EXPLICIT EQUATION FOR SAFETY FACTOR OF SIMPLE SLOPES

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ABSTRACT

The existing methods for determining the safety factor of simple homogeneous slopes are graphical in nature and may require iterations. This technical note presents a simple explicit equation for determining the safety factor for such conditions. A polynomial surface of the stability number was established based on the Taylor's chart. A dimensionless parameter and a trigonometric series approximation were then used along with the stability number surface to establish the explicit equation. The proposed equation is applicable to the case of homogeneous slopes without seepage as well as the special cases involving complete submergence, complete sudden drawdown, steady seepage, and zero boundary neutral force. Validation of the proposed equation was performed by comparing its results with those of the existing graphical and analytical methods. The results showed that the proposed equation was very accurate. As such, the proposed equation should be useful in many geotechnical applications, especially those that implement safety factor as part of a larger modeling system.

Keywords: *Explicit equation, Safety factor, Simple homogeneous slopes, Regression, Validation, Seepage.*

1. INTRODUCTION

The analysis of slope stability is frequently encountered in practical applications, see for example Smith [1], Keskin [2], Greenwoo [3], and Renaud et al. [4]. The computation of the safety factor for analyzing the stability of simple, homogeneous slopes with no tension cracks can be easily performed using stability charts developed by Taylor [5]. Taylor's charts have a theoretical basis and represent a classical method that is still being used in practical applications. The method, which is based on the friction-circle method, has been described in most geotechnical engineering textbooks, such as Murthy [6], Shroff and Shah [7], and Smith [1]. The basic method assumes a finite slope without seepage, but it may also be used to provide rough determinations and preliminary solutions for more complex cases of Taylor [8].

Taylor presented two charts, one for ϕ = 0 and limited depth that can be used to determine the safety factor directly. For the case where the internal friction angle ϕ > 0, the chart requires a trial procedure to determine the safety factor for a given slope. The reason for this is that the chart is developed in terms of the mobilized friction angle, which is a function of the safety factor. The iterative procedure was later eliminated by Baker [9] who developed a similar chart using a dimensionless parameter. Both procedures, however, relies on charts. This technical note eliminates this graphical procedure by developing an explicit equation for directly calculating the safety factor. Before presenting the development of this equation, it is useful to describe the existing Taylor-based graphical methods.

2. EXISTING GRAPHICAL METHODS

Existing graphical methods (trial or direct) are based on Taylor's stability chart. The chart presents the stability number *SN* as a function of the slope inclination angle β and the mobilized friction angle ϕ_{m}. The chart for the case ϕ > 0, which requires a trial procedure, is shown in Fig. 1. As noted from the figure, as the slope inclination angle increases the value of *SN* increases. On the other hand for a given β when the mobilized friction angle increases the stability number decreases. Taylor's chart is a general solution of slope stability, based on the friction-circle method which is solved using a numerical trial procedure.

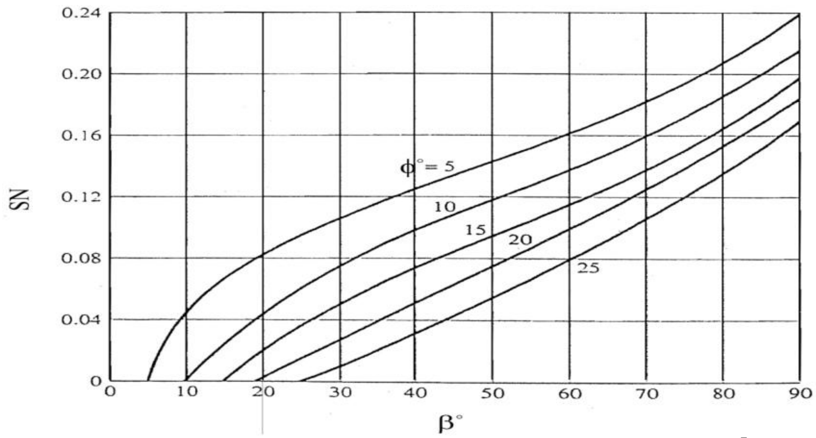

Figure 1 Taylor's chart for stability number (based on Taylor[5])

The stability number and the mobilized friction angle are given by

$$SN = \frac{c'}{\gamma H F} \tag{1}$$

$$\tan(\phi_m) = \frac{\tan(\phi)}{F} \tag{2}$$

where c' = soil cohesion (kPa), ϕ = internal friction angle (degrees), γ = unit weight of soil (kN/m^3), H = slope height (m), and F = safety factor with respect to shear strength. Figure 2 depicts the slope characteristics along with a typical critical-slip circle.

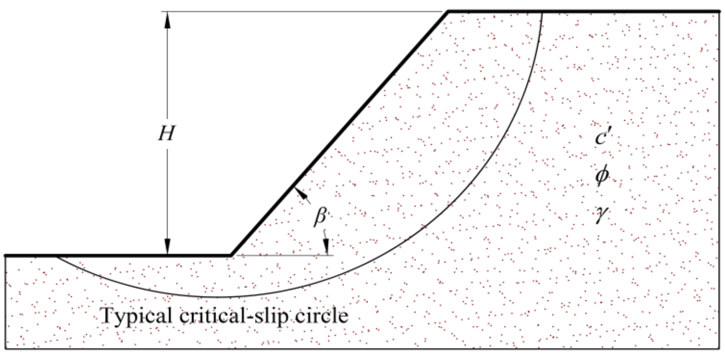

Figure 2 slope geometry along with a typical critical-slip circle

For given slope characteristics (slope and height), the safety factor is traditionally determined iteratively. The normal procedures involve first assuming a trial safety factor F', calculating ϕ_m from Eq. 2, determining SN from the chart (given β and ϕ_m), and then calculating the trial height H' from Eq. 1. Then F' is changed until H' equals or is close to H. Note that the obtained value of F corresponds to the critical slip circle.

Baker [9] presented a convenient alternative representation of Taylor's stability chart by presenting two charts involving a dimensionless parameter, originally implemented in a direct graphical procedure by Janbu [10]. The method eliminates the use of iterations, but it is still graphical in nature. The method includes three steps: (1) calculating the dimensionless parameter, (2) entering the chart to determine a quantity, and (3) using this quantity in a simple formula to determine the safety factor.

3. PROPOSED EXPLICIT EQUATION

To eliminate the trial and direct graphical procedures, a direct mathematical equation for determining the safety factor is developed in this technical note. The stability number SN is first represented mathematically as a polynomial surface in terms of ϕ_m and β. For this purpose, the data for the stability number presented by Taylor [8], which are presented for a 5°-increment of ϕ_m, were used. These data were supplemented by additional refined data for $\phi_m < 5^\circ$ by Baker [9].

To determine the shape of this surface, it is noted from Fig. 1 that for a constant ϕ_m the stability number may be represented by a cubic polynomial function of *β*. Similarly *SN* may be represented by a quadratic polynomial function of ϕ_m for a constant *β*. Thus, the developed regression surface for the stability number is given by

$$SN = 0.042186 + 0.004905\beta - 6.44\times10^{-5}\beta^2 + 4.07\times10^{-7}\beta^3 - 0.00807\phi_m + 3.41\times10^{-5}\beta\phi_m + 5.94466\times10^{-5}\phi_m^2 \quad (3)$$

A comparison of the original Taylor's data and the values estimated using Eq. (3) is shown in Fig. 3. As noted, there is close agreement between observed and estimated stability numbers.

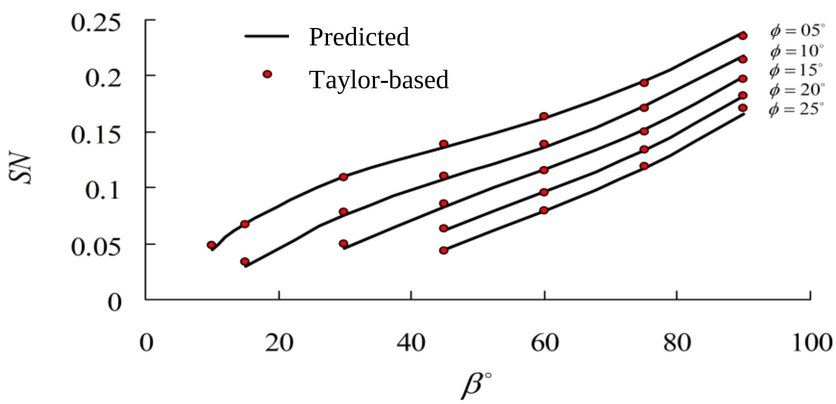

Figure 3 Comparison of Taylor-based stability numbers with those predicted using the proposed regression formula

To enable development of the direct solution, a dimensionless parameter used by Baker [9] and its inverse was originally used by Janbu [10] is used here. Dividing both sides of Eq. (1) by Eq. (2) yield the following dimensionless parameter *λ*,

$$\lambda = \frac{SN}{\tan(\phi_m)} = \frac{c'}{\gamma H \tan(\phi)} \quad (4)$$

From which

$$SN = \lambda \tan(\phi_m) \quad (5)$$

From trigonometric series,

$$\tan(\phi_m) \cong \phi_m^{\,r} \quad (6)$$

where $\phi_m^{\,r}$ is mobilized friction angle in radians. For the range of practical values used in Taylor's chart, the approximation of Eq. (6) is very accurate (Fig. 4). For example, the percent difference between the exact and approximate values range from 0.3% and 6.4% for $\phi_m = 5^o$ and 25^o, respectively. Thus, Eq. (5) can be written as

$$SN \cong \frac{\lambda \phi_m \pi}{180} \quad (7)$$

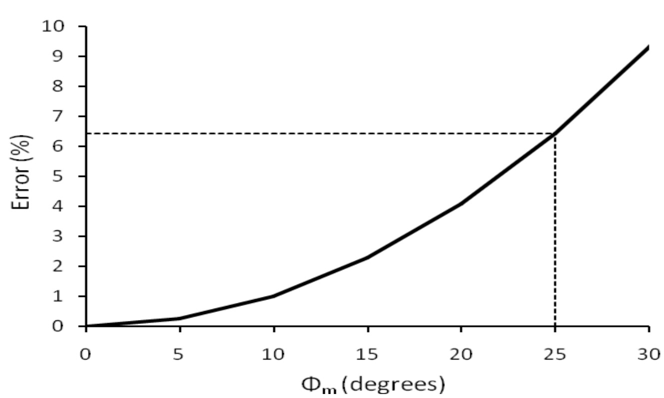

Figure 4 Error in using one term of tan(ϕ_m).

The stability number surface of Eq. (3) and the trigonometric approximation of Eq. (7) allow the development of the explicit equation for the safety factor. Equating Eqs. (3) and (7) yields

$$\lambda\phi_{\mathrm{m}}\pi/180 = 0.042186 + 0.004905\beta - 6.44\times10^{-5}\beta^2 + 4.07\times10^{-7}\beta^3 - 0.00807\phi_{\mathrm{m}} + 3.41\times10^{-5}\beta\phi_{\mathrm{m}} + 5.94466\times10^{-5}\phi_{\mathrm{m}}^2 \qquad (8)$$

which is a quadratic equation in ϕ_m. Solving Eq. (8) for ϕ_m yields

$$\phi_m = \frac{-b-(b^2-4ac)^{1/2}}{2a} \qquad (9)$$

where

$$\mathrm{a} = 5.94466\times10^{-5} \qquad (10)$$

$$b = -0.00807 + 3.41\times10^{-5}\beta - \lambda\pi/180 \qquad (11)$$

$$\mathrm{c} = 0.042186 + 0.004905\beta - 6.44\times10^{-5}\beta^2 + 4.07\times10^{-7}\beta^3 \qquad (12)$$

Note that only the minus sign before the radical of Eq. (9) is used. The plus sign is not selected because in this case Eq. (9) produces a root greater than 90$^{\mathrm{o}}$ for ϕ_m which is not practical.

Finally, the safety factor is given, based on Eq. (2) by

$$F = \frac{\tan(\phi)}{\tan\left(\dfrac{-b-(b^2-4ac)^{1/2}}{2a}\right)}, \qquad (\phi > 0) \qquad (13)$$

Equation (13) is the explicit formula for calculating the safety factor. A very accurate, but a little complex, alternative explicit equation for the safety factor, based on approximating *tan*(ϕ_{m}) of Eq. (6) using two terms, is presented in Appendix 1.

4. SPECIAL CASES

The proposed direct method is applicable to the four special cases described by Taylor [8]. In all four cases, it is assumed that the soil is completely saturated. The four cases are as follows: Case I (complete submergence), Case II (complete sudden drawdown), Case III (Steady seepage), and Case IV (zero boundary neutral force). These four cases can be analyzed to determine the safety factor using Taylor's charts, but the analysis requires a trial procedure. However, the proposed Eq. (13) can be used to accurately calculate the safety factor for these cases directly.

In all cases, Eq. (13) is applicable, but the values of the soil unit weight and friction angle will depend on the specific case considered [8. For Case I, the submerged unit weight must be used for the unit weight value in Eq. (4) along with ϕ. The submerged unit weight equals ($\gamma_t - \gamma_w$), where γ_t is the total unit weight of the essentially saturated soil and γ_w is unit weight of water. For Cases II and III, the total unit weight is used. For these cases, the weighted friction angle, ϕ_w, to be used in the analysis can be written as

$$\phi_w = \left(\frac{\gamma_{\mathrm{t}} - \mathrm{r}\gamma_{\mathrm{w}}}{\gamma_{\mathrm{t}}}\right)\phi, \qquad \text{(Cases II and III)} \qquad (14)$$

where r = ratio of the height of water surface in the soil to the total height. For Case II (complete sudden drawdown), r = 1 and for Case III (steady seepage) r will be less than 1. For Case IV, the total unit weight of the soil is used along with ϕ. Typical values of the submerged unit weight range from 50 to 70 1b/ft^3 and for the total unit weight typical values range from 110 to 130 lb/ft^3.

5. COMPARISON AND VALIDATION

To validate the proposed explicit equation, its results were compared with the results of the following methods: graphical methods, analytical methods, and the special cases discussed by Taylor.

For the graphical methods, several examples illustrating the Taylor-based trial procedure presented in textbooks were used. Six examples from the textbooks of Murthy [6], Huang [11], Smith [1], Janbu [10], and Baker [9] are used. The data for these examples and the results are shown in Table 1. As noted, the calculated safety factor using the explicit equation differs by only up to 3.3% from the graphical methods, except for one case where the difference is 12.7%. Note that the accuracy of the graphical procedure depends on the degree of refinement used in the stability chart. For example, the smallest unit for the quantity obtained from Baker's chart is 0.5, the increments of the β values are 10^o, and it is necessary to interpolate three quantities to determine the chart quantity. In general, the value of safety factor determined from the stability charts may vary by up to 0.1 to 0.2.

Table 1 Comparison of safety factors of proposed explicit equation and existing graphical Taylor-based methods

Example	c'	γ	φ^a	β	H	λ	a	b	c	ϕ_m	Safety Factor		
											Proposed	Graph	Diff (%)
Murthy [6]	550	69	20	45	39	0.56	6×10^{-5}	-0.016	0.170	10.81	**1.91**	1.90	0.4
Murthy [6]	420	121	18	23.5	50	0.21	6×10^{-5}	-0.011	0.127	12.39	**1.48**	1.51	-1.4
Huang [11]	800	100	10	30	40	1.13	6×10^{-5}	-0.027	0.142	5.37	**1.88**	1.82	3.1
Janbu [10]	280	120	17	30	23	0.33	6×10^{-5}	-0.013	0.142	11.73	**1.47**	1.43	3.0
Smith [1]	25	16	20	26.6	31	0.14	6×10^{-5}	-0.010	0.135	15.57	**1.31**	1.35	-3.3
Baker [9]	10	20	25	14.0	10	0.11	6×10^{-5}	-0.001	0.100	11.32	**2.33**	2.67	-12.7

[a] For Murthy [6], Huang [11], and Janbu [10] the data corresponds to Customary units, while for Smith [1] and Baker [9] they correspond to SI units. All angles are in degrees.

For the comparison with analytical methods (e.g. the method of slices), two examples were used. The results are shown in Table 2. In Example 1, used by Radoslaw [12], c' = 10 kPa, ϕ =20°, γ = 17 kN/m^3, β = 30°, and H = 10 m. The safety factor of the proposed equation is 1.273, compared with 1.300 for Radoslaw [12] and 1.256 for Bhattacharjya and Satish [13]. The difference in the results may be attributed to the type of the method and solution techniques used. For example, the number of slices has a considerable effect on the calculated value of the safety factor, as shown by Sun et al. [14]. In Example 2, used by Yamagami and Ueta [15], c' = 9.8 kPa, ϕ =10°, γ = 17.64 kN/m^3, β = 26.56°, and H = 5 m. The safety factor of the proposed equation is 1.321, compared with 1.321-1.324 for Sun et al. [14], 1.276 and 1.335 by Bhattacharjya and Satish [13], 1.338-1.348 by Yamagami and Ueta [15], and 1.327-1.333 by Greco [16]. The difference in the results is again due to the nature of the methods and the solution techniques implemented. Clearly, the proposed equation provides very comparable results and can accurately predict the safety factor for the critical slip circle of an earthen slope.

Table 2 Comparison of safety factors of proposed explicit Equation and existing analytical methods

Analytical Method	Safety Factor
(a) Example 1	
Proposed explicit equation	**1.273** [a]
Bhattacharjya and Satish [13]	1.256
Radoslaw [12]	1.300
(b) Example 2	
Proposed explicit equation	**1.321** [b]
Sun et al. [14]	
Genetic algorithm (line)	1.324
Genetic algorithm (spline)	1.321
Bhattacharjya and Satish[13]	
Genetic algorithms	1.276
Gradient search	1.335
Greco [16]	
Pattern search	1.327–1.330
Monte Carlo	1.327–1.333
Yamagami and Ueta [15]	1.338–1.348

[a] The corresponding results are λ = 0.16, a = 5.95E-05, b = -0.010, c = 0.142 and ϕ_m = 15.96.
[b] The corresponding results are λ = 0.63, a = 5.95E-05, b = -0.018, c = 0.135 and ϕ_m = 7.60.

For the special cases, the example used by Taylor [8] and subsequently used by Janbu [10] was used for comparison. The data of the example are c' = 600 lb/ft^2, γ_t = 130 lb/ft^3, ϕ = 20^o, β = 45^o, and H = 40 ft. The comparison results are shown in Table 3. AS noted, the safety factors calculated using the proposed equation differs from those of Taylor and Janbu by only 2-3%. Of course, there is some inaccuracy inherent in the graphical methods and the results are considered practically the same.

Table 3 Comparison of the safety factors of the proposed and existing methods for the special cases

Case	c' (lb/ft^2)	γ (lb/ft^3)	ϕ [a]	β	H (ft)	λ	a	b	c	ϕ_m	Safety Factor		
											Proposed	Taylor	Janbu
I	600	67.5	20	45	40	0.61	6×10^{-5}	-0.017	0.17	10.23	**2.02**	2.06	1.96
II	600	130	10.38	45	40	0.63	6×10^{-5}	-0.018	0.17	10.02	**1.04**	1.06	1.02
II	600	130	18.08	45	40	0.35	6×10^{-5}	-0.013	0.17	14.31	**1.28**	- [b]	1.25 [c]
IV	600	130	20	45	40	0.32	6×10^{-5}	-0.012	0.17	15.19	**1.34**	1.38	1.30

[a] All angles are in degrees.
[b] No values are calculated for this case by Taylor [8].
[c] This value is based on a water level within the slope of 8 ft for the steady seepage case.

6. CONCLUSIONS

Existing methods for determining the safety factor of simple homogeneous earthern slopes are graphical in nature and some involve iterative procedures. To eliminate the trial procedures, the stability number *SN* was first presented mathematically as a polynomial surface in terms of ϕ and β using the data presented in the literature. Then using the proposed polynomial surface a direct equation for determining the safety factor was developed. The proposed equation is applicable to slopes without seepage as well as some special cases. Different types of comparisons were made to validate the proposed explicit equation. The numerical results showed that the proposed equation provided very comparable results with other methods and can accurately predict the safety factor for the critical slip circle of an earthen slope. The proposed equation, with its simplicity and relative accuracy, should be useful in many geotechnical applications, especially those that implement the safety factor as part of analysis models.

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Appendix 1: Alternative Explicit Equation for Safety Factor

Using a trigonometric series with two terms, Eq. (5) can be written as

$$SN \cong \frac{\lambda \phi_m \pi}{180} + \frac{\lambda \phi_m^3 \pi^3}{180^3 \times 3} \tag{1A}$$

Substituting Eq. (1A) into Eq. (3) the resulting equation can be rearranged as a cubic equation in ϕ_m as follows

$$\phi_m^3 + p\phi_m^2 + q\phi_m + r = 0 \tag{2A}$$

in which

$$p = -\frac{180^3 \times 3}{\lambda \pi^3}\left(5.94466 \times 10^{-5}\right)$$

$$q = -\frac{180^3 \times 3}{\lambda \pi^3}\left(3.41 \times 10^{-5}\,\beta - \lambda\pi/180 - 0.00807\right) \tag{3A}$$

$$r = -\frac{180^3 \times 3}{\lambda \pi^3}\left(0.042186 + 0.004905\beta - 6.44 \times 10^{-5}\beta^2 + 4.07 \times 10^{-7}\,\beta^3\right)$$

The discriminant of the Eq. (3A) is positive, thus one root is real and two roots are complex conjugate roots. The real root can be obtained, based on Beyer [17], as

$$\phi_m = \sqrt[3]{R + \sqrt{Q^3 + R^2}} + \sqrt[3]{R - \sqrt{Q^3 + R^2}} - \frac{p}{3} \tag{4A}$$

in which

$$Q = \frac{3q - p^2}{9} \tag{5A}$$

$$R = \frac{9pq - 27r - 2p^3}{54} \tag{6A}$$

It is important to note that in Eq. (4A) the cubic root of a real negative number must be chosen as a negative real number, not a complex number.

Appendix 2: Notation

a, b, c = coefficients of quadratic equation;
c' = cohesion of the soil;
F = safety factor with respect to shear strength;
H = slope height;
β = slope inclination angle;
c = soil cohesion;
ϕ = internal friction angle;
ϕ_m = mobilized friction angle;
γ = unit weight of soil;
λ = dimensionless factor