ROBUST EXPONENTIAL ATTRACTORS FOR MEMORY RELAXATION OF PATTERN FORMATION EQUATIONS

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ABSTRACT

In this paper, we prove the existence of the robust exponential attractors for memory relaxation of pattern formation equations in the phase-space H_ε^0, and we improve results in [3].

Key words: *Pattern formation equations; Robust exponential attractors.*

1. INTRODUCTION

Let $\Omega=[-l,l]^3\subset\mathbb{R}^3$ be a bounded domain with smooth boundary ,we consider the pattern formation equations

$$u_t+\Delta^2 u+\omega\Delta u+\gamma u+g(u)=0 \qquad (1.1)$$

subject to the Dirichlet boundary condition

$$u(t)|_{\partial\Omega}=0\,,\ t\geq 0.$$

We assume $g\in C^2(\mathbb{R})$, with $g(0)=0,$ such that

$$|g'(r)|\leq c(1+|r|^2). \qquad \forall r\in\mathbb{R}\,, \qquad (1.2)$$

$$\liminf_{|r|\to\infty}\frac{g(r)}{r}>-\lambda_1, \qquad (1.3)$$

where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.
According to [6] the long-term dynamics between the abstract first order evolution equation

$$u_t+Au+B[u]=0 \qquad (1.4)$$

and the memory relaxation of (1.4)

$$u_t+\int_0^\infty \mu_\varepsilon(s)\big(Au(t-s)+B[u(t-s)]\big)ds=0 \qquad (1.5)$$

are close when ε is sufficiently small. Here $u=u(t):[0,+\infty]\to H$, and be a Hilbert space $A:D(A)\subset H\to H$ is a strictly positive self-adjoint linear operator with compact inverse $B:D(A^{1/2})\subset H\to H$ is a nonlinear operator. As detailed in [12] there is a complete equivalence between (1.5) and the following system

$$\begin{cases} u_t+\int_0^\infty \mu_\varepsilon(s)\eta(s)ds=0, \\ \eta_t=T_\varepsilon\eta+Au+B[u]. \end{cases} \qquad (1.6)$$

In this paper, we set $\omega=\gamma=1, A=\Delta^2$ with the domain $D(A)=H^4(\omega)\cap H_0^2(\omega)$ and

$B(u)=g(u),$ then the memory relaxation of equation (1.1) is equivalent to the following evolution system

$$\begin{cases} u_t+\int_0^\infty \mu_\varepsilon(s)\eta(s)\,ds=0, \\ \eta_t=T_\varepsilon\eta+Au-A^{1/2}u+u+g(u). \end{cases} \tag{1.7}$$

The pattern formation equation portrays the chemical reaction and the flame combustion phenomenon (see [1.2] for detail). Existence of the global attractors for the equation (1.1) has been studied by A. ION in [3]. The global attractor is compact, fully invariant and attractive for the semigroup, but may attractor trajectories at a slow rate. Conversely, an exponential attractor is a compact set, positively invariant under the action of the semigroup, which attracts exponentially fast trajectories departing from the bounded sets. Thus exponential attractors are expected to be more robust than global attractors under perturbations. In this paper, we study the existence of the robust exponential attractor for (1.7) in the phase-space H_ε^0. It is worth noting that the exponential attractors attract bounded subsets of the whole phase-space. The main results of this paper is Theorem 3.8.

2. PRELIMINARIES

Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and the norm in $H=L^2$, respectively. For $r\in\mathbb{R}$, the hierarchy of compactly nested Hilbert spaces $V^r=D(A^{(1/2+r)/2})$ with the standard inner products

$$\langle u_1,u_2\rangle_{V^r}=\left\langle A^{(1/2+r)/2}u_1,A^{(1/2+r)/2}u_2\right\rangle.$$

Notice that $H=V^{-1/2}$. Identifying H with its dual space H^*, there holds $(V^r)^*=V^{-r-1}$.

Denoting $\mathbb{R}^+=(0,\infty)$, we assume that $\mu\in W^{(1,1)}(\mathbb{R}^+)$ is nonnegative ,and that the exponential decay condition

$$\mu'(s)+\delta\mu(s)\le 0,\quad \text{for } a.e.\ s\in(\mathbb{R}^+) \tag{2.1}$$

holds for some $\delta>0.$ Furthermore ,for $\varepsilon\in(0,1],$ we set $\mu_\varepsilon(s)=\frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right),$ then we have

$$\int_0^\infty \mu_\varepsilon(s)ds=\frac{1}{\varepsilon},\qquad \int_0^\infty s\mu_\varepsilon(s)ds=1. \tag{2.2}$$

Then, we introduce the weighted Hilbert spaces $M_\varepsilon^r=L^2_{\mu_\varepsilon}(\mathbb{R}^+;V^{r-1})$, endowed with the inner products

$$\langle\eta_1,\eta_2\rangle_{M_\varepsilon^r}=\int_0^\infty \mu_\varepsilon(s)\langle\eta_1(s),\eta_2(s)\rangle_{V^{r-1}}\,ds.$$

We make use of the infinitesimal generator of the right-translation semigroup on M_ε^0, the linear Operator $T_s=-\partial s$ (∂s being the distributional derivative with respect to s) with domain

$$D(T_\varepsilon)=\{\eta\in M_\varepsilon^0:\partial_s\eta\in M_\varepsilon^0,\eta(0)=0\}.$$

On account of (2.1) , there holds (see [6,12])

$$\langle T_\varepsilon\eta,\eta\rangle_{M_\varepsilon^0}\le-\frac{\delta}{2\varepsilon}\|\eta\|^2_{M_\varepsilon^0},\quad \forall\eta\in D(T_\varepsilon). \tag{2.3}$$

For $\varepsilon\in[0,1]$, we define the product Banach spaces

$$H_\varepsilon^r=\begin{cases} V^r\times M_\varepsilon^r, & if\ \varepsilon>0, \\ V^r & if\ \varepsilon=0. \end{cases} \tag{2.4}$$

When $\varepsilon=0$, we interpret the pair (u,η) as u , and the norm reduces to the first component only. We shall need the lifting map $L_\varepsilon:H_0^0\to H_\varepsilon^0$ and the projection map Q_ε on H_ε^0, given by

$$\mathsf{L}_\varepsilon u = (u,0), \qquad \mathsf{Q}_\varepsilon(u,\eta) = \eta.$$

We recall the compact and dense injections in [10]

$$D(A^{s/2}) \hookrightarrow D(A^{r/2}), \ \forall s > r, \tag{2.5}$$

and the continuous embedding

$$D(A^{s/2}) \hookrightarrow L^{6/3-2s}, \qquad \forall s \in [0, \frac{3}{2}) . \tag{2.6}$$

If $\eta \in M_\varepsilon^r$, then from $(1.7)_1$, we have the following inequality

$$\|u_t\|_{V^{r-1}}^2 \le \frac{1}{\varepsilon}\|\eta\|_{M_\varepsilon^r}^2 . \tag{2.7}$$

For $i = 0,1$ and $R \ge 0$ we set

$$B_\varepsilon^i(R) = \{z \in H_\varepsilon^r : \|z\|_{H_\varepsilon^r} \le R\}.$$

Definition 2.1 [6] A family $\{\varepsilon_\varepsilon\}_{\varepsilon\in[0,1]}$ of compact subsets of H_ε^0 is said to be a robust family of exponential attractors if the following conditions holds

(i) Each ε_ε is positively invariant for $S_\varepsilon(t)$.

(ii) There exist $\omega > 0$ and a positive increasing function M such that, for every $R \ge 0$ there holds

$$dist_{H_\varepsilon^0}\left(S_\varepsilon(t)B_\varepsilon^0(R), \varepsilon_\varepsilon\right) \le M(R)e^{-\omega t}.$$

(iii) The fractal dimension of ε_ε in H_ε^0 is uniformly bounded with respect to ε.

(iv) There exists a continuous increasing function Θ : [0; 1] → [0;∞) with Θ(0) = 0 such that

$$dist_{H_\varepsilon^0}^{sym}(\varepsilon_\varepsilon, L_\varepsilon \varepsilon_0) \le \Theta(\varepsilon).$$

where dist and $dist^{sym}$ denote the nonsymmetric and symmetric Hausdorff distance between sets respect- tively. For a fixed ε, (i) − (iii) are nothing but the usual conditions defining an exponential attractor (but observe that, contrary to the original definition see [9] for detail) we require the attraction property on the whole phase-space), while condition (iv) characterizes the robustness property.

Given $z = (u_0, \eta_0) \in H_\varepsilon^0 (z = u_0$ if $\varepsilon = 0)$, we denote the weak solution at time t to (1.7) with initial data z as

$$S_\varepsilon(t)z = \begin{cases} (u(t), \eta^t), & if\, \varepsilon > 0, \\ u(t), & if\, \varepsilon = 0. \end{cases} \tag{2.8}$$

Conditions on B. We assume that $B : V^0 \to V^{-1}$. For every R ≥ 0 ,there exists C = C(R) ≥ 0 such that

$$\sup_{\|u_i\|_{V^1} \le R} \|B[u_1] - B[u_2]\|_{V^{-1}} \le C\|u_1 - u_2\|_{V^0}, \tag{2.9}$$

$$\sup_{\|u\|_{V^1} \le R} \|B[u]\|_{V^0} \le C. \tag{2.10}$$

Assumption (S). For every $\varepsilon \in [0,1]$ system (1.7) generates a strongly continuous semigroup $S_\varepsilon(t)$ on the phase-space H_ε^0 Moreover, for every given R ≥ 0 ,there exists K = K(R) ≥ 0 such that

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{H_\varepsilon^0} \le Ke^{Kt}\|z_1 - z_2\|_{H_\varepsilon^0}, \tag{2.11}$$

Whenever $\|z_i\|_{H_\varepsilon^0} \le R.$

Theorem 2.1 [6] Let the following assumptions hold.

$(H1)$ for $i = 0,1,$ there exists $R_i > 0$ such that $B_\varepsilon^i(R_i)$ is an absorbing set for $S_\varepsilon(t)$ on H_ε^i, uniformly with respect to ε .Namely, given any R ≥ 0, there exists a time $t_i \geq 0$,depending only on R ,such that

$$S_\varepsilon(t)\mathrm{B}_\varepsilon^i(R) \subset \mathrm{B}_\varepsilon^i(R_i), \qquad \forall t \geq t_i.$$

Moreover, for every R ≥ 0 ,there exists $C_i = C_i(R) \geq 0,$ such that

$$\sup_{\mathrm{PzP}_{H_\varepsilon^i} \leq R} \left\| S_\varepsilon(t) z \right\|_{H_\varepsilon^i} \leq C_i, \qquad \forall t \geq 0.$$

($H2$) There exist $R_2 > 0$ such that $\mathrm{B}_\varepsilon^1(R_2)$ satisfies

$$dist_{H_\varepsilon^0}\left(S_\varepsilon(t)\mathrm{B}_\varepsilon^0(R_0), \mathrm{B}_\varepsilon^1(R_2)\right) \leq Me^{-\kappa t}.$$

for some $M > 0$ and $\kappa > 0$.

($H3$)Given any R > 0 there exist $\beta \in (0,1]$ and $\lambda, \Lambda : [0,\infty) \to ¡^{+}$ (possibly depending on R) with $\lambda(t) < 1/2$ for t large enough such that for every $z_{1,}z_2 \in \mathrm{B}_\varepsilon^1(r)$, the map $S_\varepsilon(t)$ admits the decomposition

$$S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2 = L_\varepsilon[t](z_1, z_2) + N_\varepsilon(t)(z_1, z_2),$$

Where $L_\varepsilon(t)$ and $N_\varepsilon(t)$ satisfy

$$\left\| L_\varepsilon(t)(z_1, z_2) \right\|_{H_\varepsilon^0} \leq \lambda(t) \left\| z_1 - z_2 \right\|_{H_\varepsilon^0},$$

$$\left\| N_\varepsilon(t)(z_1, z_2) \right\|_{H_\varepsilon^\beta} \leq \Lambda(t) \left\| z_1 - z_2 \right\|_{H_\varepsilon^0}.$$

Besides $\overline{\eta}^t = ¤_\varepsilon(t) N_\varepsilon(t)(z_1, z_2)$ fulfils the Cauchy problem

$$\begin{cases} \overline{\eta}_t = T_\varepsilon \overline{\eta} + \overline{\omega}, t > 0, \\ \qquad \overline{\eta}^0 = 0, \end{cases}$$

for some $\overline{w}$ satisfying, for all T > 0,

$$\left\| \overline{w}(t) \right\|_{V^{-1-\beta}} \leq \Lambda(t) \left\| z_1 - z_2 \right\|_{H_\varepsilon^0}, \quad \forall t \in (0,T).$$

Then,there exists a family $\{\varepsilon_\varepsilon\}$ of robust exponential attractors.

Remark 2.2 Condition (H3) with $\beta = 1$ is actually implied by the stronger condition

(H4) For every R ≥ 0, there exists C = C(R) ≥ 0 ,such that

$$\sup_{\mathrm{P}u_i\mathrm{P}_{V^1} \leq R} \left\| B[u_1] - B[u_2] \right\|_{V^0} \leq C \left\| u_1 - u_2 \right\|_{V^0}.$$

Lemma 2.3[4] Let X be a Banach space ,and let $¢ \subset C(¡^{+}, X)$ Let E : X→ ¡ be a function such that

$$\sup_{t \in ¡^{+}} E(z(t)) \geq -m \quad \text{and} \quad E(z(0)) \leq M,$$

for some m, M ≥ 0 and every z ∈¢ . In addition, assume that for every z∈ ¢ the function t a $E(z(t))$ is continuously differentiable and satisfies the differential inequality

$$\frac{d}{dt} E(z(t)) + \delta \left\| z(t) \right\|_X^2 \leq k,$$

For some $\delta > 0$ and $k > 0$ both independent of z∈ ¢ then

$$E(z(t)) \leq \sup_{\varsigma \in X} \left\{ E(\varsigma) : \delta \left\| \varsigma \right\|_X^2 \leq 2k \right\}, \quad \forall t \geq \frac{m+M}{k}.$$

Lemma 2.4 [5] Let Φ be an absolutely continuous positive function on $\mathbb{R}^+$ which satisfies for some $\varepsilon > 0$ the differential inequality

$$\frac{d}{dt}\Phi(t) + 2\varepsilon\Phi(t) \le g(t)\Phi(t) + h(t),$$

for almost every $t \in \mathbb{R}^+$, where g and h are functions on $\mathbb{R}^+$, such that

$$\int_\tau^t |g(y)|dy \le m_1\left(1 + (t-\tau)^\mu\right), \qquad \forall t \ge \tau \ge 0,$$

for some $m_1 \ge 0$ and $\mu \in [0,1)$, and

$$\sup_{t\ge 0}\int_t^{t+1} |h(y)|dy \le m_2,$$

for some $m_2 \ge 0.$ Then

$$\Phi(t) \le \beta\,\Phi(0)e^{-\varepsilon t} + \rho, \;\; \forall t \in \mathbb{R}^+,$$

for some $\beta = \beta(m_1,\mu) \ge 1$ and $\rho = \beta m_2 e^{\varepsilon} / \left(1 - e^{-\varepsilon}\right).$

Lemma 2.5 [7] Let $\kappa_1, \kappa_2, \kappa_3$ be subsets of X such that

$$dist_X\left(S(t)\kappa_1, \kappa_2\right) \le L_1 e^{-v_1 t}, \quad dist_X\left(S(t)\kappa_2, \kappa_3\right) \le L_2 e^{-v_2 t},$$

for some $v_1, v_2 > 0$ and $L_1, L_2 \ge 0$. Assume also that for all $z_1, z_2 \in \bigcup_{t\ge 0} S(t)K_j$ $(j = 1,2,3)$ there holds

$$\|S(t)z_1 - S(t)z_2\|_X \le L_0 e^{v_0 t}\|z_1 - z_2\|_X$$

for some $v_0 \ge 0$ and some $L_0 \ge 0.$ Then it follows that

$$dist_X\left(S(t)\kappa_1, \kappa_3\right) \le Le^{-vt}, \text{ where } v = \frac{v_1 v_2}{v_0 + v_1 + v_2} \text{ and } L = L_0 L_1 + L_2.$$

3. ROBUST EXPONENTIAL ATTRACTORS IN H_ε^0

We need to verify conditions on B, assumptions (S), and (H1), (H2), (H3) in theorem 2.1. In fact, B[u] = g(u) is easily seen to fulfil conditions (2.9),(2.10) and (H4). By means of the Galerkin scheme adapted to systems with memory (see [8] for detail), one can show that, for every $\varepsilon \in [0,1)$, system (1.7) generates a strongly continuous semigroup $S_\varepsilon(t)$ on the phase-space H_ε^0.

Lemma 3.1 For every given R ≥ 0 ,there exist K = K(R) ≥ 0 such that

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{H_\varepsilon^0} \le Ke^{Kt}\|z_1 - z_2\|_{H_\varepsilon^0}, \tag{3.1}$$

Whenever $\|z_i\|_{H_\varepsilon^0} \le R.$

Proof : Set $\tilde{u} = u_1 - u_2, \tilde{\eta}(s) = \eta_1(s) - \eta_2(s),$ then we get

$$\begin{cases} \tilde{u}_t + \int_0^\infty \mu_\varepsilon(s)\tilde{\eta}(s)\mathrm{d}s = 0, \\ \tilde{\eta}_t = T_\varepsilon\tilde{\eta} + A\tilde{u} - A^{1/2}\tilde{u} + \tilde{u} + g(u_1) - g(u_2). \end{cases} \tag{3.2}$$

Multiplying $(3.2)_1$ by $A\tilde{u}$ in V^{-1} and $(3.2)_2$ by $\tilde{\eta}$ in M_ε^0, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{u}\|^2_{V^0} + \|\tilde{\eta}\|^2_{M^0_\varepsilon}) - 2\langle T_\varepsilon\tilde{\eta}, \tilde{\eta}\rangle_{M_\varepsilon^0} = -2\langle A^{1/2}\tilde{u}, \tilde{\eta}\rangle_{M_\varepsilon^0} + 2\langle \tilde{u}, \tilde{\eta}\rangle_{M_\varepsilon^0} + 2\langle g(u_1) - g(u_2), \tilde{\eta}\rangle_{M_\varepsilon^0}. \tag{3.3}$$

Because of $(1.2),(2.3)$ and $(2.5),$ we have the following estimates

$$-2\langle A^{1/2}\tilde{u},\tilde{\eta}\rangle_{M_\varepsilon^0} \le \int_0^\infty \mu_\varepsilon(s)\|-A^{1/2}\tilde{u}\|^2_{V^{-1}}ds + \int_0^\infty \mu_\varepsilon(s)\|\tilde{\eta}\|^2_{V^{-1}}ds \le c(\|\tilde{u}\|^2_{V^0}+\|\tilde{\eta}\|^2_{M_\varepsilon^0}),$$

$$2\langle \tilde{u},\tilde{\eta}\rangle_{M_\varepsilon^0} \le c(\|\tilde{u}\|^2_{V^0}+\|\tilde{\eta}\|^2_{M_\varepsilon^0}),$$

$$\begin{aligned}
&2\langle g(u_1)-g(u_2),\tilde{\eta}\rangle_{V^{-1}} \le c\|g(u_1)-g(u_2)\|_{V^{-1}}\|\tilde{\eta}\|_{V^{-1}}\\
&\le c(\|g(u_1)-g(u_2)\|^2_{V^{-1}}+\|\tilde{\eta}\|^2_{V^{-1}})\\
&\le c(\|\tilde{u}\|^2_{V^{-1}}+\|\tilde{\eta}\|^2_{V^{-1}}),\\
&2\langle g(u_1)-g(u_2),\tilde{\eta}\rangle_{M_\varepsilon^0} \le c(\|\tilde{u}\|^2_{V^0}+\|\tilde{\eta}\|^2_{M_\varepsilon^0}),
\end{aligned}$$

Hence

$$\frac{d}{dt}(\|\tilde{u}\|^2_{V^0}+\|\tilde{\eta}\|^2_{M_\varepsilon^0}) \le c(\|\tilde{u}\|^2_{V^0}+\|\tilde{\eta}\|^2_{M_\varepsilon^0}). \tag{3.4}$$

By Gronwall lemma ,we complete the proof.

Lemma 3.2 Condition (H1) holds for i = 0.

Proof : For any given $z=(u,v)\in H_\varepsilon^0$, we consider the functional

$$E(z)=\|z\|^2_{H_\varepsilon^0}+\|u\|^2_{V^{-\frac{1}{2}}}+\|u\|^2_{V^{-1}}+2\langle G(u),1\rangle_{V^{-1}}-2\alpha\varepsilon\langle u,\eta\rangle_{M_\varepsilon^0},$$

where G(r) = $\int_0^r g(\rho)d\rho,$ for some $\alpha\ge 0$ which will be chosen small enough so that the following estimates hold. From (1.3) and $(2.2),$ there exists $\lambda\in(0,1)$ such that

$$\langle g(u),u\rangle_{V^{-1}} \ge -(1-\lambda)\|u\|^2_{V^0}-c\ , \tag{3.5}$$

$$2\langle G(u),1\rangle_{V^{-1}} \ge -(1-\lambda)\|u\|^2_{V^0}-c, \tag{3.6}$$

$$2\alpha\varepsilon\left|\langle \mu,\eta\rangle_{M_\varepsilon^0}\right| \le \frac{\lambda}{4}\|u\|^2_{V^0}+\frac{1}{2}\|\eta\|^2_{M_\varepsilon^0}. \tag{3.7}$$

The last two inequalities, together with $(1.2),$ yield

$$\frac{\lambda}{2}\|\eta\|^2_{H_\varepsilon^0}-c\le E(z)\le c(1+\|z\|^3_{H_\varepsilon^0}). \tag{3.8}$$

We now fix $z\in H_\varepsilon^0$ with $\|z\|_{H_\varepsilon^0}\le R,$ and we denote $(u(t),\eta^t)=S_\varepsilon(t)z.$ Multiplying $(1.7)_1$ by $Au-A^{1/2}u+u+g(u)$ in V^{-1} and $(1.7)_2$ by η in M_ε^0, we obtain

$$\langle u_t,Au\rangle_{V^{-1}}-\left\langle u_t,A^{1/2}u\right\rangle_{V^{-1}}+\langle u_t,u\rangle_{V^{-1}}+\langle u_t,g(u)\rangle_{V^{-1}}+\langle \eta_t,u\rangle_{M_\varepsilon^0}-\langle T_\varepsilon\eta,\eta\rangle_{M_\varepsilon^0}=0.$$

Hence ,we get

$$\frac{d}{dt}(\|u\|^2_{V^0}+\|\eta\|^2_{M_\varepsilon^0}+\|u\|^2_{V^{-\frac{1}{2}}}+\|u\|^2_{V^{-1}}+2\langle G((u),1\rangle_{V^{-1}})-\int_0^\infty u_\varepsilon{}'(s)\|\eta(s)\|^2_{V^{-1}}\,ds=0\,. \tag{3.9}$$

Besides, recalling (2.2), and multiplying $(1.7)_2$ by u in M_ε^0, it is straightforward to obtain

$$\frac{d}{dt}\langle u,\eta\rangle_{M_\varepsilon^0}=\langle u_t,\eta\rangle_{M_\varepsilon^0}+\langle T_\varepsilon\eta,u\rangle_{M_\varepsilon^0}+\frac{1}{\varepsilon}\|u\|_{V^0}^2+\frac{1}{\varepsilon}\|u\|_{V^{-\frac{1}{2}}}^2+\frac{1}{\varepsilon}\|u\|_{V^{-1}}^2+\frac{1}{\varepsilon}\langle g(u),u\rangle_{V^{-1}}. \quad (3.10)$$

Therefore, $E=E(S_\varepsilon(t)z)$ satisfies

$$\frac{d}{dt}E-\int_0^\infty \mu_\varepsilon{}'(s)\|\eta\|_{V^{-1}}^2\,ds=-2\alpha\|u\|_{V^{-1}}^2-2\alpha\|u\|_{-\frac{1}{2}}^2-2\alpha\|u\|_{V^0}^2$$
$$-2\alpha\langle g(u),u\rangle_{V^{-1}}-2\alpha\varepsilon\int_0^\infty \mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{-1}}\,ds+2\alpha\varepsilon\left\|\int_0^\infty \mu_\varepsilon(s)\eta(s)\right\|_{V^{-1}}^2 \quad (3.11)$$

Using $(3.5)-(3.7)$, we obtain

$$\frac{d}{dt}(E(S_\varepsilon(t)z))+2\alpha\lambda\|u\|_{V^0}^2-\int_0^\infty \mu_\varepsilon'(s)\|\eta(s)\|_{V^{-1}}^2 ds$$
$$\leq -2\alpha\|u\|_{V^{-\frac{1}{2}}}^2-2\alpha\|u\|_{V^{-1}}^2-2\alpha\varepsilon\int_0^\infty \mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{-1}}\,ds$$
$$+2\alpha\varepsilon\left\|\int_0^\infty \mu_\varepsilon(s)\eta(s)\right\|_{V^{-1}}^2+\alpha c. \quad (3.12)$$

Because of $(1.7)_1$ and (2.7), the terms on the right-hand side can be controlled as

$$-2\alpha\|u\|_{V^{-\frac{1}{2}}}^2-2\alpha\|u\|_{V^{-1}}^2\leq\frac{2\alpha\lambda}{3}\|u\|_{V^0}^2,$$
$$-2\alpha\varepsilon\int_0^\infty \mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{-1}}ds\leq\frac{\alpha\lambda}{3}\|u\|_{V^0}^2-\frac{1}{2}\int_0^\infty \mu_\varepsilon'(s)\|\eta(s)\|_{V^{-1}}^2\,ds+\alpha c,$$
$$2\alpha\varepsilon\left\|\int_0^\infty \mu_\varepsilon(s)\eta(s)\right\|_{V^{-1}}^2\leq\frac{\delta}{4\varepsilon}\|\eta\|_{M_\varepsilon^0}^2+\alpha c.$$

By the force of (2.1), and choose α small enough we get

$$\frac{d}{dt}E(S_\varepsilon(t)z)+\alpha\lambda\|S_\varepsilon(t)z\|_{H_\varepsilon^0}^2+\frac{\delta}{8\varepsilon}\|\eta\|_{M_\varepsilon^0}^2\leq\alpha c. \quad (3.13)$$

In view of (3.8), from lemma 2.3, there exists $t_0=t_0(R)$ such that

$$E(S_\varepsilon(t)z)\leq\sup_{\varsigma\in H_\varepsilon^0}\{E(\varsigma):\|\varsigma\|_{H_\varepsilon^0}^2\leq c\},\qquad \forall t>t_0.$$

Using (3.8), and subsequently integrating (3.13) on $(0,t_0)$, we meet the claim.

Remark 3.3 In view of (2.7), and integrating (3.13) with $\alpha=0$ on ¡ $^+$, we find the following Integral estimate

$$\int_0^\infty \left\|u_t(y)dy\right\|_{V^{-1}}^2\leq\Lambda_0, \quad (3.14)$$

for some $\Lambda_0=\Lambda_0(R)\geq 0.$

Lemma 3.4 Under the above assumptions ,there exist $t_0>0$ such that

$$\|u\|_{V^{1/2}}=\left\|A^{1/2}u\right\|\leq C,\ \forall t\geq 0.$$

Proof : Multiply (1.1) with Au in $V^{-1/2}$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2+\|Au\|^2+\left\|A^{1/2}u\right\|=-\langle g(u),Au\rangle+\left\langle A^{1/2}u,Au\right\rangle. \quad (3.15)$$

we have the following controls

$$-\langle g(u),Au\rangle=\int_\Omega|Au||g(u)|dx\le\|Au\|(\int_\Omega|g(u)|^2dx)^{1/2}\le\frac{\|Au\|^2}{4}+\int_\Omega|g(u)|^2dx,$$

$$\left\langle A^{1/2}u,Au\right\rangle=\int_\Omega\left|A^{1/2}u\right||Au|dx\le\|Au\|\left\|A^{1/2}u\right\|\le\frac{\|Au\|^2}{4}+\left\|A^{1/2}u\right\|^2.$$

Hence we have

$$\frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2\le\int_\Omega|g(u)|^2dx, \quad (3.16)$$

because of (1.2), we get

$$\int_\Omega|g(u)|^2dx\le C\int_\Omega(1+|u|^6)dx\le C+C\ \|u\|_{L^6}^6.$$

We set $\tilde{A}=-\Delta$ subject to the Dirichlet boundary condition then we have

$$D(\tilde{A})=H_0^2(\Omega)\subset H_0^{\frac{2}{3}}(\Omega)=D(\tilde{A}^{\frac{1}{3}})\subset L^2(\Omega)=D(\tilde{A}^0),$$

Lemma 3.5 There exist $\beta>0$ and an increasing function ϑ such that, if $\|z\|_{H_\varepsilon^\sigma}\le R$ for some $\sigma\in(0.1]$, then

$$\|S_\varepsilon(t)z\|_{H_\varepsilon^\sigma}\le M_\sigma e^{-\beta t}+\vartheta(\|z\|_{H_\varepsilon^0})$$

And

$$\int_\tau^t\|u_t(y)\|_{V^{\sigma-1}}^2dy\le\Lambda_\sigma\left[1+(t-\tau)^{\frac{1}{2}}\right]$$

For some $M_\sigma(R)\ge0$ and $\Lambda_\sigma=\Lambda_\sigma(R)\ge0.$

Proof :For $\alpha\ge0$ small enough we introduce the energy functional

$$E(t)=\|S_\varepsilon(t)z\|_{H_\varepsilon^\sigma}^2+\|u\|_{\sigma-\frac{1}{2}}^2+\|u\|_{V^{\sigma-1}}^2+2\langle g(u),u\rangle_{V^{\sigma-1}}-2\alpha\varepsilon\langle u,\eta\rangle_{M_\varepsilon^\sigma}+c.$$

Choose the constant c appearing in $E(t)$ large enough it is apparent that

$$\frac{1}{2}\|S_\varepsilon(t)z\|_{M_\varepsilon^\sigma}^2\le E(t)\le2\|S_\varepsilon(t)z\|_{H_\varepsilon^\sigma}^2+c\,. \quad (3.17)$$

Multiplying $(1.7)_1$ by $Au-A^{\frac{1}{2}}u+u+g(u)$ in $V^{\sigma-1},-2\alpha\varepsilon\eta$ in M_ε^σ and $(1.7)_2$ by η in $M_\varepsilon^\sigma,-2\alpha\varepsilon\mu$ in M_ε^σ, this yields

$$\frac{d}{dt}E(t)-\int_0^\infty\mu_\varepsilon'(s)\|\eta(s)\|_{V^{\sigma-1}}^2ds+2\alpha\|u\|_{V^\sigma}^2$$
$$+2\alpha\|u\|_{V^{\sigma-\frac{1}{2}}}^2+2\alpha\|u\|_{V^{\sigma-1}}^2+2\alpha\langle g(u),u\rangle_{V^{\sigma-1}}$$
$$=2\langle g'(u)u_t,u\rangle_{V^{\sigma-1}}+2\alpha\varepsilon\left\|\int_0^\infty\mu_\varepsilon(s)\eta(s)\right\|_{V^{\sigma-1}}^2$$
$$-2\alpha\varepsilon\int_0^\infty\mu_\varepsilon'(s)\langle u,\eta(s)\rangle_{V^{\sigma-1}}ds. \quad (3.18)$$

Arguing as in the previous proof and taking into account lemma $3.2,$ we have

$$2\alpha\varepsilon\left\|\int_0^\infty \mu_\varepsilon(s)\eta(s)\right\|^2_{V^{\sigma-1}} - 2\alpha\varepsilon\int_0^\infty \mu'_\eta(s)\langle u,\eta(s)\rangle_{V^{\sigma-1}}ds$$
$$\leq 2\alpha\|\eta\|^2_{M^\sigma_\varepsilon} - \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|\eta(s)\|^2_{V^{\sigma-1}} + \alpha c. \qquad (3.19)$$

Hence ,we obtain

$$\frac{d}{dt}E(t) + \alpha E(t) - \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|\eta(s)\|^2_{V^{\sigma-1}}ds$$
$$\leq 2\langle g'(u)u_t,u\rangle_{V^{\sigma-1}} + 2\alpha\|\eta\|^2_{M^\sigma_\varepsilon} - 2\alpha^2\varepsilon\langle u,\eta\rangle_{M^\sigma_\varepsilon} + \alpha c. \qquad (3.20)$$

Moreover, using the embedding $V^\sigma = D(A^{\sigma+\frac{1}{2}/2}) \hookrightarrow L^{\frac{3}{1-\sigma}}(\Omega)$. we have the controls

$$2\langle g'(u)u_t,u\rangle_{V^{\sigma-1}} \leq c(1+\|u\|_{L^{3/1-\sigma}})\|u_t\|_{L^{\frac{3}{2}}}\left\|A^{\frac{2\sigma-1}{2}}u\right\|_{L^{\frac{3}{\sigma}}}$$
$$\leq c(1+\|u\|_{V^\sigma})\|u_t\|_{V^{-1}}\left\|A^{(\sigma+\frac{1}{2})/2}u\right\|$$
$$\leq c(1+\|u\|_{V^\sigma})\|u_t\|_{V^{-1}}\|u\|_{V^\sigma} \qquad (3.21)$$

Exploiting (2.1) and

$$-2\alpha^2\varepsilon\langle u,\eta\rangle_{M^0_\varepsilon} \leq \alpha\|\eta\|^2_{M^\sigma_\varepsilon} + \alpha c,$$

provided that α is small enough we end up with

$$\frac{d}{dt}E + \alpha E + \frac{\delta}{8\varepsilon}\|\eta\|^2_{M^0_\varepsilon} \leq \alpha c + c\|u_t\|_{V^{-1}} + c\|u_t\|_{V^{-1}}E\ . \qquad (3.22)$$

Because of (3.14), we apply lemma 2.4 to get

$$E(t) \leq cE(0)e^{-(\alpha/2)t} + c.$$

Finally, integrating the differential inequality (3.20) with $\alpha = 0$ on (τ,t), and using (2.7), we complete the proof.

Remark 3.6 In particular, for $\sigma = 1,$ setting $R_1 = 2\vartheta(R_0)$ and $C_1 = M_1 + \vartheta(C_0),$ we see that condition (H1) is satisfied for i = 1 as well.

Remark 3.7 From the estimates of (3.19) we know that the sobolev embedding is the maximal but how to control a higher growth condition than (1.2) is open.

In order to prove (H2) analogously to what observed in [11] a function $g \in C^2(R)$ such that $(1.2),(1.3)$ admits a decomposition $g = g_0 + g_1$ satisfying

$$|g_0'(r)| \leq c(1+|r|),\quad g_0'(0)=0,\quad g_0(r)r \geq 0,\quad |g_1'(r)| \leq c.\ \ \forall r \in \mathbb{R} \qquad (3.23)$$

Next, for any fixed $z \in B^0_\varepsilon(R_0),$ we make the decomposition

$$S_\varepsilon(t)z = D_\varepsilon(t)z + K_\varepsilon(t)z$$

where $D_\varepsilon(t)z = (v(t),\xi^t)$ and $K_\varepsilon(t)z = (w(t),\varsigma^t)$ are the solutions to the problems

$$\begin{cases} v_t + \int_0^\infty \mu_\varepsilon(s)\xi(s)ds = 0, \\ \xi_t = T_\varepsilon \xi + Av - A^{1/2}v + v + g_0(v), \\ D_\varepsilon(0)z = z \end{cases} \tag{3.24}$$

and

$$\begin{cases} w_t + \int_0^\infty \mu_\varepsilon(s)\varsigma(s)ds = 0, \\ \varsigma_t = T_\varepsilon \varsigma + Aw - A^{1/2}w + w + g(u) - g_0(v), \\ K_\varepsilon(0)z = 0 \end{cases} \tag{3.25}$$

By a slight modification of lemma 3.2, it can be shown that there exist $M_0 \geq 0$ and $\kappa_0 > 0$ such that

$$\|D_\varepsilon(t)z\|_{H_\varepsilon^0} \leq M_0 e^{-\kappa_0 t}, \tag{3.26}$$

Lemma 3.8 Given $\sigma \in [0, \frac{3}{4}]$, if $\|z\|_{H_\varepsilon^\sigma} \leq R,$ there exists $\prod_\sigma = \prod_\sigma(R) \geq 0$ such that

$$\|K_\varepsilon(t)\|_{H_\varepsilon^{\frac{1+4\sigma}{4}}} \leq \prod_\sigma, \quad \forall t \geq 0.$$

Proof : From the previous results we know $\|K_\varepsilon(t)z\|_{H_\varepsilon^0} \leq c.$ For $\alpha > 0$ small enough We introduce the energy functional

$E(t) = \|K_\varepsilon(t)z\|^2_{H_\varepsilon^{\frac{1+4\sigma}{4}}} + \|w\|^2_{V^{\frac{4\sigma-1}{4}}} + \|w\|^2_{V^{\frac{4\sigma-3}{4}}} + 2\langle g(u) - g_0(v), w\rangle_{V^{\frac{4\sigma-3}{4}}} - 2\alpha\varepsilon\langle w, \varsigma\rangle_{M_\varepsilon^{\frac{1+4\sigma}{4}}} + c.$ Choose the constant c appearing in $E(t)$ large enough it is apparent that

$$\frac{1}{2}\|K_\varepsilon(t)z\|^2_{H_\varepsilon^{\frac{1+4\sigma}{4}}} \leq E(t) \leq 2\|K_\varepsilon(t)z\|^2_{H_\varepsilon^{\frac{1+4\sigma}{4}}} + c. \tag{3.27}$$

Arguing as in the previous proof we obtain

$$\begin{aligned} \frac{d}{dt}E(t) + \alpha E(t) + \frac{\delta}{4\varepsilon}\|\varsigma\|^2_{M_\varepsilon^{\frac{1+4\sigma}{4}}} &\leq 2\langle (g_0'(u) - g_0'(v))u_t, w\rangle_{V^{\frac{4\sigma-3}{4}}} \\ &+ 2\langle g_1'(u)u_t, w\rangle_{V^{\frac{4\sigma-3}{4}}} + 2\langle g_0'(v)w_t, w\rangle_{V^{\frac{4\sigma-3}{4}}} + c. \end{aligned} \tag{3.28}$$

Due to $(3.21), (2.5)$ and $V^{\sigma-1} = D(A^{(\sigma-\frac{1}{2})/2}) \hookrightarrow L^{\frac{3}{2-\sigma}}(\Omega),$ we get

$$\begin{aligned} 2\langle (g_0'(u) - g_0'(v))u_t, w\rangle_{V^{\frac{4\sigma-3}{4}}} &\leq c\|w\|_{L^4}\|u_t\|_{L^{\frac{3}{2-\sigma}}}\|A^{\frac{4\sigma-1}{\sigma}}w\|_{L^{\frac{12}{4\sigma+1}}} \\ &\leq c\left\|A^{\frac{3}{8}}w\right\|\|u_t\|_{V^{\sigma-1}}\|A^{\frac{3+4\sigma}{8}}w\| \\ &\leq c\|u_t\|_{V^{\sigma-1}}\|A^{\frac{3+4\sigma}{8}}w\|^2 \end{aligned}$$

$$\le c\|u_t\|_{V^{\sigma-1}}\|w\|^2_{\frac{1+4\sigma}{4}}$$
$$\le c\|u_t\|_{V^{\sigma-1}}E,$$

on account of $(3.21),(2.5),$ we obtain

$$2\langle g_1(v)u_t,w\rangle_{V^{\frac{4\sigma-3}{4}}}\le c\|u_t\|_{L^{\frac{3}{2-\sigma}}}\|A^{\frac{4\sigma-1}{\sigma}}w\|_{L^{\frac{12}{4\sigma+1}}}$$
$$\le c\|u_t\|_{V^{\sigma-1}}\|A^{\frac{3+4\sigma}{8}}w\|^2$$
$$\le c\|u_t\|_{V^{\sigma-1}}+c\|u_t\|_{V^{\sigma-1}}E,$$

because of $(2.5),(2.7),(3.21),$ this yields

$$2\langle g_0'(v)w_t,w\rangle_{V^{\frac{4\sigma-3}{4}}}\le c\|v\|_{L^{\frac{3}{2}}}\|w_t\|_{L^{\frac{12}{3-4\sigma}}}\|A^{\frac{4\sigma-1}{\sigma}}w\|_{L^{\frac{12}{4\sigma+1}}}$$
$$\le c\|v\|_{V^0}\|w_t\|_{V^{\frac{1+4\sigma}{4}}}\|w\|_{V^{\frac{1+4\sigma}{4}}}$$
$$\le\frac{\delta}{4\varepsilon}\|\varsigma\|^2_{M_\varepsilon^{\frac{1+4\sigma}{4}}}+c\|v\|_{V^0}E.$$

Hence

$$\frac{d}{dt}E(t)+\alpha E(t)\le hE+h+c,$$

where $h=c\|v(t)\|_{V^0}+c\|u_t\|_{V^{\sigma-1}}$. Due to (3.24) lemma 3.4 and the inequality, we get

$$\int_\tau^t h(y)dy\le c_\sigma[1+(t-\tau)^{3/4}],$$

For some $c_\sigma\ge 0.$
According to lemma 2.4 ,we meet the claim.

Lemma 3.9 Condition (H2) holds.

Proof : Successive applications of lemma 3.6 for $\sigma=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},$ together with the exponential decay $(3.24),$ we construct a sequence of five balls starting from $B_\varepsilon^0(R_0),$ each exponentially attracted by the next one. Thus on account of the continuous dependence estimate $(3.1),$ the transitivity of the exponential attraction lemma 2.5 entails the desired property (H2).

Our main theorem reads as follows.

Theorem 3.10 Assume that $g\in C^2(\mathbb{R}),$ and satisfies $(1.2),(1.3)$ with $g(0)=0.$ Then there exist robust exponential attractors $\{\varepsilon_\varepsilon\}\subset H_\varepsilon^0$ for the semigroup of operators $S_\varepsilon(t)$ generated by system $(1.7).$

As a straightforward consequence we have the following corollary.

Corollary 3.11 The semigroup $S_\varepsilon(t)$ acting on the phase-space H_ε^0 possesses a connected global attractor $A_\varepsilon\subset\varepsilon_\varepsilon.$ In particular the fractal dimension of A_ε in H_ε^0 is uniformly bounded with respect to ε .

REFERENCES

[1]. H. Haken Advanced Synergetirs Instability hierarchies of self-organizing systems and devices Springer-Verlag. Berlin (1983);

[2]. A. C. Newell T. Passot J. Lega Order Parameter Equations for Patterns Annual Review of Fluid Mechanics. 25 (1993);

[3]. A. Ion A. Georgescu On the existence and on the fractal and hausdorff dimensions of some global attractor Nonlinear Analysis TMA 8 (1997):5527-5532.

[4]. V. Belleri V. Pata Attractors for semilinear strongly dampedwave equation on R^3 Discrete Contin. Dynam. Systems 7 (2001):719-735.

[5]. M. Grasselli V. Pata Asymptotic behavior of a parabolic-hyperbolic system Commun. Pure Appl. Anal. 38 (2004):49-81.

[6]. S. Gatti M. Grasselli A. Miranville and V. Pata Memory relaxation of first order evolution equations Nonlinearity.18 (2005):1859-1883.

[7]. Fabrie P. Galusinski C. Miranville A. Zelik S. Uniform exponential attractors for a singularly perturbed damped wave equation. Discrete Contin. Dynam. Systems 10 (2004):211-238.

[8]. Grasselli M and Pata V. Existence of a universal attractor for a fully hyperbolic phase-field system. J. Evol. Eqns 4 (2004):27-51.

[9]. A.EdenC.FoiasB.NicolaenkoR.Temam Exponential Attractors for Dissipative Evolution Equations New York:Masson.Paris:Wiely(1994).

[10]. R.Temam Infinite-dimensional dynamical systems in mechanics and physic Springer New York(1997).

[11]. Arrieta J. Carvalho A.N. Hale J.K. A damped hyperbolic equation with critical exponent Comm. Partial Differ. Eqs.17 (1992):841-866.

[12]. Grasselli M. Pata V. Uniform attractors of nonautonomous systems with memoryEvolution Equations Semigroups and Functional Analysis (Progress in Nonlinear Differential Equations and their Applications vol 50) ed A Lorenzi and B Ruf (Boston MA:Birkhauser)(2002):155-178.