising from processing image frequency in superheterodyne systems which if neglected causes interference with other communication systems at close frequencies.

Keywords: *Lowpass IIR Digital Filter Design Using MATLAB.*

1. INTRODUCTION

Mitra (2002) defined a filter as a frequency selective device that functions by accepting certain frequency input signals and attenuating others referred to as unwanted components. Digital filters are the basic building block of any Digital Signal Processor (DSP) as reported by Kaiser (1974). Rabiner and Gold, (1975) classified filters into two groups-Analog filters and Digital filters, which are categorized into lowpass, highpass, bandpass and bandstop. A lowpass filter passes low frequencies, from dc (zero frequency) to ω_o (lower cut off frequency) and attenuates all other frequencies outside this range. An Elliptic filter is an electronic filter with equalized ripple behavior (equirippled) in both the passband and stopband but with a unique sharp skirt in its transition region. The gain of lowpass elliptic filter, given as a transfer function of its angular frequency may be represented as (Smith, 2006; Smith, 2004; Smith, 2005).

$$G_n(\omega) = \frac{1}{\sqrt{1 + \varepsilon^2 R_n^2\left(\xi, \frac{\omega}{\omega_o}\right)}} \quad (1)$$

R_n is the n^{th} order of elliptic rational function and ω_o is the cutoff frequency, ε is the ripple factor, ξ is the selectivity factor while ω is the centre frequency. Filters are designed in accordance to what kind of application is envisaged.

A first order lowpass IIR digital filters has a transfer function given by (Mitra, 1998):

$$H_{lp}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{(1 - \alpha)z^{-1}} \quad (2)$$

where $\alpha < 1$ for stability. The transfer function in equation (2) has a zero $z = -1$, where $\omega = \pi$, which is in the stop band of the filter. It has a real pole at $z = \alpha$. As ω increases from 0 to π , the magnitude of the zero vector decreases from 1 to α . The maximum value of the magnitude function is unity at $\omega = 0$, and the minimum value is zero at π , which may be represented as (Mitra, 2002; Mitra, 1998)

$$H_{lp}(\lambda^{j\omega}) = 1, \quad H_{lp}(\lambda^{j\pi}) = 0 \quad (3)$$

$H_{lp}(\lambda^{j\omega})$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$. The squared magnitude function can be derived as (Mitra, 2002; Mitra, 1998):

$$H_{lp}(\lambda^{j\omega})^2 = \frac{(1 - \alpha)^2 (1 + \cos \omega)}{2(1 + \alpha^2 - 2\alpha \cos \omega)} \quad (4)$$

The derivative (Mitra, 2002; Mitra, 1998):

$$\frac{H_{lp}(\lambda^{j\omega})^2}{d\omega} = \frac{-1(1-\omega)^2(1+2\omega)^2 \sin \omega}{2(1+\alpha^2-2\alpha \cos \omega)} \quad (5)$$

This is non-positive in the range $0 \leq \omega \leq \pi$ verifying again the monotonically decreasing behaviour of the magnitude function. To determine the 3dB cut off frequency ω_c , we set $H_{lp}(e^{j\omega})^2 = \frac{1}{2}$ in equation (3) so that

$$(1-\alpha)^2(1+\cos \omega_c) = 1 \text{ or } (1-\alpha^2)(1+\cos \omega_c) = 1+\alpha^2-2\alpha \cos \omega_c \quad (6)$$

This when solved yields (Mitra, 2002; Mitra, 1998):

$$\cos \omega_c = \frac{2\alpha}{1+\alpha^2} \quad (7)$$

The above quadratic equation can be solved for ω yielding two solutions. The solution resulting in the stable transfer function $H_{lp}(z)$ is given by (Mitra, 2002; Mitra, 1998)

$$H_{lp}(z) = \frac{1 - \sin \omega_c}{\cos \omega_c} \quad (8)$$

This research employs the MATLAB toolbox to design, manipulate and analyze digital filter efficacy. The need to have a versatile and flexible tool in the design and implementation of digital filters and the rapid evolution in the field of computing and communications due to the availability of various hardwares at relatively cheap costs with high performances translates to advantages that may be derived especially when using DSP.

2. DIGITAL FILTER RESPONSE

In most frequency domain applications frequency response is most important because it includes the passband, stopband and transition band. The impulse response is used in implementing a digital filter. To create an infinite impulse response (IIR), the impulse response $y(n)$ is convolved with the input signal $x(n)$ (Kale, 2004; Mitra, 2002).

$$Y(n) = \sum_{i=0}^m b_i X(n-i) - \sum_{j=1}^m a_j Y(n-j) \quad (9)$$

where x is the input signal, y is the output signal and the constants $b_{ij} = 0, 1, 2, \dots, m^+$, $a_{ij} = 0, 1, 2, \dots$. These are called the coefficients.

Impulse response representation of a recursive filter as an LTI system is given as (Soderstorm, 1983).

$$Y(n) = X(n) + \frac{1}{2} Y(n-1) \quad (10)$$

$$Y(n) = X(n) + \frac{1}{2} X(n-1) + \frac{1}{4} X(n-2) + \frac{1}{8} X(n-3) + \dots \quad (11)$$

This gives rise to the general transfer function $H(z)$ which contains polynomials in both the numerator and denominator. The roots of the denominator determine the pole locations of the filter, and the roots of the numerator determine the zero locations. The transfer function of IIR filter is given by as (Smith, 2006; Smith, 2004; Smith, 2003).

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{j=0}^P b_j z^{-j}}{\sum_{j=1}^Q a_j z^{-j}} \quad (12)$$

P is the feed forward filter order, b is the feed forward filter coefficient, Q is the feedback filter order, a the feedback filter coefficient and z^{-1} represents unit delay.

Smith (2004) expressed the frequency response as a ratio of DTFTs.

$$H(\lambda^{j\omega T}) \triangleq \frac{\text{DTFT}_{\omega T}(B) \sum_{M=0}^M \chi(n) \lambda^{-j\omega n T}}{\text{DTFT}_{\omega T}(A) \sum_{n=0}^A a(n) \lambda^{-j\omega n T}} \quad (13)$$

3. DESIGN APPROACH, SPECIFICATIONS AND METHOD

In the design of a IIR digital filter, an analog IIR filter is converted to its digital form by applying discretization technique such as bilinear transformation method. This is done by convolving the input signal with the impulse response of the appropriate filter, (Wu, 2004). It is then implemented by transforming the transfer function of the analog prototype filter into the system function of a digital filter with similar characteristic.

The bilinear transformation from the s-plane to the z-plane is derived by applying the trapezoidal numerical integration approach to the differential equation of $H_a(s)$ that leads to the difference equation representation of $G(z)$. The parameter T represents the step size in the numerical integration.

$$G(z) = H_a(s) = \frac{2}{T} \left[\frac{1 + z^{-1}}{1 - z^{-1}} \right] \quad (14)$$

The inverse bilinear transformation is applied to the digital filter specifications to obtain the desired transfer function $G(z)$ from the analog transfer function $H_a(z)$. Using the transformation and noting that $s = j\Omega_0$, we may write

$$z = \frac{1 + j\Omega_0 T}{1 - j\Omega_0 T} \quad (15)$$

This has a unity magnitude and this implies that a point on the linearity axis in the s-plane is mapped onto a point on the unit circle in the z-plane. In the general case for $s = -\sigma_0 + j\Omega_0$

$$z = \frac{1 + (\sigma_0 + j\Omega_0)T}{1 - (\sigma_0 + j\Omega_0)T} = \frac{(1 + \sigma_0) + j\Omega_0 T}{(1 + \sigma_0) - j\Omega_0 T} \quad (16)$$

Therefore,

$$z^2 = \frac{(1 + \sigma_0)^2 + \Omega_0^2 T^2}{(1 - \sigma_0)^2 + \Omega_0^2 T^2} \quad (17)$$

A point on the $j\Omega$ axis in the s-plane ($\sigma_0 = 0$) is mapped onto a point on the unit circle in the z-plane as $z = 1$. In the mapping of the s-plane into the z-plane via the bilinear transformation as illustrated, there is no aliasing due to the one to one mapping. A computer program is developed for the design of the IIR lowpass Elliptic filter based on MATLAB program. The digital filter coefficients, which represent the physical values of resistors, capacitors and inductors, are determined by computation. These filter coefficients are used with the sampled data values to perform the filter calculations.

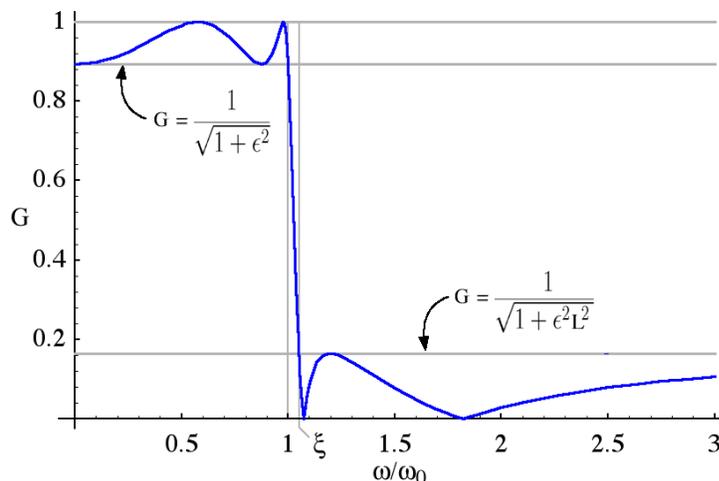


Fig 1. A typical Elliptic filter response (Smith, 2006; Smith, 2004)

In the first step of the design, the filter order N and the frequency scaling factor ω_n are determined from the given specifications. The parameters obtained in step 1, are used to determine the coefficients of the transfer function using the following function statements.

$$[N, \omega_n] = \text{Ellip ord}(\omega_p, \omega_s, R_p, R_s)$$

$$[N, \omega_n] = \text{Cheb1 ord}(\omega_p, \omega_s, R_p, R_s)$$

$$[N, \omega_n] = \text{Cheb2 ord}(\omega_p, \omega_s, R_p, R_s) \tag{18}$$

R_p and R_s are passband ripple and minimum stopband attenuation respectively. Both are specified in dB. ω_n is the frequency scaling factor. ω_p and ω_s are the normalized passband and stopband frequencies respectively. These frequency points are between 0 and 1 and sampling is equal to 2. If the sampling frequency F_s , the passband edge frequency F_{pass} are specified in Hz, then

$$\omega_p = \frac{2F_p}{F_t} \text{ and } \omega_s = \frac{2F_s}{F_t} \tag{19}$$

Specifications:

- Passband edge frequency, $F_{pass} = 10\text{Hz}$
- Stopband edge frequency, $F_{stop} = 337\text{Hz}$
- Peak ripple value at bandpass, $A_{pass} = 1\text{Hz}$
- Peak ripple value at bandstop, $A_{stop} = 0.45\text{Hz}$
- Sampling rate, $F_s = 400\text{Hz}$

These specifications were used with the M-files in the signal processing tool of the MATLAB software. The LPF has its own M-files and specifications which include;

$$\begin{aligned} [z, p, k] &= \text{iirellip}(n, R_p, R_s, \omega_p) \\ [z, p, k] &= \text{iirellip}(n, R_p, R_s, \omega_p, \text{'ftype'}) \\ [b, a] &= \text{iirellip}(n, R_p, R_s, \omega_p) \\ [b, a] &= \text{iirellip}(n, R_p, R_s, \omega_p, \text{'ftype'}) \end{aligned} \tag{20}$$

Table 1. Parameters of the Designed Lowpass Elliptic Digital Filter

Filter specifications	Bandpass frequency Hz F_{stop}	Normalized angular frequency rad/s ω_o	ω_o/π	Gain (G)
Upper 3dB frequency	380	0.0010	0.0003	0.9102
	370	0.2512	0.0799	0.9231
Lower 3dB frequency	340	0.5024	0.1599	0.9841
$F_s = 400\text{Hz}$	330	0.9514	0.3028	0.9720
	330	1.0212	0.3251	1.0381
$A_{pass} = 1\text{Hz}$	330	1.0423	0.3318	0.2231
	330	1.2333	0.3926	0.3432
$A_{stop} = 0.45\text{Hz}$	330	1.5042	0.4788	0.0124
	330	2.0143	0.6412	0.1950
$F_{pass} = 10\text{Hz}$	330	2.5236	0.8033	0.2104
	330	3.0146	0.9596	0.2231
BW = 85Hz				

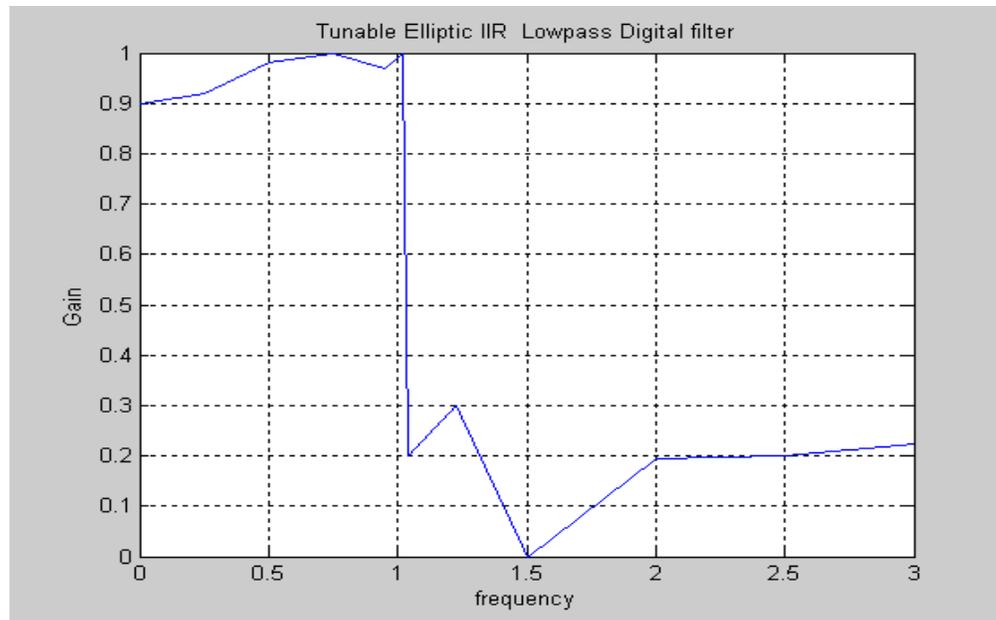


Fig 2. The designed Elliptic filter response

4. DISCUSSION AND CONCLUSION

This work was primarily focused on the design and implementation of IIR Elliptic digital lowpass filter. Program to design the filter was developed and implemented using the MATLAB toolbox. The program was written based on the necessary equations underlying the operation of the filter in question. The bilinear transformation method was used to develop the program which was tried out severally to ascertain its flexibility and accuracy before presentation. For transfer function, $[numd, dend = \text{bilinear num, den, } f_s, f_p]$ in the call functions, converts s-domain transfer function given by num (i.e numerator) and den (i.e denominator) to a discrete equivalent in the z-domain. Row vectors num and den specify the coefficients of the numerator and denominator, respectively in descending powers of s so that at f_s of 400Hz, A_{pass} of 1Hz and A_{stop} of 0.45Hz, the Elliptic LPF designed has its upper 3dB point at 85Hz and its lower 3dB point at 0Hz, giving a BW of 85Hz. It has an equirrippled passband and stopband and a sharp roll-off rate which is in line with predicted theory. This is quite perfect for baseband signals that are pulse coded in format. The Elliptic lowpass filter is also suitable for use where degraded phase responses can be tolerated. The designed IIR elliptic LPF has an equirrippled passband and stopband and a sharp roll-off rate. This is in line with predicted theory. This type of filter is attractive in communications such as adaptive equalization, echo cancellation, noise reduction, speech analysis and synthesis in PCM applications.

5. REFERENCES

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