QUICK GLANCE ON DIFFERENT WAVELETS AND THEIR OPERATIONAL MATRIX PROPERTIES: A REVIEW

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ABSTARCT

In the present work an attempt has been taken to understand the basic behind the different kind of Wavelets with their important properties, characteristic and applications, main emphasize is on the orthonormal function. The manuscript is basically divided in to seven sections. In first section we start with introduction methodology of the corresponding work and in the second section we give a glance on the basic definition of particular wavelets transform and kind of wavelet transform with approximating approach. In part third we define operational matrix, its role and importance in the above field. In fourth section we look at the Legendre Wavelets and its difference between Wavelets along with that we gives the properties of the Legendre Wavelets along with the application, its quite interesting to see their approximation also in the above work. In the section five we give a detail of CAS wavelets, its properties application and its approximation advantages with approximation detail along with that, a comparison with others also shown in brief. In the next section we move on to the Sine-cosine wavelets, its properties, summary application and there role in the particular field. Finally we conclude in the last section that how the wavelet transform is useful tool in present work.

Keywords: *Legendre wavelets, CAS wavelets, Sine-Cosine wavelets, Operational matrix.*

1. INTRODUCTION

The theory of approximation and transformation plays an important role in real and applied field. In mathematics**, approximation theory** is concerned with how functions can best be approximated with simpler functions, and with quantitatively characterizing the errors introduced thereby. Note that what is meant by *best* and *simpler* will depend on the application. The objective is to make the approximation as close as possible to the actual function, typically with accuracy while the transformation is means translation of the particular task, it might have a some similar properties but different characteristics. The advantage of these technique is found while solving the some complicated problem of Mathematics(Non-linear equations, ODE, PDE, and etc)

Here our attention is given to some particular type of function approximation and there properties. Wavelet theory is relatively new and an emerging area in mathematical research. In recent years, wavelets have found their way in different fields of science & engineering. Wavelets permit the accurate representation of a variety of functions &operators. Orthogonal functions (OFs) and polynomial series have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. The approach is based on converting the underlying differential equations into integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix P of integration, to eliminate the integral operations.The matrix P is given by

$$\int_0^t \phi(t')dt' \approx P\phi(t),$$

Where $\phi(t)=[\phi_0,\phi_1,....,\phi_{n-1}]^T$ and the matrix P can be uniquely determined on the basis of the particular orthogonal functions. The elements $\phi_0,\phi_1,.....,\phi_{n-1}$ are the basis functions, orthogonal on a certain interval $[a,b]$ .Special attention has been given to applications of Walsh functions (**Chen and Hsiao** 1975),block-pulse functions (**Chen***et al.*1977),Laguerre series (**Hwang and Shih** 1983),Legendre polynomials (**Chang and Wang** 1983),Chebyshev polynomials (**Horng and Chou** 1985),Fourier series(**Razzaghi and Razzaghi** 1988) and Bessel series (**Paraskevopoulo s***et al.* 1990).Among these orthogonal functions, the shifted Legendre, which is obtained from Legendre polynomials by shifting the defining interval $[-1,1]$ to $[0,\alpha]$, is computationally more effective. This is because

1. the defining domain is finite,
2. the operational matrix of integration is tridiagonal,
3. the weight function of orthogonality is unity,
4. the convergence rate is rapid.

There are three classes of sets of OFs which are widely used. The first includes sets of piecewise constant basis functions (PCBFS) (eg,Walsh,block-pulse,etc).The second consists of sets of polynomials(OPs)(eg,Laguerre,Legendre,Chebyshev,etc).The third is the widely used sets of sine-cosine functions(SCFs) in Fourier series. While OPs and SCFs together form a class of continuous basis function, PCBFs have inherent discontinuities or jumps.

In 2001, **M.Razzaghi & S.Yousefi** presented an operational matrix of integration P based on Legendre wavelets. A general procedure for forming this matrix was given with examples.

In 2000, **M.Razzaghi & S.Yousefi** introduced Legendre wavelets operational matrix P,together with the integration of the product of two Legendre wavelets vectors, & then used to solve the variational problems. It was also shown that the Legendre wavelets provide an exact solution for the heat conduction problem.

According to the Translation property of Legendre wavelets, another method for computing operational matrix of integration P was presented through integrating on subintervals, **Xiao-Yang Zhang, Xiao-Fan Yang, Young-Wu** (2008)

The CAS wavelet operational matrix P of integration was first presented by **HanDanfu, ShangXuFeng**(2007),which was then utilised to reduce the integro-differential equation to the algebraic equations.

In 2006, **S. Yousefi, A. Banifatemi** presented CAS wavelet approximation method for solving Fredholm integral equations which was then utilized to reduce the Fredholm integral equations to the solution of algebraic equations.

In 2002, an operational matrix of integration P based on Sine-cosine wavelets was presented .This matrix can be used to solve problems such as identification, analysis and optimal control like those of other orthogonal function. It was also shown that SCW provide an exact solution for the heat conduction problem considered in **Chen & Hsiao**(1975).

Wavelet analysis possesses several useful properties, such as orthogonality, compact support, exact representation of polynomials to a certain degree, and ability to represent functions at different levels of resolution.

1.1 Definitions

Definition1.1.

Wavelets constitute a family of functions constructed from dilation and translation of a single function called Mother Wavelet. When the dilation parameter a and translation parameter b vary continuously we have the following family of continuous wavelets as (**GU and Jiang** 1996)

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \qquad a,b\in\Re,\ a\neq 0.$$

If we restrict the parameters $a\,\&\,b$ to discrete values of $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0\ \&\ n,$ and k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t)=|a_0|^{\frac{k}{2}}\ \psi\left(a_0^k t-nb_0\right),$$

Where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$.

In particular, when $a_0=2$ and $b_0=1$, then $\psi_{k,n}(t)$ forms an Orthonormal basis,(**GU and Jiang**1996).

Definition1.2.

Legendre polynomials $L_m(t)$ on the interval [-1, 1]

$$L_0(t)=1,$$
$$L_1(t)=t,$$
$$L_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)tL_m(t)-\left(\frac{m}{m+1}\right)L_{m-1}(t) \qquad m=1,2,3.............$$

Set $\{L_m(t): m=0,1,......\}$ in Hilbert space $L^2[-1,1]$ is a complete orthogonal set.

Definition 1.3:

A **translation operator** is defined by

$$T(i)\,f : f(x)\rightarrow g(x)$$

Where the operator $T(i)$ denotes the function $g(x)$ derived from the function $f(x)$ by translating i times on the right and left. While the $T_{right}(i)$ and $T_{left}(i)$ denote the function $g(x)$ derived from the function $f(x)$ by translating i times on the right and left, respectively.

Operational Matrix:

The basic idea of this technique is as follows:

1. The differential equation is converted to an integral equation via multiple integration.
2. Subsequently, the various signals involved in the integral equation are approximated by representing them as linear combinations of the orthonormal basis functions and truncating them at optimal levels.
3. Finally, the integral equation is converted to an algebraic equation by introduction the operational matrix of integration of the basis functions.

Using operational matrix of an orthonormal system of functions to perform integration for solving, identifying & optimizing a linear dynamic system has several advantages:

- Method is computer oriented, thus solving higher order differential equation becomes a matter of dimension increasing.
- Solution is multiresolution type.
- Solution is convergent, even though the size increment may be large.

Approximations by orthonormal family of functions have played a vital role in development of physical sciences, engineering & tech. in general & mathematical analysis in particular since long in last three decades.

2. PROPERTIES OF LEGENDRE WAVELETS

2.1 *Wavelets and Legendre Wavelets:*

Legendrewavelets $\psi_{n,m}(t)=\psi(k,\hat{n},m,t)$ have four arguments; $\hat{n}=2n-1, n=1,2,3,.....,2^{k-1}$, k can assume any positive integer, m is the order for Legendre polynomials and t is normalized time.

They are defined on the interval [0, 1) as:

$$\psi_{n,m}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\;2^{\frac{k}{2}}L_m\left(2^k t-\hat{n}\right), & for\;\; \frac{\hat{n}-1}{2^k}\le t\le\frac{\hat{n}+1}{2^k}, \\ 0 & otherwise\end{cases} \quad (2.1)$$

Where $m=0,1,.........,M-1, n=1,2,3,.........,2^k-1.$ In eq(2.1) ,the coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a=2^{-k}$ and translation parameter is $b=\hat{n}\;2^{-k}$. Here $L_m(t)$ are the well-known Legendre polynomials of order m. The set of Legendre wavelets are an Orthonormal set.

2.2 Function Approximation:

A function $f(t)$ defined over [0, 1) may be expanded as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\,\psi_{n.m}(t), \qquad (2.2)$$

Where $C_{nm}=(f(t),\psi_{n,m}(t)),$ in which (.,.) denotes the inner product. (2.3)
If the infinite series in eq(2.2) is truncated, theneq(2.2) can be written as

$$f(t)\cong\sum_{n=1}^{2^k-1}\sum_{m=0}^{M-1} C_{nm}\,\psi_{nm}(t)=C^T\psi(t), \qquad (2.4)$$

Where $C\,\&\,\psi(t)$ are $2^{k-1}M\times 1$ matrices given by

$$C=[C_{10},C_{11},.....,C_{1M-1},C_{20},.....,C_{2M-1},......C_{2^{k-1}0},.....,C_{2^{k-1}M-1}]^T\ , \qquad (2.5)$$

$$\psi(t)=[\psi_{10}(t),\psi_{11}(t),.....\psi_{1M-1}(t),\psi_{20}(t),.....,\psi_{2M-1}(t),.....,\psi_{2^{k-1}0}(t),.....,\psi_{2^{k-1}M-1}(t)]^T, \qquad (2.6)$$

Using (2.6), numerical computation can be done by wavelet method.

Actually,Legendre wavelets approximate a function by piecewise Legendre polynomials.Consequently,Legendre wavelets that are defined on the interval [0, 1) can be obtained through a translation operator transformation on Legendre wavelets, defined on the subintervals.Here,a concept of translation operator is presented.

Lemma 1: Legendre wavelets, defined on the interval [0, 1), can be obtained through the translation operator $T_{right}(m)$, which transforms on Legendre wavelets,defined on the subinterval $[0,\frac{1}{2^n})$ and satisfies

$$\psi_{nm}^{k}(x)=\begin{cases}\left(k+\frac{1}{2}\right)^{\frac{1}{2}}2^{\frac{n+1}{2}}L_k\left(2^{n+1}x-2m-1\right) & \frac{m}{2^n}\le x\le\frac{m+1}{2^n} \\ 0 & otherwise\end{cases} \qquad (2.7)$$

Where $m=0,1,........,2^n-1.$

Proof: Legendre wavelets,defined on the interval [0, 1), can be obtained through translation Legendre wavelets,defined on the subinterval $[0,\frac{1}{2^n}),\quad i(i=1,2,.....,2^n-1)$ times on the right.

Taking advantage of the translation property, the definition interval of Legendre wavelets can be extended. Thus, a concept of the extended Legendre wavelets was presented later on.

2.3 Legendre wavelets operational matrix of integration:

In this section the operational matrix of integration P will be derived.First,we find the matrix P with $M=3$ and $k=2.$The six basis functions are given by

$$\left.\begin{array}{l}\psi_{10} = 2^{\frac{1}{2}} \quad \square \\ \psi_{11} = 6^{\frac{1}{2}}(4t-1)\ , \\ \psi_{12} = (10)^{\frac{1}{2}}[\frac{3}{2}(4t-1)^2 - \frac{1}{2}],\end{array}\right\} \quad 0 \le t < \frac{1}{2}, \qquad (2.8)$$

and

$$\left.\begin{array}{l}\psi_{20} = 2^{\frac{1}{2}}\ , \\ \psi_{21} = 6^{\frac{1}{2}}(4t-3), \\ \psi_{22} = (10)^{\frac{1}{2}}\left[\frac{3}{2}(4t-3)^2 - \frac{1}{2}\right],\end{array}\right\} \quad \frac{1}{2} \le t < 1\ , \qquad (2.9)$$

By integrating (2.8) and (2.9) from 0 to t and using (2.3),we obtain

$$\int_0^t \psi_{10}(t)\,dt = \begin{cases} 2^{\frac{1}{2}} t & 0 \le t < \frac{1}{2} \\ \frac{2^{\frac{1}{2}}}{2} & \frac{1}{2} \le t < 1 \end{cases}$$

$$= \frac{1}{4}\psi_{10} + \frac{2^{\frac{1}{2}}}{4 \times 6^{\frac{1}{2}}}\psi_{11} + \frac{1}{2}\psi_{20} \quad = \left[\frac{1}{4}, \frac{2^{\frac{1}{2}}}{4 \times 6^{\frac{1}{2}}}\ , 0\ , \frac{1}{2}\ 0\ ,\ 0\right]^T \psi_6(t)\ ,$$

$$\int_0^t \psi_{11}(t)\,dt = \begin{cases} 2 \times 6^{\frac{1}{2}} t^2 - 6^{\frac{1}{2}} t & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

$$= -\frac{3^{\frac{1}{2}}}{12}\psi_{10} + \frac{3^{\frac{1}{2}}}{12 \times 5^{\frac{1}{2}}}\ \psi_{12} = \left[-\frac{3^{\frac{1}{2}}}{12}\ , 0\ , \frac{3^{\frac{1}{2}}}{12 \times 5^{\frac{1}{2}}}\ , 0\ , 0\ ,0\right]^T \psi_6(t)$$

Similarly we have

$$\int_0^t \psi_{12}(t)\,dt = -\frac{5^{\frac{1}{2}}}{20 \times 3^{\frac{1}{2}}}\ \psi_{11}\ ,$$

$$\int_0^t \psi_{20}(t)\ dt = \int_0^t 2^{\frac{1}{2}}\ L_0(4t-3)\,dt = \frac{1}{4}\ \psi_{20} + \frac{2^{\frac{1}{2}}}{4 \times 6^{\frac{1}{2}}}\ \psi_{21}$$

$$\int_0^t \psi_{21}(t)\,dt = -\frac{3^{\frac{1}{2}}}{12}\psi_{20} + \frac{3^{\frac{1}{2}}}{12 \times 5^{\frac{1}{2}}}\ \psi_{22}$$

$$\int_0^t \psi_{22}(t)\, dt = -\frac{5^{\frac{1}{2}}}{20 \times 3^{\frac{1}{2}}}\,\psi_{21}$$

Thus

$$\int_0^t \psi_{6\times 1}(t)\,dt = P_{6\times 6}\,\psi_6(t)\,, \qquad (2.10)$$

Where

$$\psi_6(t) = [\,\psi_{10},\psi_{11},\psi_{12},\psi_{20},\psi_{21},\psi_{22}]^T,$$

and

$$P_{6\times 6} = \frac{1}{4}\begin{bmatrix} 1 & \frac{2^{\frac{1}{2}}}{6^{\frac{1}{2}}} & 0 & 2 & 0 & 0 \\ -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}}{3\times 5^{\frac{1}{2}}} & 0 & 0 & 0 \\ 0 & -\frac{5^{\frac{1}{2}}}{5\times 3^{\frac{1}{2}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2^{\frac{1}{2}}}{6^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}}{3\times 5^{\frac{1}{2}}} \\ 0 & 0 & 0 & 0 & -\frac{5^{\frac{1}{2}}}{5\times 3^{\frac{1}{2}}} & 0 \end{bmatrix}$$

In (2.10) the subscript of $P_{6\times 6}$ and $\psi_6(t)$ denote the dimensions.In (2.10) the matrix $P_{6\times 6}$ can be written as

$$P_{6\times 6} = \begin{bmatrix} L_{3\times 3} & F_{3\times 3} \\ 0_{3\times 3} & L_{3\times 3} \end{bmatrix},$$

Where

$$L_{3\times 3} = \begin{bmatrix} 1 & \frac{2^{\frac{1}{2}}}{6^{\frac{1}{2}}} & 0 \\ -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}}{3\times 5^{\frac{1}{2}}} \\ 0 & -\frac{5^{\frac{1}{2}}}{5\times 3^{\frac{1}{2}}} & 0 \end{bmatrix}$$

and

$$F_{3\times 3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In general,we have

$$\int_0^t \psi(t)\,dt = P\psi(t) \tag{2.11}$$

where $\psi(t)$ is given in eq(2.6) and P is a $\left(2^{k-1}M\right)\times\left(2^{k-1}M\right)$ matrix given by

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & . & F \\ 0 & L & F & . & F \\ . & 0 & \ddots & \ddots & . \\ . & . & . & . & F \\ 0 & 0 & . & 0 & L \end{bmatrix}$$

Where F and L are $M \times M$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix}
1 & \frac{1}{3^{\frac{1}{2}}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}}{3\times 5^{\frac{1}{2}}} & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{5^{\frac{1}{2}}}{5\times 3^{\frac{1}{2}}} & 0 & \frac{5^{\frac{1}{2}}}{5\times 7^{\frac{1}{2}}} & \ddots & 0 & 0 & 0 \\
0 & 0 & -\frac{7^{\frac{1}{2}}}{7\times 5^{\frac{1}{2}}} & 0 & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\frac{(2M-3)^{\frac{1}{2}}}{(2M-3)(2M-5)^{\frac{1}{2}}} & 0 & \frac{(2M-3)^{\frac{1}{2}}}{(2M-3)(2M-1)^{\frac{1}{2}}} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{(2M-1)^{\frac{1}{2}}}{(2M-1)(2M-3)^{\frac{1}{2}}} & 0
\end{bmatrix}$$

The integration of the product of two Legendre wavelet function vector is obtained as

$$I = \int_0^1 \psi(t)\psi^T(t)dt, \tag{2.12}$$

The Legendre wavelets operational matrix P ,together with the integration of the product of two Legendre wavelets vectors, areused to solve the variational problems.The present method reduces a variational problem into a set of algebraic equations. The integration of the product of two Legendre wavelet function vectors is an identity matrix, hence making Legendre wavelets computationally very attractive.

2.4 Advantages of using Legendre Wavelets Method:

1. The operational matrix P contains many zeros which play an important rule in simplifying the performances index.
2. The Gaussian integration formula is exact for polynomials of degree not exceeding $2s+1$.
3. Only a small number of k, N, S and M are needed to obtain very satisfactory results.

3. PROPERTIES OF CAS WAVELETS

3.1 Wavelets and CAS Wavelets:

CAS wavelets $\psi_{nm}(t)=\psi(k,n,m,t)$ involve four arguments n,k,m *and* t, *where* $n=0,1,...,2^k-1$, k is assumed any non-negative integer, m is any integer and t is the normalized time. Recently, **Yousefi&Banifatemi**(2006) introduced the CAS wavelets which are defined by

$$\psi_{nm}(t)=\begin{cases} 2^{\frac{k}{2}}CAS_m(2^k t-n), & \text{for } \frac{n}{2^k}\le t<\frac{n+1}{2^k} \\ 0 & \text{otherwise} \end{cases} \qquad (3.1)$$

Where

$$CAS_m(t)=\cos(2m\pi t)+\sin(2m\pi t)$$

It is clear that the set of CAS wavelets also form an orthonormal basis for $L^2([0,1])$.

3.2Function Approximation:

A function $f(t)$ defined over [0, 1) may be expanded as

$$f(t)=\sum_{n=0}^{\infty}\sum_{m\in z}C_{nm}\psi_{n.m}(t), \qquad (3.2)$$

Where $C_{nm}=(f(t),\psi_{n,m}(t))$, in which (.,.) denotes inner product. (3.3)

If the infinite series in eq(3.2) is truncated, then eq(3.2) can be written as

$$f(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}C_{nm}\psi_{nm}(t)=C^T\psi(t), \qquad (3.4)$$

Where C & $\psi(t)$ are $2^k(2M+1)\times1$ matrices given by

$$C=[C_{0(-M)},C_{0(-M+1)},.....,C_{0M},C_{1(-M)},.....,C_{1M},......C_{(2^{k-1})(-M)},.....,C_{2^{k-1}M}]^T \;, (3.5)$$

$$\psi(t)=[\psi_{0(-M)},\psi_{0(-M+1)},.....\psi_{0M},\psi_{1(-M)},.....,\psi_{1M},.....,\psi_{(2^k-1)(-M)},.....,\psi_{(2^{k-1})M}]^T, \quad (3.6)$$

Using (3.6), numerical computation can be done by wavelet method.

3.3 CAS wavelets operational matrix of integration:

For constructing 6×6 matrix P taking $M=1$ and $k=1$, six basis functions are given by

$$\left.\begin{aligned} &\psi_{0(-1)}(t)=2^{\frac{1}{2}}(\cos(4\pi t)-\sin(4\pi t)) \\ &\psi_{00}(t)=2^{\frac{1}{2}}\;, \\ &\psi_{01}(t)=2^{\frac{1}{2}}(\cos(4\pi t)+\sin(4\pi t)) \end{aligned}\right\}\quad 0\le t<\frac{1}{2}, \qquad (3.7)$$

and

$$\left.\begin{aligned} &\psi_{1(-1)}(t)=2^{\frac{1}{2}}(\cos(4\pi t)-\sin(4\pi t)) \\ &\psi_{10}(t)=2^{\frac{1}{2}}, \\ &\psi_{11}(t)=2^{\frac{1}{2}}(\cos(4\pi t)+\sin(4\pi t)) \end{aligned}\right\} \quad \frac{1}{2}\le t<1\,, \tag{3.8}$$

By integrating (3.7) and (3.8) from 0 to t and representing it to the matrix form,we obtain

$$\int_0^t \psi_{0(-1)}(t')\,dt' = \begin{cases} \frac{\sqrt{2}}{4\pi}\left(\cos(4\pi t)+\sin(4\pi t)-1\right) & 0\le t<\frac{1}{2} \\ 0 & \frac{1}{2}\le t<1 \end{cases}$$

$$=\frac{1}{4\pi}(-\psi_{00}(t)+\psi_{01}(t)) \quad =\left[\,0\,,-\frac{1}{4\pi},\frac{1}{4\pi},\,0\,,\,0,0\right]\ \psi_6(t)\,,$$

$$\int_0^t \psi_{00}(t')\,dt' = \begin{cases} \sqrt{2t} & 0\le t<\frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2}\le t<1 \end{cases}$$

$$=\frac{1}{4\pi}\psi_{0(-1)}(t)+\frac{1}{4}\psi_{00}(t)-\frac{1}{4\pi}\psi_{01}(t)+\frac{1}{2}\psi_{10}(t)$$

$$=\left[\frac{1}{4\pi}\,,\frac{1}{4},-\frac{1}{4\pi},\ 0\,,\frac{1}{2}\,,\,0\right]\psi_6(t)$$

Similarly we have

$$\int_0^t \psi_{01}(t')dt' = \frac{1}{4\pi}\left(\psi_{0(-1)}(t)+\psi_{00}(t)\right)=\left[\frac{1}{4\pi},\frac{1}{4\pi},0,0,0,0\right]\psi_6(t)$$

$$\int_0^t \psi_{1(-1)}(t')\,dt' = \frac{1}{4\pi}(-\psi_{10}(t)+\psi_{11}(t) = \left[0,0,0,0,-\frac{1}{4\pi},\frac{1}{4\pi}\right]\psi_6(t).$$

$$\int_0^t \psi_{10}(t')\,dt' = \frac{1}{4\pi}\psi_{1(-1)}(t)+\frac{1}{4}\,\psi_{10}(t)-\frac{1}{4\pi}\psi_{11}(t)$$

$$=\left[0,0,0,\frac{1}{4\pi},\frac{1}{4},-\frac{1}{4\pi}\right]\psi_6(t)$$

$$\int_0^t \psi_{11}(t')\,dt' = \frac{1}{4\pi}\left(\psi_{1(-1)}(t)+\psi_{10}(t)\right)=\left[0,0,0,0,\frac{1}{4\pi},\frac{1}{4\pi}\right]\psi_6(t)$$

Thus

$$\int_0^t \psi_6(t')\,dt' = P_{6\times 6}\,\psi_6(t)\,, \tag{3.9}$$

Where

$$\psi_6(t) = \left[\, \psi_{0(-1)}, \psi_{00}, \psi_{01}, \psi_{1(-1)}, \psi_{10}, \psi_{11} \right]^T$$

and

$$P_{6\times6} = \frac{1}{4}\begin{bmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} & 0 & 0 & 0 \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} & 0 & 2 & 0 \\ \frac{1}{\pi} & \frac{1}{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ 0 & 0 & 0 & \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ 0 & 0 & 0 & 0 & \frac{1}{\pi} & \frac{1}{\pi} \end{bmatrix}$$

In (3.9) the subscript of $P_{6\times6}$ and $\psi_6(t)$ denote the dimensions. In fact the matrix $P_{6\times6}$ can be written as

$$P_{6\times6} = \begin{bmatrix} L_{3\times3} & F_{3\times3} \\ 0_{3\times3} & L_{3\times3} \end{bmatrix},$$

Where

$$L_{3\times3} = \begin{bmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ \frac{1}{\pi} & \frac{1}{\pi} & 0 \end{bmatrix}$$

and

$$F_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In general,we have

$$\int_0^t \psi(t')\,dt' = P\psi(t),$$

where $\psi(t)$ is given in eq(3.6) and P is a $\left(2^k(2M+1)\right)\times\left(2^k(2M+1)\right)$ matrix given by

$$P = \frac{1}{2^{k+1}}\begin{bmatrix} L & F & F & . & F \\ 0 & L & F & . & F \\ . & 0 & \ddots & \ddots & . \\ . & . & . & . & F \\ 0 & 0 & . & 0 & L \end{bmatrix}$$

Where F and L are $(2M+1)\times(2M+1)$ matrices given by

$$F = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

And

$$L=\begin{bmatrix} 0 & 0 & \dots & 0 & -\frac{1}{M\pi} & 0 & \dots & 0 & \frac{1}{M\pi} \\ 0 & 0 & \dots & 0 & -\frac{1}{(M-1)\pi} & 0 & \dots & \frac{1}{(M-1)\pi} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -\frac{1}{\pi} & \frac{1}{\pi} & \dots & 0 & 0 \\ \frac{1}{\pi} & \frac{1}{\pi} & \dots & \frac{1}{\pi} & 1 & \frac{1}{\pi} & \dots & \frac{1}{\pi} & \frac{1}{\pi} \\ 0 & 0 & \dots & \frac{1}{\pi} & \frac{1}{\pi} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{1}{(M-1)\pi} & \dots & 0 & \frac{1}{(M-1)\pi} & 0 & \dots & 0 & 0 \\ \frac{1}{M\pi} & 0 & \dots & 0 & \frac{1}{M\pi} & 0 & \dots & 0 & 0 \end{bmatrix}$$

The integration of the product of two CAS function vectors is obtained as

$$I=\int_0^1 \psi(t)\psi(t)^T\,dt \qquad (3.10)$$

Where I is an identity matrix.

3.4 Advantages of using CAS Wavelets Method:

Since, the integration of the product of two CAS wavelet function vectors is an identity matrix and the matrix of algebraic equations is sparse, CAS wavelet method is computationally attractive.

4. PROPERTIES OF SINE-COSINE WAVELETS

4.1 Wavelets and Sine-cosine wavelets:

Sine-cosine wavelets $\psi_{n,m}(t)=\psi(n,k,m,t)$ have four arguments: $n=0,1,2,....,2^k-1, k=0,1,2,....$, the values of m are given in Eq.(4.2) and t is the normalized time. They are defined on the interval [0, 1) as

$$\psi_{n,m}(t)=\begin{cases} 2^{\frac{k+1}{2}} f_m(2^k t-n) & \text{for } \frac{n}{2^k}\le t<\frac{n+1}{2^k}, \\ 0 & \text{otherwise} \end{cases} \qquad (4.1)$$

with

$$f_m(t)=\begin{cases} \frac{1}{\sqrt{2}}, & m=0 \\ \cos(2m\pi t), & m=1,2,.........., L, \\ \sin(2(m-L)\pi t), & m=L+1, L+2,........., 2L, \end{cases} \qquad (4.2)$$

Where L is any positive integer. The set of SCW are an orthonormal set.

4.2 Function Approximation

A function $f(t)$ defined over [0, 1) may be expanded as

$$f(t)=\sum_{m=0}^{\infty}\sum_{n=0}^{2^k-1} c_{nm}\,\psi_{nm}(t), \qquad (4.3)$$

Where $C_{nm}=(f(t),\psi_{n,m}(t)),$ in which (.,.) denotes inner product. (4.4)

By truncating the infinite series (4.3) at levels $m=2L$ and $n=2^k-1$, we obtain an approximate representation for $f(t)$ as

$$f(t)\cong\sum_{m=0}^{2L}\sum_{n=0}^{2^k-1}c_{nm}\,\psi_{nm}(t)=C^T\,\psi(t), \qquad (4.5)$$

where the matrices C and $\psi(t)$ are $2^k(2L+1)\times 1$ matrices given by

$$C=[c_{0,0},\,c_{0,1},\,,\ldots,\,c_{0,2L},\,c_{10},\ldots,c_{1,2L},\ldots,c_{2^k-1,0},\ldots,c_{2^k-1,2L}]^T \quad (4.6)$$

$$\psi(t)=[\psi_{0,0}(t),\,\psi_{0,1}(t),\ldots,\psi_{0,2L}(t),\psi_{1,0}(t),\ldots,\psi_{1,2L}(t),\psi_{2^k-1,0}(t),\ldots,\psi_{2^k-1,2L}(t)]^T. \qquad (4.7)$$

4.3 Sine-cosine wavelets operational matrix of integration:

Here,the operational matrix of integration will be derived.First matrix P with $L=2$ and $k=1.$ was found. The 10 basis function are given by

$$\left.\begin{aligned}\psi_{0,0}(t)&=\sqrt{2}\\ \psi_{0,1}(t)&=2\cos(4\pi t)\\ \psi_{0,2}(t)&=2\cos(8\pi t)\\ \psi_{0,3}(t)&=2\sin(4\pi t)\\ \psi_{0,4}(t)&=2\sin(8\pi t)\end{aligned}\right\}\qquad 0\le t<\frac{1}{2} \qquad (4.8)$$

and

$$\left.\begin{aligned}\psi_{1,0}(t)&=\sqrt{2}\\ \psi_{1,1}(t)&=2\cos(2\pi(2t-1))\\ \psi_{1,2}(t)&=2\cos(4\pi(2t-1))\\ \psi_{1,3}(t)&=2\sin(2\pi(2t-1))\\ \psi_{1,4}(t)&=2\sin(4\pi(2t-1))\end{aligned}\right\}\qquad \frac{1}{2}\le t<1 \qquad (4.9)$$

By integrating equation (4.8) from 0 to t and using equation (4.4)

$$\int_0^t\psi_{0,0}(t)\,dt=\frac{1}{2}\left[\frac{1}{2},0,0,-\frac{1}{\pi},-\frac{1}{2\pi},1,0,0,0,0\right]^T\psi_{10}(t),$$

$$\int_0^t\psi_{0,1}(t)\,dt=\frac{1}{2}\left[0,0,0,\frac{1}{2\pi},0,0,0,0,0,0\right]^T\psi_{10}(t),$$

$$\int_0^t\psi_{0,2}(t)\,dt=\frac{1}{2}\left[0,0,0,0,\frac{1}{4\pi},0,0,0,0,0\right]^T\psi_{10}(t),$$

$$\int_0^t\psi_{0,3}(t)\,dt=\frac{1}{2}\left[\frac{1}{2\pi},-\frac{1}{2\pi},0,0,0,0,0,0,0,0\right]^T\psi_{10}(t),$$

$$\int_0^t\psi_{0,4}(t)\,dt=\frac{1}{2}\left[\frac{1}{4\pi},0,-\frac{1}{4\pi},0,0,0,0,0,0,0\right]^T\psi_{10}(t),$$

And similarly for (4.9)

Thus

$$\int_0^t\psi_{10}(t)\,dt=P_{10\times 10}\,\psi_{10}(t)\,, \qquad (4.10)$$

Where

$$\psi_{10}(t) = \left[\psi_{0,0}, \psi_{0,1}, \psi_{0,2}, \psi_{0,3}, \psi_{0,4}, \psi_{1.0}, \psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4} \right]^T, \tag{4.11}$$

and

$$P_{10\times 10} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{\pi} & -\frac{1}{2\pi} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\pi} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\pi} & 0 & -\frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{\pi} & -\frac{1}{2\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4\pi} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\pi} & 0 & -\frac{1}{4\pi} & 0 & 0 \end{bmatrix}$$

In (4.10) the subscript of $P_{10\times 10}$ and $\psi_{10}(t)$ denote the dimensions. IN general we have:

$$\int_0^t \psi(t')dt' = P\psi(t) \tag{4.12}$$

With $\psi(t)$ given in equation (4.7) and P is a $2^k(2L+1)\times 2^k(2L+1)$ matrix given by

$$P = \frac{1}{2^{k+1/2}}\begin{bmatrix} F & S & S & \dots & S \\ 0 & F & S & \dots. & S \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & & & & S \\ 0 & 0 & \dots & 0 & F \end{bmatrix}$$

Where F and S are $(2L+1)\times(2L+1)$ matrices given by

$$S = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$F = \begin{bmatrix} \frac{1}{2} & 0 & \dots & 0 & -\frac{1}{\pi} & -\frac{1}{2\pi} & \dots & -\frac{1}{L\pi} \\ 0 & 0 & \dots & 0 & \frac{1}{2\pi} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{4\pi} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{1}{2L\pi} \\ \frac{1}{2\pi} & -\frac{1}{2\pi} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{1}{2L\pi} & 0 & \dots & -\frac{1}{2L\pi} & 0 & 0 & \dots & 0 \end{bmatrix}$$

The integration of the product of two SCW function vectors is obtained as

$$I = \int_0^1 \psi(t)\psi^T(t)dt, \quad (4.13)$$

Where I is an identity matrix.

4.3 Advantages of using SCW Method:

The advantage of owing to use sinusoidal functions are, since they are well known **Endow**(1989).The SCW operational matrix of integration can be used to solve problems such as identification, analysis and optimal control, like that of other orthogonal functions. The integration of the product of two SCW function vectors is an identity matrix, hence making SCW computationally attractive.

5. CONCLUSIONS

From the above details and work we have conclude that the Wavelets and Wavelets transform is the useful tool for the latest trendical work. It has a lot of advantage and application in the real and applied field. The major advantage of the wavelet transform is found in the field of Engineering and wave mechanics. Since integration of the product of two orthonormal function vectors is an identity matrix, hence wavelet method is computationally more attractive. Also different kind of wavelets ishaving number of advantages in the various field.

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