ON SOME GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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ABSTRACT
In this paper we define some generalized sequenc spaces defined by a sequence of moduli. The results here in proved are analogous to those by ASMA BEKTAS Cigdem (2006)[Journal of Zhejiang University Science A (2006),7(12) 2093-2096] .

Keywords: *Difference sequence space , Sequence of moduli, Strongly almost convergent .*

1. INTRODUCTION

Let ℓ^0 be the set of all complex sequences and l_∞, c and c_0 be the sets of all bounded, convergent and null sequences $x=(x_k)$ with complex terms, respectively , normed by

$$\| x \|_\infty = \sup_k | x_k |, \ \ where \ \ k \in I \quad N = \{1,2,\cdots\}.$$

The idea of difference sequence space was introduced by Kizmaz (1981). In 1981, Kizmaz defined the sequence spaces : [1]

$$l_\infty(\Delta) = \{x=(x_k) \in \ell^0 : (\Delta x_k) \in l_\infty\},$$

$$c(\Delta) = \{x=(x_k) \in \ell^0 : (\Delta x_k) \in c\},$$

and

$$c_0(\Delta) = \{x=(x_k) \in \ell^0 : (\Delta x_k) \in c_0\},$$

where $\Delta x = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$\| x \|_\Delta = | x_1 | + \| \Delta x \|_\infty .$$

After then R. Colak and M. Et (1995) defined the sequence spaces :

$$l_\infty(\Delta^m) = \{x=(x_k) \in \ell^0 : (\Delta^m x_k) \in l_\infty\},$$

$$c(\Delta^m) = \{x=(x_k) \in \ell^0 : (\Delta^m x_k) \in c\},$$

and

$$c_0(\Delta^m) = \{x=(x_k) \in \ell^0 : (\Delta^m x_k) \in c_0\},$$

where $m \in I \quad N$,

$$\Delta^0 x = (x_k),$$

$$\Delta x = (x_k - x_{k+1}),$$

$$\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}),$$

and so that

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} x_{k+i}.$$

and show that these are Banach spaces with the norm

$$\| x \|_\Delta = \sum_{i=1}^{m} | x_i | + \| \Delta^m x \|_\infty .$$

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Definition 1.1. A function $f:[0,\infty)\rightarrow[0,\infty)$ is called a modulus if

1. $f(t)=0$ if and only if $t=0,$
2. $f(t+u)\le f(t)+f(u)$ *for all* $t,u\ge 0,$
3. f is increasing, and
4. f is continuous from the right of 0.

Let X be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f)=\{x=(x_k)\in\ell^0:(f(|x_k|))\in X\}$$

for a modulus f(Maddox 1986 and Ruckle 1973).

Kolk (1993, 1994) gave an extension of $X(f)$ by considering a sequence of moduli $F=(f_k)$ i.e.

$$X(F)=\{x=(x_k)\in\ell^0:(f_k(|x_k|))\in X\}.$$

A sequence $x\in l_\infty$ is said to be almost convergent (Lorentz, 1984) if all Banach limits of x coincide. Lorentz(1984) proved that

$$\hat{c}:=\{x=(x_k)\in\ell^0:\lim_n\frac{1}{n}\sum_{k=1}^{n}x_{k+s},\ uniformly\ in\ s\}.$$

Maddox (1967; 1978) has defined x to be strongly almost convergent to L if

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}|x_{k+s}-L|=0,\ uniformly\ in\ s,\ for\ some\ L>0.$$

Let $p=(p_k)$ be a sequence of strictly positive real numbers . Nanda (1984) defined

$$[\hat{c},p]:=\{x=(x_k)\in\ell^0:\lim_n\frac{1}{n}\sum_{k=1}^{n}|x_{k+s}-L|^{p_k}=0,\ uniformly\ in\ s,\ for\ some\ L>0\},$$

$$[\hat{c},p]_0:=\{x=(x_k)\in\ell^0:\lim_n\frac{1}{n}\sum_{k=1}^{n}|x_{k+s}|^{p_k}=0,\ uniformly\ in\ s,\},$$

$$[\hat{c},p]_\infty:=\{x=(x_k)\in\ell^0:\sup_n\frac{1}{n}\sum_{k=1}^{n}|x_{k+s}-L|^{p_k}=0,\ uniformly\ in\ s,\}.$$

2. MAIN RESULTS

Let $F=(f_k)$ be a sequence of moduli, $u=(u_k)$ be any sequence such that $u_k\neq 0$ for all k and $p=(p_k)$ be any sequence space of strictly positive real numbers then we define the following sequence spaces :

$$[\hat{c},F,p](\Delta_u^m):=\{x=(x_k)\in\ell^0:\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k\Delta^m x_{k+s}-L|)]^{p_k}=0,\ uniformly\ in\ s$$

$$,\ for\ some\ L>0\},$$

$$[\hat{c},F,p]_0(\Delta_u^m):=\{x=(x_k)\in\ell^0:\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k\Delta^m x_{k+s}|)]^{p_k}=0,\ uniformly\ in\ s\},$$

$$[\hat{c},F,p]_\infty(\Delta_u^m):=\{x=(x_k)\in\ell^0:\sup_{s,n}\frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k\Delta^m x_{k+s}|)]^{p_k}<\infty,\ uniformly\ in\ s\},$$

If $f_k(x)=x$ for every k , then $[\hat{c},F,p](\Delta_u^m)=[\hat{c},p]$, $[\hat{c},F,p]_0(\Delta_u^m)=[\hat{c},p]_0(\Delta_u^m)$ and $[\hat{c},F,p]_\infty(\Delta_u^m)=[\hat{c},p]_\infty(\Delta_u^m)$ .We denote $[\hat{c},F,p](\Delta_u^m)$, $[\hat{c},F,p]_0(\Delta_u^m)$ and $[\hat{c},F,p]_\infty(\Delta_u^m)$ by $[\hat{c},F](\Delta_u^m)$,$[\hat{c},F]_0(\Delta_u^m)$ and $[\hat{c},F]_\infty(\Delta_u^m)$, when $p_k=1$ for all k .

Theorem 2.1. For a sequence $F=(f_k)$ of moduli , the following statements are equivalent:

1. $[\hat{c},p]_\infty(\Delta_u^m)\subseteq[\hat{c},F,p]_\infty(\Delta_u^m),$
2. $[\hat{c},p]_0(\Delta_u^m)\subseteq[\hat{c},F,p]_0(\Delta_u^m),$
3. $\sup_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}<\infty \qquad (t>0).$

Proof. (i)$\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$(iii): Let $[\hat{c},p]_0(\Delta_u^m)\subseteq[\hat{c},F,p]_0(\Delta_u^m)$. Suppose that (iii) is not true. Then for some t

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=\infty \qquad .$$

and there exists a sequence (n_i) of positive integers such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i} f_k(\frac{1}{i})>i \quad for \quad i=1,2,\cdots(1)$$

Now we define $x=\{x_k\}$ by

$$x_k=\begin{cases}\frac{1}{i} & , \ if \quad 1\le k\le n_i, \quad i=1,2,\cdots, \\ 0 & , \ (k>n_i).\end{cases}$$

Then $x\in[\hat{c},p]_0(\Delta_u^m)$ but by Eqn (1), $x\notin[\hat{c},F,p]_\infty(\Delta_u^m)$ which contradicts (ii).

Hence (iii) is true.

(iii)$\Rightarrow$ (i) :

Let (iii) is true and $x\in[\hat{c},p]_\infty(\Delta_u^m).$ If we suppose that $x\notin[\hat{c},F,p]_\infty(\Delta_u^m)$. Then

$$\sup_{s,n} \frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k\Delta^m x_{k+s}|)]^{p_k}=\infty.(2)$$

If we take $t=|u_k\Delta^m x_{k+s}|$ for each k and fixed s, then by Eqn(2)

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=\infty,$$

which contradicts (iii). Hence $[\hat{c},p]_\infty(\Delta_u^m)\subseteq[\hat{c},F,p]_\infty(\Delta_u^m)$.

Theorem 2.2. Let $1\le p_k\le\sup_k p_k<\infty$. For a sequence of moduli $F=(f_k)$ the following statements are equivalent:

1. $[\hat{c},F,p]_0(\Delta_u^m)\subseteq[\hat{c},p]_0(\Delta_u^m),$
2. $[\hat{c},F,p]_0(\Delta_u^m)\subseteq[\hat{c},p]_\infty(\Delta_u^m),$
3. $\inf_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}>0 \qquad (t>0).$

Proof. (i)$\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$(iii): Let $[\hat{c},F,p]_0(\Delta_u^m)\subseteq[\hat{c},p]_\infty(\Delta_u^m).$ Suppose that (iii) does not hold. Then

$$\inf_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=0 \ \ (t>0),(3)$$

and there exists a sequence (n_i) of positive integers such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[f_k(i)]^{p_k} < \frac{1}{i}, \quad for \quad i=1,2,\cdots.$$

Now define the sequence $x=\{x_k\}$ by

$$x_k = \begin{cases} i & , \ if \quad 1\le k\le n_i, \quad for \quad i=1,2,\cdots, \\ 0 & , \ k > n_i. \end{cases}$$

By Eqn.(3), $x\in[\hat{c},F,p]_0(\Delta_u^m)$ but $x\notin[\hat{c},p]_\infty(\Delta_u^m),$ which contradicts (ii).

Hence (iii) is true.

(iii)$\Rightarrow$ (i) :

Let (iii) is true and $x\in[\hat{c},F,p]_0(\Delta_u^m)$ i.e.

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k\Delta^m x_{k+s}|)]^{p_k} = 0, \ uniformly\ in\ s\}.(4)$$

Suppose that $x\notin[\hat{c},p]_0(\Delta_u^m)$ .Then for some number $\varepsilon_0>0$ and positive integer n_0 we have $|u_k\Delta^m x_{k+s}|\ge\varepsilon_0$ for some $s\ge s'$ and $1\le k\le n_0$. Therefore

$$[f_k(\varepsilon_0)]^{p_k} \le [f_k(|u_k\Delta^m x_{k+s}|)]^{p_k}$$

and hence $\lim_n \sum_{k=1}^{n}[f_k(\varepsilon_0)]^{p_k} = 0$, which contradicts (iii). Thus $[\hat{c},F,p]_0(\Delta_u^m)\subseteq[\hat{c},p]_0(\Delta_u^m)$.

Theorem 2.3.

Let $1\le p_k\le\sup_k p_k<\infty$. The inclusion $[\hat{c},F,p]_\infty(\Delta_u^m)\subseteq[\hat{c},p]_0(\Delta_u^m),$ holds if and only if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k} = \infty \quad for \quad t>0.(5)$$

Proof. Let $[\hat{c},F,p]_\infty(\Delta_u^m)\subseteq[\hat{c},p]_0(\Delta_u^m),$. Suppose Eqn(5) does not hold. Then there exists a number $t_0>0$ and a sequence (n_i) of positive integers such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[f_k(t_0)]^{p_k} \le M<\infty, \ i=1,2,\cdots.(6)$$

Noe we define the sequence $x=(x_k)$ by

$$x_k = \begin{cases} t_0 & , \ if \quad 1\le k\le n_i \quad for \quad i=1,2,\cdots, \\ 0 & , \ k > n_i. \end{cases}$$

Thus by Eqn (6), $x\in[\hat{c},F,p]_\infty(\Delta_u^m)$ but $x\notin[\hat{c},p]_0(\Delta_u^m)$.So that Eqn (5) must hold.

Conversely let Eqn (5) hold. If $x\in[\hat{c},F,p]_\infty(\Delta_u^m)$, then for each s and n

$$\frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k\Delta^m x_k|)]^{p_k} \le M<\infty.(7)$$

Suppose that $x\notin[\hat{c},p]_0(\Delta_u^m)$. Then for some number $\varepsilon_0>0$ and positive integer s_0 and index n_0 we have $|u_k\Delta^m x_{k+s}|\ge\varepsilon_0$ for $s\ge s_0$. Therefore

$$[f_k(\varepsilon_0)]^{p_k} \le [f_k(|u_k\Delta^m x_{k+s}|)]^{p_k},$$

and hence for each k and s we get

$$\frac{1}{n}\sum_{k=1}^{n}[f_k(\varepsilon_0)]^{p_k} \leq M < \infty,$$

for some $M > 0$, by Eqn (7) which contradicts Eqn (5). Hence

$$[\hat{c}, F, p]_\infty(\Delta_u^m) \subseteq [\hat{c}, p]_0(\Delta_u^m)$$

.

Theorem 2.4.

Let $1 \leq p_k \leq \sup_k p_k < \infty$. The inclusion $[\hat{c}, p]_\infty(\Delta_u^m) \subseteq [\hat{c}, F, p]_0(\Delta_u^m),$ holds if and only if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t_0)]^{p_k} = 0 \quad for \quad t > 0.(8)$$

Proof. Suppose that $[\hat{c}, p]_\infty(\Delta_u^m) \subseteq [\hat{c}, F, p]_0(\Delta_u^m),$ but (8) does not hold. Then for some $t_0 > 0$

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t_0)]^{p_k} = L \neq 0.(9)$$

Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{\nu=0}^{k-\nu}(-1)^m \begin{pmatrix} m+k-\nu-1 \\ k-\nu \end{pmatrix}$$

for $k = 1,2,\cdots.$ Then $x \notin c_0(F, p, \Delta_\nu^m)$, by Eqn (6).Hence Eqn (5) must hold.

Conversely let $x \in [\hat{c}, p]_\infty(\Delta_u^m)$, and suppose that Eqn (8) holds. Then for every k and s

$$|u_k \Delta^m x_{k+s}| \leq M < \infty.$$

Therefore

$$[f_k(|u_k \Delta^m x_{k+s}|)]^{p_k} \leq [f_k(M)]^{p_k},$$

and

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(|u_k \Delta^m x_{k+s}|)]^{p_k}) \leq \lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(M)]^{p_k} = 0, by \quad Eqn(8).$$

Hence $x \in [\hat{c}, F, p]_0(\Delta_u^m)$.

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