ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT
We shall obtain an upper bound for the solution of integro-differential equation. Also the stability of solution in investigated mathematical subject classification: 34D99, 45M10, 45j05.

KEYWORDS: *Boundedness, Stability Integrodifferential Equations.*

1. INTRODUCTION
Differential and integral inequalities play a prominent role in the study of existence uniqueness, boundedness, stability and other qualitative properties of solutions of differential, and integral and integrodifferential equation and volterra integral equations $[see(1-8)]$.

2. BOUNDEDNESS OF SOLUTION
In this section we shall assume that

i) h: $R_+ = [0,\infty) \to R$, $k: {R^2}_+ \times R \to R$, $F: R_+ \times R^2 \to R$ are continuous functions

ii) $|K(t,s,x(s))| \le f(t)g(s)|x(s)|$

iii) $|F(t,x(s),v)| \le f(t)|x(t)| + |v|$

Where f, g, v are non negative continuous functions on R

We need the following Lemma:

LEMMA [1]: let x, f, g be nonnegative continuous function defined on R_+ , $c > 0$ *is a constant*. if $x^2(t) \le c^2 + 2\int_0^t \left[f(s)x(s)\left(x(s) + \int_0^s g(r)x(r)dr\right) + h(s)x(s)\right]ds$

Then

$x(t) \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s [f(r) + g(r)]\right)dr\right]ds$

Where

$$p(t) = |x_0| + \int_0^t |h(s)|ds, \qquad t \in R_+$$

THEOREM- 1

The solution of the integrodifferential equation

$$x'(t) = b(t) + F\left(t, x(t), \int_0^t k(t,s,x(s)ds)\right), n(o) = x_0 \qquad (1)$$

Has an upper bound

$$|x(t)| \le p(t)\left[1 + \int_0^t f(s)exp\left(\int_0^s f(r) + g(r)dr\right)ds\right]$$

Proof: multiplying both sides of (1) by $x(t)$ and integrating from 0 to t, we get

$$x^2 = {x^2}_0 + 2\int_0^t \left[x(s)F\left(s, x(s), \int_0^s k(s,r,x(s))dr\right] + [h(s)x(s)]ds \qquad (2)$$

From assumptions $i \to iii$ and lemma we get

$$|x(t)| \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s f(r) + g(r)dr\right)\right]ds$$

This compete the proof.

3. STABILITY
In [4] Massera established the existence of liapunov function when a system of ordinary differential equations is uniformly asymptotically stable. Miller [5] has been shown the existence of a liapunov function when the linear system

$$\frac{d}{dt}\underline{x}(t) = A(t)\underline{x}(t) + \int_{t_0}^t B(t,s)f\left(s, \underline{x}(s)\right)ds, \quad \underline{x}(t_0) = \underline{x}_0 \qquad (3)$$

Is uniformly asymptotically stable.

In this paper we investigate the existence of a liapunov function when the nonlinear integrodifferential system

$$\frac{d}{dt}\underline{x}(t) = \int_{t_0}^t k(t,s)f\left(s, \underline{x}(s)\right)ds, \quad \underline{x}(t_0) = \underline{x}_0 \qquad (4)$$

Is generalized exponentially stable. In the above equations A (t), B (t, s) and k (t, s) are $n \times n$ matrices defined for $t_0 \leq t < \infty \quad and \quad t_0 \leq s \leq t < \infty \qquad t_0 \geq 0$
Let J denote $[t_0, \infty)$ $and\ R_+\ denote\ half\ line$ $[0, \infty)$.

DEFINITION: 1 A function $g \in [R, R]$ is said to belong to the class K if g (o) =0, g (t) is monotonic increasing in t. For a vector $\underline{x} \in R^n$,the norm of $\underline{x}$ is defined by $\|X\| = \sum_{t=1}^{n} |X_i|$, for a $n \times n$ matrix A, the norm is defined by $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$, S(r) denotes the open sphere S(r) = { $x \in R^n$; $\|X\| < r$ }.
Consider the integrodifferential equation

$$\frac{d}{dt} R(t) = A(t)R(t) + \int_{t_0}^{t} B(t,s)R(s)ds \,, \; R(t_0) = I \qquad (5)$$

Where I is identity matrix the proof of the lemma stated below is similar to the argument given in [ ] for the matrix system $\frac{d}{dt} R(t) = A(t)R(t)$, $R(t_0) = I$ and so we omit the proof.

LEMMA 2: the matrix equation (5) has a unique solution R (t), $t \in J$, if the following conditions are satisfied

i. A (t) is continuous on J.
ii. Sup $\left\{ \int_{s}^{t} \|B(u,S)\| \, du : 0 \leq s \leq t < \infty \right\}$ is bounded further, $\underline{x}(t) = R(t)\, \underline{x}_0$ satisfies(1) uniquely

DEFINITION 2: The trivial solution of (1) is generalized exponentially asymptotically stable (GEAS) if $\|\underline{x}(t, t_0, \underline{x}_0)\| \leq K(t) \|\underline{x}_0\| \exp[p(t_0) - p(t)]$ Where $K(t) > 0$ is continuous for $t \in J$, $P \in K$ and $P(t) \to \infty \;\; as\ t \to \infty\ if\ K(t) \equiv K > 0$

$P(t) = a\, t, with\ a > 0$ We have the exponential stability.

$X(t, t_0, \underline{x}_0)$ Represents the solution of (1) evaluated at $t \in J$ with the initial condition $\underline{x}(t_0) = \underline{x}_0$.

We assume in the following, that the solution of (1) exists in S(r).

There exists a function V (t, $\underline{x}$) satisfying the following

i. $V \in C\ [J \times S_r, R_+]\ and\ \left|V(t, \underline{x}) - V(t, \underline{y})\right| \leq k(t)\|x - y\|, t \in J, \underline{x}, \underline{y} \in S_r$
ii. $\|x\| \leq V(t, \underline{x}) \leq K(t)\|\underline{x}\|\,, (t, x) \in J \times s_r$
iii. $D^{+}V(t,x) \leq \; -p'(t)V(t,x), (t,x) \in J \times s_r$

THEOREM (2):
Assume that the solution $\underline{x} \equiv 0$ of (3) is GEAS .further let $p'(t)$ exist and be continuous for t, J. Then there exist a V (t, X) satisfying the above the conditions $(i) \to (iii)$.
Proof: Denote $x(t, t_0, x_0)\ by\ x. Define\ V(t,x) by\ the\ relation$

$$V(t,x) = sup_{s\geq 0}\|x(t+s,t,\underline{x})\| \exp p(t+s)p(t) \quad s \geq 0 \qquad (6)$$

Before proceeding further we let
e (t) =exp p (t), d (t) =exp (-p (t))
From (4) it follows that (ii) is satisfied .let $x, y \in s_r$ then

$$V(t,x) - V(t,y) = |sup_{s\geq 0}\|X(t+s,t,x)\|e(t+s)d(t) - sup_{s\geq 0}\|x(t+s,t,y\|e(t+s)d(t)|$$
$$\leq sup\|x(t+s,t,x) - x(t+s,t,y)\|e(t+s)d(t)$$
$$\left|V(t,\underline{x}) - v\left(t,\underline{y}\right)\right| \leq sup_{s\geq 0}\|X(t+s,t,x)\|e(t+s)d(t) \leq k(t)\|x - y\|$$

Which established (i). It is now shown that v (t, $\underline{x}$) is continuous in (t, $\underline{x}$) let δ>0 be a real number. Then

$$\left|V(t+\delta,\underline{x}) - V(t,\underline{x})\right|$$
$$\leq \left|V(t+\delta,\underline{x}) - V(t+s,\underline{x}) + \left|V(t+\delta,\underline{x}) - V(t+\delta, x(t+\delta,t,\underline{x})\right|\right.$$
$$+ V(t+\delta,\underline{x}), x(t+\delta,t,x) - V(t,x)|$$

The first two terms on the right side if the preceding inequalities are small if $\|x - \underline{x}\|$ are sufficiently small because v (t, x) is Lipschian in a and $\underline{x}(t+\delta,t,x)$ is continuous in δ hence, because of uniqueness of solution we have

$$\left|v(t+\delta, x(t+\delta,t,x)) - v(t,x)\right| = |sup_{s\geq\delta}\|x(t+s,t,x)\|dt. e(t+s-\delta) - sup_{s\geq\delta}\|x(t+s,t,x)\|e(t+s)dt|$$

Then $a(\delta) \to a(0)\ as\ \delta \to 0$ and so on we have

$$\left|v(t+\delta, X(t+\delta,t,x)) - v(t,x)\right| = |a(\delta) - a(0)| \to 0\ as\ \delta \to 0$$

Which implies that V (t, x) is continuous in (t, $\underline{x}$) the proof of the completed if (iii) is verified consider

$$D^{+}V(t,x(t)) = \lim_{h\to o^{+}} sup \frac{1}{h}[sup_{s\geq 0}\|x(t+s,t,x)\|e(t+s)d(l+h) - sup_{s\geq 0}\|x(t+s,t,x)\|e(t+s)d(t)]$$
$$\leq -p'(t)V(t,\underline{x})$$

By setting

$g\left(t, \underline{x}(t)\right) = A(t)\,\underline{x}(t) + \int_{t_0}^{t} B(t,s)\,\underline{x}(s)ds$, We have

$$V(t+h, x+hg(t,x) - V\big(t, \underline{x}\big) \le K(t)\big\|\underline{x}\big(t+h, t, \underline{x}\big) - x - hg(t,x)\big\| + V(t+h, x(t+h,t,x) - V(t, \underline{x})$$

And now it easily follows that $D^{+}V(t,x) \le p^{'}(t)V(t,x)$

Along the solution of (1)

Remark: it is assumed that the solution of (4) lies in S_r

THEOREM (3):

Let $x(t) = x(t, t_0, x_0)$ and $y(t, t_0, y_0)$represents two solutions of (2) passing through (t_0, x_0) and (t_0, y_0) respectively. Further assumes that the conditions

a) $f(t,x) satisfies$

$$\|f(t,x) - f(t,y)\| \le L(t,r)\|x - y\|, x, y \in S_r, t \in J.$$

Where $L(t,r) > 0$ $such\ that$ $\int_{t_0}^{\infty} L(S,r)ds = M$

b) $K(t,s) satisfies$

$$sup_{s\ge 0}\left\{\int_{s}^{t} \|k(u,s)\|\, ds : 0 \le s \le t < \infty\right\} = N$$

Then $\|x(t) - y(t)\| \le \|x_0 - y_0\| \exp(MN)$

Proof:

The proof is a straight forward application of a well known integral inequality. Consider

$$x(t) - y(t) = x_0 - y_0 + \int_{t_0}^{t}\left(\int_{s}^{t} K(u,s)du\ \ (f(s,x(s)) - f(s,y(s))\right)ds$$

(Which follows by change of order of integration) and so we get

$$\|x(t) - y(t)\| \le \|x_0 - y_0\| \int_{t_0}^{t} NL\|x(s) - y(s)\|ds$$

By the usual integral inequality we thus obtain

$$\|x(t) - y(t)\| \le \|x_0 - y_0\| \exp\left(\int_{t_0}^{t} NL(s,r)\right)ds$$
$$\le \|x_0 - y_0\| \exp(MN)$$

THEOREM 4:

Assume that the trial solution of (2) is GEAS and satisfying condition (a) and (b) in theorem (2) then there exists a function V (t, x) fulfilling the conditions:

i. $V \in C[J \times S_r, R_+],\ 0 < r_0 < r$

$$\left|V(t, \underline{x}) - V\left(t, \underline{y}\right)\right| \le \exp(MN) \sup[\exp(t+s)\ dt\|x - y\|$$

ii. $\|\underline{x}\| \le V(t, \underline{x}) \le K(t)\|\underline{x}\|, (t, \underline{x}) \in J \times S_r$

iii. $D^{+}V(t, \underline{x}) \le -p^{'}(t)V(t, \underline{x}), (t,x) \in J \times S_r$

Proof: We define liapunov function by

$V(t, \underline{x}) = sup_{s\ge 0}\|x(t+s,t,x)\|exp[p(t+s) - p(t)]$ (7)

$V(t, \underline{x})$ $is\ defined\ for$ $(t, \underline{x}) \in (J, S_{r_0})$ $because$ $K(t)$ Is assumed to be bounded and $\in K$. Here $r_0 = r/M$ where $M = \sup\{k(t), t \in J\}$ $so\ we\ have$

$$|V(t,x) - V(t,y)| \le sup_{s\ge 0}[\|x(t+s,t,x) - x(t+s,t,y)\|] \exp(t+s)d$$

And by theorem 2 we have

$|V(t,x) - V(t,y)| \le \exp(MN)\, sup_{s\ge 0}[\exp(t+s)\, d(t)]\|x - y\|$.

This establishes (ii) the statements (i) (iii) of theorem 3 can be as done in theorem 1 and hence the proof is complete

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