PROJECTION-ITERATION METHOD FOR SOLVING NONLINEAR INTEGRAL EQUATION OF MIXED TYPE

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ABSTRACT

In this paper, the existence of a unique solution of Volterra-Hammerstein integral equation of the second kind (V-HIESK) is proved by using Banach fixed point theorem (BFPT) in the space $L_2(\Omega)\times C[0,T]$, where Ω represents the domain of integration of the variable space and T is the time. Then, different kinds of projection-iteration methods (PIMs) for solving this integral equation in the space $L_2(\Omega)\times C[0,T]$ are introduced. Finally, we deduced that: this method is quick convergent and the estimating error is better than the approximate error in the method of successive approximation for solving the integral equation numerically.

Keywords: *Volterra-Hammerstein integral equation; Projection operator; Projection- iteration methods, Banach fixed point theorem.*

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1. INTRODUCTION

Many problems of mathematical physics, theory of elasticity, viscodynamic fluids and mixed problems of mechanics lead to the nonlinear integral equations, see [1-4]. Therefore many different methods are used to obtain the solutions of these types. Abdalkhani [5] obtained a numerical solution of nonlinear Volterra integral equation of the second kind (NVIESK) when the kernel taken Abel's function form. The product Nystrom method is used to obtain the solution of NVIE when its kernel takes a logarithmic and Carleman forms, see Orsi [6]. Also, Kauthen [7] obtained numerical solution of NVIE by applying the linear multistep method.

In [8], Kilbas and Saigo used an asymptotic method to obtain numerical solution of nonlinear Abel-VIE. The solution of two dimensional of NVIE by collocation method and iterated collocation method is obtained by Guoqiang et al. [9]. In [10], Badr introduced a family of methods depending on a few parameters to obtain the solution of NVIESK with logarithmic kernel. A new quadrature method for solving NVIESK could be found, see Tao and Young [11]. On the other hand, Abdou et al., [12] obtained the solution of nonlinear integral equation of type Hammerstein–Volterra of the second kind (HVIESK). In [13], Abdou et al., used the Toeplitz matrix method to obtain the solution of nonlinear integral equation of Hammerstein of the second kind. Consider the V-HIESK in n-dimensional,

$$\mu\phi(x,t)=g(x,t)+\lambda\int_0^t\int_\Omega f(t,\tau)k(x,y)\gamma(\tau,y,\phi(y,\tau))\,dy\,d\tau \tag{1.1}$$

Here, $g(x,t)$ and $\gamma(t,x,\phi(x,t))$ are two given functions in the space $L_2(\Omega)\times C[0,T]$, where Ω is a closed bounded set depends on the vector of position and the time $t\in[0,T], T<\infty$. While, the function $\phi(x,t)$ is unknown and will be discussed in $L_2(\Omega)\times C[0,T]$.The two kernels $k(x,y)$, $f(t,\tau), t,\tau\in[0,T], T<\infty$ are continuous with their derivatives with respect to position and time respectively.

Write the formula (1.1) in the integral operator form

$$\overline{W}\,\phi(x,t)=\frac{1}{\mu}g(x,t)+W\,\phi(x,t) \tag{1.2}$$

where:

$$W\,\phi(x,t)=\frac{\lambda}{\mu}\int_0^t\int_\Omega f(t,\tau)k(x,y)\gamma(\tau,y,\phi(y,\tau))\,dy\,d\tau\,. \tag{1.3}$$

Also, we assume the following conditions:

i. The kernel of position $k(x,y)$ satisfies the discontinuity condition:

$\{ \int_\Omega \int_\Omega |k(x,y)|^2 dxdy \}^{1/2} \le c$, c is constant.

ii. The kernel of time $f(t,\tau) \in C[0,T]$ and satisfies $|f(t,\tau)| \le M$, M is constant, $\forall t,\tau \in [0,T], 0 \le \tau \le t \le T < \infty$.

iii. The given function $g(x,t)$ with its partial derivatives are continuous in the space $L_2(\Omega) \times C[0,T]$ and its norm is defined as,

$$\|g(x,t)\| = \max_{0\le t\le T} \left| \int_0^t \{ \int_\Omega g^2(x,\tau)dx \}^{1/2} d\tau \right| \le G, \quad G \text{ is constant.}$$

iv. The known continuous function $\gamma(t,x,\phi(x,t))$ for the constants $q > \sigma$ and $q > q_1$, satisfies the following conditions:

(a) $\max_{0\le t\le T} \left| \int_0^t \{ \int_\Omega |\gamma(\tau,x,\phi(x,\tau)|^2 dx \}^{1/2} d\tau \right| \le q_1 \|\phi(x,t)\|$

(b) $|\gamma(t,x,\phi_1(x,t)) - \gamma(t,x,\phi_2(x,t))| \le N(t,x)|\phi_1(x,t) - \phi_2(x,t)|$

where:

$$\|N(t,x)\| = \max_{0\le t\le T} \left| \int_0^t \{ \int_\Omega N^2(\tau,x)dx \}^{1/2} d\tau \right| \le \sigma < \infty$$

In this work, we concentrate our heed to the V-HIE, where the existence and uniqueness of the solution is considered. Then, we introduce different kinds of projection-iteration methods for solving the integral equation in the space $L_2(\Omega) \times C[0,T]$.

For this, we use the operator effect to write Eq. (1.1) in the form:

$$\phi = \frac{1}{\mu} g + \frac{\lambda}{\mu} F\, K\, \Gamma(\phi) \quad , \quad \mu \ne 0 \tag{1.4}$$

where:

$$\Gamma(\phi) = \gamma(\tau,y,\phi(y,\tau)) \ , \tag{1.5}$$

$$K\,\Gamma(\phi) = \int_\Omega k(x,y)\gamma(\tau,y,\phi(y,\tau))dy \tag{1.6}$$

$$F\,K\,\Gamma(\phi) = \int_0^t \int_\Omega f(t,\tau)k(x,y)\gamma(\tau,y,\phi(y,\tau))dy\,d\tau \tag{1.7}$$

2 . THE EXISTENCE AND UNIQUENESS OF V-HIESK

In this section, we use BFPT, see [14], to prove the existence of a unique solution of Eq.(1.1). For this, we state the following theorem:

Theorem 1:

The formula (1.1) has a unique solution $\phi(x,t)$ in the space $L_2(\Omega) \times C[0,T]$ under the condition,

$$\frac{|\lambda|}{|\mu|} M\, c\, q\, T < 1 \quad , \quad \mu \ne 0 \tag{2.1}$$

Proof:

To prove the existence of a unique solution of Eq.(1.1) using BFPT, we must prove the normality and continuity of the integral operator Eq.(1.2).

To the normality of the integral operator, we have,

$$\|\overline{W}\,\phi(x,t)\| \le \frac{1}{|\mu|}\|g(x,t)\| + \frac{|\lambda|}{|\mu|} \left\| \int_0^t \int_\Omega |f(t,\tau)||k(x,y)||\gamma(\tau,y,\phi(y,\tau))|dy\,d\tau \right\|$$

Using the conditions (i) and (iv-a), and applying Cauchy-Schwarz inequality, we get

$$\|\overline{W}\,\phi(x,t)\| \le \frac{G}{|\mu|} + \beta\|\phi(x,t)\| \quad , \quad \beta = \frac{|\lambda|}{|\mu|} M\, cqT\,.$$

Hence, $\overline{W}$ is a norm operator.

For the continuity of the integral operator $\overline{W}$, we assume the two potential functions $\phi_1(x,t), \phi_2(x,t)$ in the space $L_2(\Omega) \times C[0,T]$ satisfies the formula (1.1) then,

$$\left\|\overline{W}(\phi_1-\phi_2)\right\| \le \frac{|\lambda|}{|\mu|}\left\|\int_0^t \int_\Omega \left| f(t,\tau)\right|\left|k(x,y)\right|\left|\gamma(\tau,y,\phi_1(\tau,y))-\gamma(\tau,y,\phi_2(\tau,y))\right| dy\, d\tau\right\|$$

Using the conditions (i) and (iv-b), and applying Cauchy-Shwarz inequality we get

$$\left\|\overline{W}(\phi_1-\phi_2)\right\| \le \beta\left\|\phi_1(x,t)-\phi_2(x,t)\right\| .$$

So, $\overline{W}$ is a continuous operator in the space $L_2(\Omega)\times C[0,T]$. Moreover, since $\beta<1$, then $\overline{W}$ is a contraction operator. Hence, by BFPT, $\overline{W}$ has a unique fixed point which is the unique solution of Eq. (1. 1).

3. PROJECTION-ITERATION METHODS

In this section, we present various variants of the projection-iteration methods for Eq.(1.4), in the space $L_2(\Omega)\times C[0,T]$, independent on the following property.

Property

There exists an orthogonal projection operator Q for the projection operator P satisfying the following

$$\|Px+Qy\|^2=\|Px\|^2+\|Qy\|^2 \qquad (Q=I-P) \tag{3.1}$$

Now, we will construct three different kinds of alogarithm based on projection-iteration method for an approximate solution of Eq.(1.4).

1-The first algorithm can be constructed in the form,

$$\phi_n=\frac{1}{\mu}g+\frac{\lambda}{\mu}F\,K\,(\Gamma(\phi_{n-1})+B_n) \quad , \quad (\phi_0\in L_2(\Omega)\times C[0,T],\, n=1,2,...) \tag{3.2}$$

where:
$$B_n=P(\Gamma(\phi_n)-\Gamma(\phi_{n-1})) \quad , \quad n=1,2,... \tag{3.3}$$

Substituting Eq. (3.3) in Eq. (3.2) and then using (3.1) we obtain,

$$\phi_n=\frac{1}{\mu}g+\frac{\lambda}{\mu}F\,K\,P\,\Gamma(\phi_n)+\frac{\lambda}{\mu}F\,K\,Q\,\Gamma(\phi_{n-1}) \quad , \quad n=1,2,... \tag{3.4}$$

Theorem 2:

Suppose that the two operator $P\Gamma$ and $Q\Gamma$ having the Lipschitz condition with the constants D_1 and D_2 respectively in the space $L_2(\Omega)\times C[0,T]$, the condition,

$$\frac{|\lambda|}{|\mu|}\|FK\|\sqrt{D_1^2+D_2^2}<1 \tag{3.5}$$

is satisfied, then the sequence $\{\phi_n\}$ of a unique solutions of Eq. (3.4) in the space $L_2(\Omega)\times C[0,T]$ converges to the unique solution $\overline{\phi}$ of Eq. (1.4) and the estimating error is given by the relation,

$$\left\|\overline{\phi}-\phi_n\right\| \le \frac{\varepsilon_1^n}{1-\varepsilon_1}\|\phi_1-\phi_0\| \quad , \quad \varepsilon_1<1 \tag{3.6}$$

where:
$$\varepsilon_1=\frac{\left|\lambda/\mu\right|\|FK\|D_2}{\sqrt{1-(\left|\lambda/\mu\right|\|FK\|D_1)^2}} \tag{3.7}$$

Proof:

After using Eq. (3.4) and relation (3.1), we get

$$\|\phi_n-\phi_{n-1}\|^2=\left|\frac{\lambda}{\mu}\right|^2\|FK\|^2\left\{\|P(\Gamma(\phi_n)-\Gamma(\phi_{n-1}))\|^2+\|Q(\Gamma(\phi_{n-1})-\Gamma(\phi_{n-2}))\|^2\right\}.$$

Applying the Lipschtiz condition for $P\Gamma$ and $Q\Gamma$ respectively we have

$$\|\phi_n-\phi_{n-1}\|^2\le\left|\frac{\lambda}{\mu}\right|^2\|FK\|^2\left\{D_1^2\|\phi_n-\phi_{n-1}\|^2+D_2^2\|\phi_{n-1}-\phi_{n-2}\|^2\right\}$$

which can be adapted in the form

$$\|\phi_n-\phi_{n-1}\|\le\varepsilon_1\|\phi_{n-1}-\phi_{n-2}\| \ . \tag{3.8}$$

With the same steps we can prove that

$$\|\phi_n-\phi_{n-1}\|\le\varepsilon_1^{n-1}\|\phi_1-\phi_0\| \ . \tag{3.9}$$

Now, we have

$$\|\phi_{n+p}-\phi_n\| = \|\phi_{n+p}-\phi_{n+p-1}+\phi_{n+p-1}-\phi_n\|.$$

Applying the properties of the norm to have

$$\|\phi_{n+p}-\phi_n\| \le \frac{\varepsilon_1^n}{1-\varepsilon_1}\|\phi_1-\phi_0\|.$$

The last inequality shows that $\{\phi_n\}$ is Cauchy sequence. Since the space $L_2(\Omega)\times C[0,T]$ is a complete space, then there exists $\bar{\phi}$ such that $\|\bar{\phi}-\phi_n\|\to 0$. Then $\bar{\phi}$ is the unique solution of Eq.(1.4) in the space $L_2(\Omega)\times C[0,T]$. Also, if $p\to\infty$, the estimate error can be obtained.

Now, if the Lipschitz condition is not verified in the whole space then we prove that $\|\bar{\phi}-\phi_n\|\to 0$ as $n\to\infty$ in a certain ball of Banach space.

Theorem 3:

Assume, $$\|P\Gamma(\phi)\| \le V_1(r), \qquad (3.10)$$

And $$\|Q\Gamma(\phi)\| \le V_2(r), \qquad (3.11)$$

are satisfied in a certain ball $H(\|\phi\|\le r)$ with a radius $r>0$, where V_1 and V_2 are functions of positive values. If the condition (3.5) is valid and for $r'>0$, we have the inequality

$$\frac{1}{|\mu|}\|g\| + \left|\frac{\lambda}{\mu}\right|\|FK\|\sqrt{V_1^2(r)+V_2^2(r)} \le r\ . \qquad (3.12)$$

Then, we can determine the sequence $\{\phi_n\}$ in the ball $H(\|\phi\|\le r')$, the unique solutions of Eq.(3.4) converges to the unique solution $\bar{\phi}$ of Eq.(1.4) in that ball for every $\phi_0\in H(\|\phi\|\le r')$ and the estimating error holds.

Proof:

By taking the norm of Eq.(3.12) and using (3.1), (3.10) and (3.11) we have,

$$\|\phi_n\| \le \frac{1}{|\mu|}\|g\| + \left|\frac{\lambda}{\mu}\right|\|FK\|\sqrt{V_1^2(r)+V_2^2(r)}$$

Hence, from (3.12), we get $\|\phi_n\|\le r$.

Since the above inequality is valid for r', there $\phi_n\in H(\|\phi\|\le r')$ for every $\phi\in H(\|\phi\|\le r')$. Then Eq. (3.4) has a unique solution in that ball and the condition (3.5) is valid.

2- The second algorithm can be written in the form,

$$\phi_n = \frac{1}{\mu}g + \frac{\lambda}{\mu}[FK\Gamma(\phi_{n-1})+C_n] \qquad (3.13)$$

Where $$C_n = P(FK\Gamma(\phi_n) - FK\Gamma(\phi_{n-1})). \qquad (3.14)$$

Using Eq. (3.14) in Eq. (3.13) we get

$$\phi_n = \frac{1}{\mu}g + \frac{\lambda}{\mu}PFK\Gamma(\phi_n) + \frac{\lambda}{\mu}(I-P)FK\Gamma(\phi_{n-1}). \qquad (3.15)$$

With the aid of relation (3.1), Eq. (3.15) becomes

$$\phi_n = \frac{1}{\mu}g + \frac{\lambda}{\mu}[PFK\Gamma(\phi_n) + QFK\Gamma(\phi_{n-1})] \quad , \quad \phi_0\in L_2(\Omega)\times C[0,T],\ n=1,2,\ldots \qquad (3.16)$$

Theorem 4:

If the condition,

$$\left|\frac{\lambda}{\mu}\right|\|FK\|\sigma<1,$$

is satisfied, then the sequence $\{\phi_n\}$ of solutions of Eq. (3.16) converges to the unique solution $\bar{\phi}$ of Eq.(1.4) and the estimate error is given by the relation,

$$\|\bar{\phi}-\phi_n\| \le \frac{\varepsilon_2^n}{1-\varepsilon_2}\|\phi_1-\phi_0\| \quad , \qquad (3.17)$$

where: $$\varepsilon_2 = \frac{\|QFK\|\sigma}{\sqrt{1-(\left|\lambda/\mu\right|\|PFK\|\sigma)^2}} \tag{3.18}$$

Proof:

By using Eq.(3.16), we get $\|\phi_n - \phi_{n-1}\| = \left|\frac{\lambda}{\mu}\right| \|PFK\Gamma(\phi_n) + QFK\Gamma(\phi_{n-1}) - PFK\Gamma(\phi_{n-1}) - QFK\Gamma(\phi_{n-2})\|$.

Using Eq. (3.1), and squaring both sides, we have

$$\|\phi_n - \phi_{n-1}\|^2 \le \left|\frac{\lambda}{\mu}\right|^2 [\|PFK\sigma(\phi_n - \phi_{n-1})\|^2 + \|QFK\sigma(\phi_{n-1} - \phi_{n-2})\|^2].$$

Adapting the above formula, we obtain $\|\phi_n - \phi_{n-1}\| \le \varepsilon_2^{n-1}\|\phi_1 - \phi_0\|$.

Finally, we get $\|\phi_{n+p} - \phi_n\| \le \frac{\varepsilon_2^n}{1-\varepsilon_2}\|\phi_1 - \phi_0\|$.

Theorem 5:

If the two conditions, $\left|\frac{\lambda}{\mu}\right|\|FK\|\sigma < 1$, $\|\Gamma(\phi)\| \le V(r)$,

are satisfied in a certain ball $H(\|\phi\| \le r)$ where V is a positive values function. Let $r' > 0$ presents the inequality,

$$\frac{1}{|\mu|}\|g\| + \left|\frac{\lambda}{\mu}\right|\|FK\|V(r) \le r \tag{3.19}$$

Then, the sequence $\{\phi_n\}$ in the ball $H(\|\phi\| \le r')$ of unique solutions of Eq. (3.16) converges to the unique solution of Eq.(1.4) in this ball for all $\phi_0 \in H(\|\phi\| \le r')$ and the estimating error holds.

Proof:

From Eq. (3.16) and relation (3.1), we get

$$\|\phi_n\| \le \frac{1}{|\mu|}\|g\| + \left|\frac{\lambda}{\mu}\right|\sqrt{\|PFK\Gamma(\phi_n)\|^2 + \|QFK\Gamma(\phi_{n-1})\|^2},$$

By using the second condition of the last theorem, we get

$$\|\phi_n\| \le \frac{1}{|\mu|}\|g\| + \left|\frac{\lambda}{\mu}\right|\sqrt{\|PFK\|^2 + \|QFK\|^2}\, V(r),$$

which can be adapted as, $\|\phi_n\| \le \frac{1}{|\mu|}\|g\| + \left|\frac{\lambda}{\mu}\right|\|FK\|V(r) \le r$.

If we fined $r' > 0$ for which the above inequality is satisfied for all $\phi_n \in H(\|\phi\| \le r')$ then in the considered ball Eq.(3.16) has a unique solution. One can complete the proof by following the same way of theorem 2.

3- The third algorithm

Consider the following equation,

$$\phi_n = \frac{1}{\mu}g + \frac{\lambda}{\mu}FK\Gamma(\phi_{n-1} + A_n) \quad , \phi_0 \in L_2(\Omega)\times C[0,T],\ n = 1,2,... \tag{3.20}$$

where: $$A_n = P(\phi_n - \phi_{n-1}). \tag{3.21}$$

Substituting Eq. (3.21) in Eq. (3.20), we get

$$\phi_n = \frac{1}{\mu}g + \frac{\lambda}{\mu}FK\Gamma(\phi_{n-1} + P(\phi_n) - P(\phi_{n-1})) \quad .$$

Finally we have,

$$\phi_n = \frac{1}{\mu}g + \frac{\lambda}{\mu}FK\Gamma(P(\phi_n) + Q(\phi_{n-1})) \quad , \phi_0 \in L_2(\Omega)\times C[0,T],\ n = 1,2,... \tag{3.22}$$

Theorem 6:

If the condition, $\left|\frac{\lambda}{\mu}\right|\|FK\|\sigma < 1$, is satisfied, then the sequence $\{\phi_n\}$ of unique solutions of Eq.(3.22) in the space $L_2(\Omega)\times C[0,T]$ converges to the unique solution $\bar{\phi}$ of Eq.(1.4) and the estimating error is given by the relation,

$$\|\bar{\phi}-\phi_n\| \le \frac{\left|\frac{\lambda}{\mu}\right|\|FK\|\sigma}{\sqrt{1-(\left|\frac{\lambda}{\mu}\right|\|PFK\|\sigma)^2}}\frac{\varepsilon_2^{n-1}}{1-\varepsilon_2}\|Q(\phi_1-\phi_0)\| \qquad (3.23)$$

Proof:

From Eq. (3.22), we have $\|\phi_n-\phi_{n-1}\| = \left|\frac{\lambda}{\mu}\right|\|FK\Gamma(P(\phi_n)+Q(\phi_{n-1}))-FK\Gamma(P(\phi_{n-1})+Q(\phi_{n-2}))\|$.

By using the condition (iv-b), and the relation (1.4), we get

$$\|\phi_n-\phi_{n-1}\| \le \left|\frac{\lambda}{\mu}\right|\|FK\|\sigma\sqrt{\|P(\phi_n-\phi_{n-1})\|^2+\|Q(\phi_{n-1}-\phi_{n-2})\|^2}$$

Inserting the effect of the operator P, and then after some calculus, we have

$$\|P(\phi_n-\phi_{n-1})\| \le \frac{\left|\frac{\lambda}{\mu}\right|\|PFK\|\sigma}{\sqrt{[1-(\left|\frac{\lambda}{\mu}\right|\|PFK\|\sigma)^2]}}\|Q(\phi_{n-1}-\phi_{n-2})\|.$$

Hence, we have

$$\|\phi_n-\phi_{n-1}\| \le \frac{\left|\frac{\lambda}{\mu}\right|\|FK\|\sigma}{\sqrt{[1-(\left|\frac{\lambda}{\mu}\right|\|PFK\|\sigma)^2]}}\|Q(\phi_{n-1}-\phi_{n-2})\|.$$

Also, inserting the effect of operator Q, we obtain

$$\|\phi_n-\phi_{n-1}\| \le \frac{\left|\frac{\lambda}{\mu}\right|\|FK\|\sigma}{\sqrt{[1-(\left|\frac{\lambda}{\mu}\right|\|PFK\|\sigma)^2]}}(\varepsilon_2)^{n-2}\|Q(\phi_{n-1}-\phi_{n-2})\|.$$

The complete proof can be obtained by following the same prove of theorem 2. Also we proof that the estimating error is given by the relation (3.23).

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