GOURSAT FUNCTIONS OF THE THERMO-ELASTIC PROBLEM OF AN INFINITE PLATE WITH HYPITROCHOIDAL HOLE

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ABSTRACT

Complex variable methods are used to solve the thermo-elastic problem of the infinite isotropic homogeneous plate with a hypitrochoidal hole with multi round corners conformally mapped on the domain outside a unit circle by means of a rational mapping function .The thermo-elastic problem is equivalent to finding two analytic functions(Goursat functions)at any point $z = x + \iota y$ within the region of the plate . The problem is transformed to solve an integrodifferential equation, in the complex plane, with singular kernel. Closed expressions for the Goursat function and consequently the tangential thermo-elastic stresses on the boundary of the hole are obtained in quadrature in the presence of a uniform heat stream.

Keywords: *Complex variable methods, thermo-elastic problem, Goursat functions, integrodifferential equation.*

1. INTRODUCTION

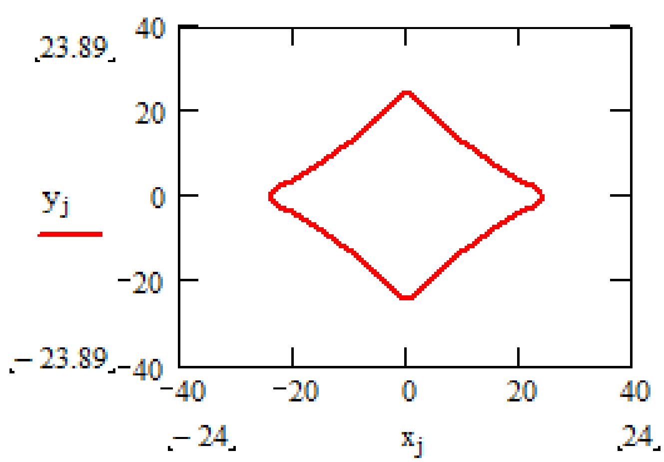
The thermo-elastic plane problem has been investigated by many author's [1-5].Exadaktylos and Stavropoulou [6] and Exadaktylos et al. [7] considered simple rational mapping functions with complex parameters that conformally map the holes inside a unit circle and using Laurant's method for finding Goursat functions .

Savin [8] apply the series method of solution to obtain the solution of thermo-elastic problem for infinite plate bounded internally by the elliptic ,triangle and square hole by using the simple transformations respectively.

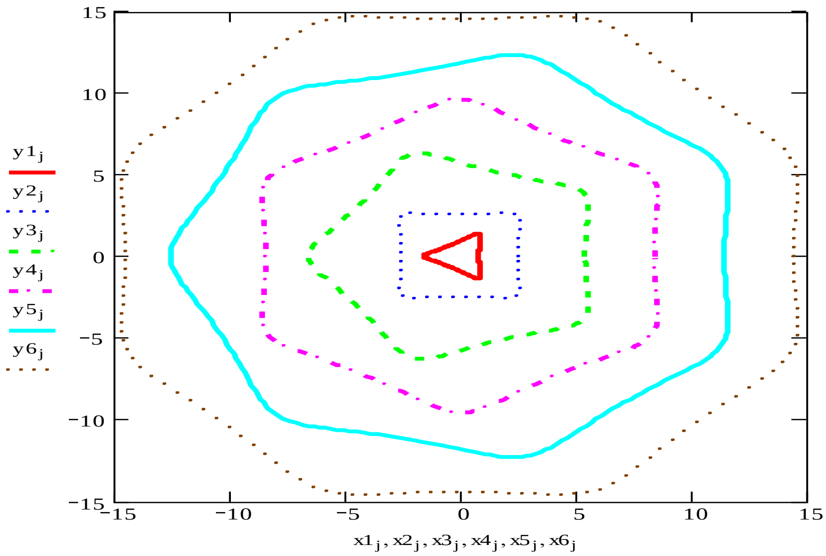
$$z = c(\zeta + m\zeta^{-1}) \ , z = c\ (\zeta + m\zeta^{-2}) \ , \ z = c\ (\zeta + m\zeta^{-3})$$

In this manuscript , complex variable methods and the mapping functions

$$z = c(\zeta + m\zeta^{-p}) \, , p \geq 1 \tag{1.1}$$

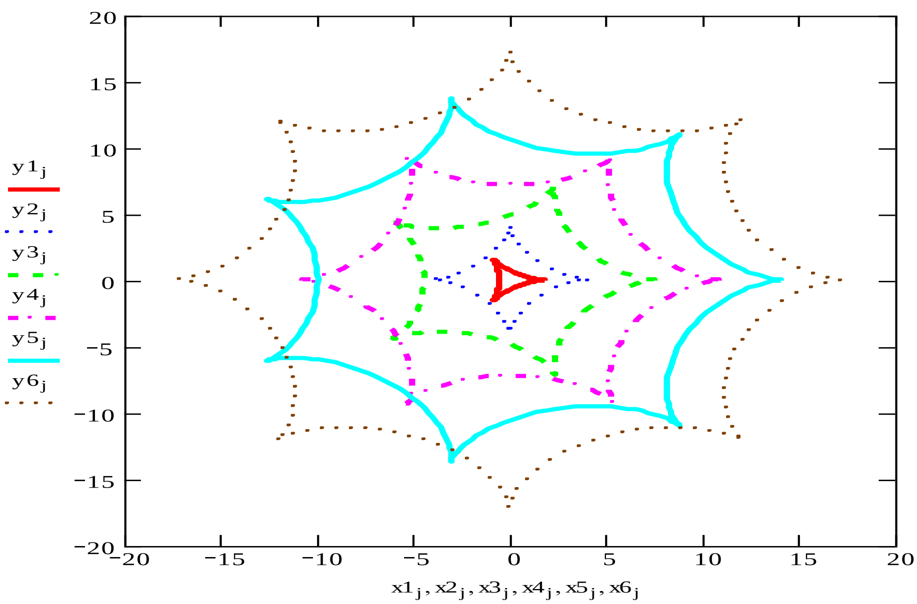


p=3,m=.2
``Figure 1'' Square hole.



p=2,3,…,7 m=-2/[p(p+1)]

``Figure 2" Hypitrochoidal holes with multi round corners

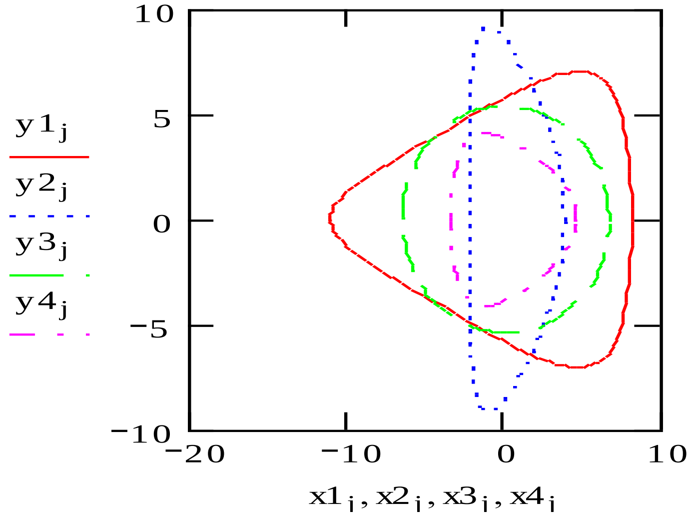


p=2,3,…,7 m=1/p

``Figure 3" Holes with star-shaped hypocycloid with p+1 cusps.

$$z = c(\zeta + m\zeta^{-1} + n\zeta^{-2}), \qquad (1.2)$$

$c > 0$, $z'(\zeta) \neq 0 or \infty$ outside the unit circle $\gamma(|\zeta|=1)$, are used to obtain the solution of thermo-elastic problem in the domain of the plate outside hypitrochoidal hole with multi round corners or curvilinear hole . The problem is transformed to solve an integro-differential equation, in complex plane, with singular kernel. The elliptic, triangle , square holes (p=1,2,3)are included as special cases.



m1=.2 n1=-.2 m2=-.5 n2=.12 m3=.1 n3=0 m4=0 n4=.12

``Figure 4" Curvilinear holes of the second mapping function.

For $m = -\frac{2}{p(p+1)}$ $(p \geq 2)$ we have good approximations to the infinite plate with a cut-out in the form of a regular rectilinear polygon of p+1 sides while for $m = \frac{1}{p}$ the hole is a star-shaped hypocycloid with p+1 cusps.

Various figures are sketched to show the shapes of the hole of the plate (see ``Figure1"-``Figure 4") . Also graphs of the distribution of tangential stresses on the boundary of the hole are plotted (see ``Figure5"-``Figure 7").

2. METHOD OF SOLUTION

The first fundamental problem in the plane theory of thermo-elasticity is equivalent to finding two analytic functions φ_o (z) and ψ_0 (z) , satisfying the boundary condition

$$\varphi_0(t)+t\,\overline{\varphi_0'(t)}+\overline{\psi_0(t)} = \frac{1}{2}\,\alpha_T\,E\int\Theta dz\,|_{z=t} + f(t), \tag{2.1}$$

where α_T is the coefficient of temperature expansion ,E is Young's modules ,Θ is the temperature function and f(t) is a given function of elastic stresses. The components of the stresses are given by

$$\sigma_x+\sigma_y = 4Re\{\varphi_0'(z)\} - \alpha_T\, E\Theta\,, \tag{2.2a}$$

$$\sigma_y - \sigma_x + 2\iota\tau_{xy} = 2[\bar{z}\,\varphi_0''(z)+\psi_0'(z)] - \alpha_T\, E\int\frac{\partial\Theta}{\partial z}d\bar{z} \tag{2.2b}$$

The temperature function Θ may be take the form [8]

$$\Theta(\zeta,\bar{\zeta}) = 2Re\{\Theta_0(\zeta)\}, \tag{2.3}$$

where $$\Theta_0(\zeta) = \frac{1}{2}\,qc[e^{i\alpha}\zeta^{-1} + e^{-i\alpha}\zeta]\,, \tag{2.4}$$

and q is a uniform heat stream making an angle α with the horizon, we note that in the ζ-plane

$$\nabla^2\Theta = 0\,,\ \frac{\partial\Theta}{\partial\rho} = 0\ at\ \rho = 1$$

2.1. First Mapping Function

The complex potential $\phi_0(z)$ and $\psi_0(z)$ can be written as

$$\phi_0(z) = \frac{1}{2}\, a\zeta^2 + \phi(\zeta) \ , \quad \psi_0(z) = -a_* \ln \zeta + \psi(\zeta) \tag{2.5}$$

where $\phi(\zeta)$ and $\psi(\zeta)$ are single valued analytic functions within the region outside $|\zeta| = 1$,and the constants a , a_* will be determined. Using the mapping function (1.1), the boundary condition (2.1) for the thermo-elastic problem takes the form

$$\phi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi'(\sigma)} + \overline{\psi(\sigma)} = F(\sigma) + H(\sigma) \quad on\ \gamma \tag{2.6}$$

Where $$F(\sigma) = \frac{1}{2}\, \alpha_T\, E \int \Theta dz\,|_{\zeta=\sigma}\ , \tag{2.7a}$$

$$H(\sigma) = -[\frac{1}{2}\, a\sigma^2 + \frac{\bar{a}\omega(\sigma)}{\sigma\overline{\omega'(\sigma)}} + \bar{a}_* \ln \sigma] \tag{2.7b}$$

and $\sigma = e^{i\theta}$ is a point on γ.

Now the function F(σ)+H(σ) can be take the form

$$F(\sigma) + H(\sigma) = \frac{1}{4} q\alpha_T c^2 E \left[e^{i\alpha}\left\{1 + \frac{mp}{p+1}\sigma^{-p-1}\right\} + e^{-i\alpha}\left\{\sigma^2 + m\left(1 + \frac{p}{p-1}\delta(p)\right)\sigma^{-p+1}\right\}\right] - \overline{a\sigma\beta(\sigma)} \tag{2.8}$$

Where $\delta(p) = \begin{cases} 1, & p \geq 2 \\ 0, & p = 1 \end{cases}$

The system of stresses (2.2) can be written in the form

$$\sigma_\theta + \sigma_\rho = 4Re\{\Phi_1(\zeta)\}\,,$$
$$\sigma_\theta - \sigma_\rho + 2\iota\, \tau_{\rho\theta} = \frac{2\zeta^2}{\rho^2\, \overline{\omega'(\zeta)}}\ [\overline{\omega(\zeta)}\ \Phi_1'(\zeta) + \omega'(\zeta)\Psi(\zeta)] \tag{2.9}$$

Where $\Phi_1(\zeta) = \Phi(\zeta) - \frac{1}{2}\alpha_T E\Theta_0(\zeta)$,

$$\Phi(\zeta) = \phi_0'(z) = \frac{\phi'(\zeta) + a\zeta}{z'(\zeta)}\,, \qquad \Psi(\zeta) = \psi_0'(z) = \frac{-a_*\zeta^{-1} + \psi'(\zeta)}{z'(\zeta)}\ . \tag{2.10}$$

Now the constants a , a_* can be chosen such that the stresses must be vanish at infinity and the function $F(\sigma) + H(\sigma)$ must be a single valued function,

i.e. , $$a = \frac{1}{4}\, qE\alpha_T c^2 e^{-i\alpha} \quad , \ a_* = \frac{1}{4} q\alpha_T c^2 \mathrm{E}[e^{-i\alpha} + e^{i\alpha} m(\delta(p) - 1)] \tag{2.11}$$

The first fundamental problem (2.1) can also take the form [7]

$$\Phi_1(\sigma) + \overline{\Phi_1(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}[\overline{\omega(\sigma)}\ \Phi_1'(\sigma) + \omega'(\sigma)\Psi(\sigma)] = N - \iota T \tag{2.12}$$

where N and T are the normal and tangential stresses at the boundary . When the edge of the hole is subject to the effect of the temperature only, $N = T = 0$.

The potential function $\Psi(\zeta)$ can be obtained, from (2.12), in the form

$$\Psi(\zeta) = \frac{\bar{\omega}'(\zeta^{-1})}{\zeta^2\omega'(\zeta)}[\Phi_1(\zeta) + \bar{\Phi}_1(\zeta^{-1})] - \frac{\bar{\omega}(\zeta^{-1})}{\omega'(\zeta)}\Phi_1'(\zeta). \tag{2.13}$$

Consider the function $\frac{\omega(\zeta^{-1})}{\omega'(\zeta)} = \alpha(\zeta^{-1}) + \beta(\zeta)$, Where $\alpha(\zeta) = m\zeta^{-p}$, $\beta(\zeta) = \frac{(1+m^2p)\zeta^p}{\zeta^{p+1}-mp}$ (2.14)

It can be seen that the function $\beta(\zeta)$ is a regular function for $|\zeta| > 1$.The boundary condition (2.6) takes the form

$$\phi(\sigma) + \alpha(\sigma)\overline{\phi'(\sigma)} + \overline{\psi_*(\sigma)} = F(\sigma) + H(\sigma) \quad on\ \gamma, \tag{2.15}$$

Where $\psi_*(\sigma) = \psi(\sigma) + \beta(\sigma)\phi'(\sigma)$,

Multiplying both sides of (2.15) by $\frac{1}{2\pi i}.\frac{d\sigma}{\sigma-\zeta}$ and integrating with respect to σ on γ , we get

$$-\phi(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)}{\sigma-\zeta}\overline{\phi'(\sigma)}\, d\sigma + G(\zeta), \tag{2.16}$$

Where $G(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)+H(\sigma)}{\sigma-\zeta} d\sigma$.

or $$G(\zeta) = -\frac{1}{4}qc^2mE\alpha_T\left[\frac{p}{p+1}e^{i\alpha}\zeta^{-(p+1)} + \frac{2p-1}{p-1}e^{-i\alpha}\zeta^{-(p-1)}\right] \tag{2.17}$$

Using (2.14) we have

$$\frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)}{\sigma-\zeta}\overline{\phi'(\zeta)}\, d\sigma = -\,m\sum_{\mu=0}^{p-1}\frac{\alpha_{p,\mu}}{\zeta^{\mu+1}} , \tag{2.18}$$

Where $\alpha_{p,\mu}$,$\mu = 0,1,2,\dots,p-1$ are constants to be determined.

Hence $$-\phi(\zeta) = G(\zeta) + m\sum_{\mu=0}^{p-1}\frac{\alpha_{p,\mu}}{\zeta^{\mu+1}} \tag{2.19}$$

Using (2.17)- (2.19), we obtain after some calculations

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{G'(\sigma)}}{\sigma^p(\sigma-\zeta)}d\sigma = 0 \tag{2.20}$$

$$\sum_{\mu=0}^{p-1}\frac{\alpha_{p,\mu}}{\zeta^{\mu+1}} = m\sum_{\mu=0}^{p-1}\frac{(\mu+1)\overline{\alpha_{p,\mu}}}{\zeta^{p-\mu-2}} \tag{2.21}$$

i.e., $\alpha_{p,\mu} = 0$, $\mu = 0,1,\dots,p-1$,

and consequently $$\phi(\zeta) = -G(\zeta) \tag{2.22}$$

Thus from (2.10) we have the Goursat function $\Phi(\zeta)$ in the form

$$\Phi(\zeta) = \frac{c\,qE\,\alpha_T}{4(\zeta^{p+1}-mp)}[-mp\,e^{i\alpha}\zeta^{-1} + e^{-i\alpha}\zeta\{m(1-2p)\delta(p) + \zeta^{p+1}\}] . \tag{2.23}$$

Now, the other Goursat function $\Psi(\zeta)$ can be obtained after some calculations by (2.13) in the form

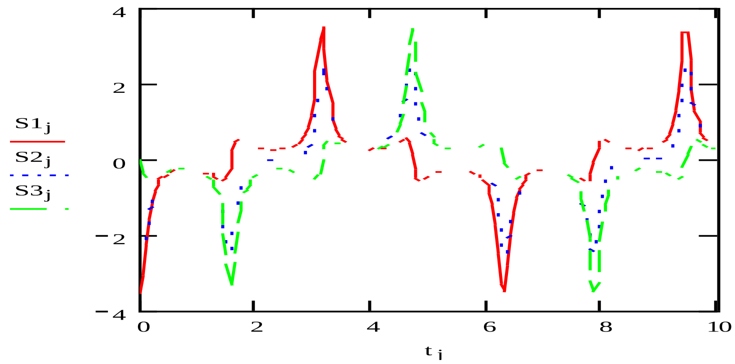
$$\Psi(\zeta) = \frac{cqE\alpha_T\zeta^p}{4(\zeta^{p+1}-mp)^3}\Big[\zeta^{p-1}\{\zeta^{2p+2}m(1-2p)(\delta(p)-1) - 2\zeta^{p+1}\big((1+m^2p^2) + pm^2\varepsilon(p)\big) + mp(1-p+ m^2p\varepsilon(p))\}e^{i\alpha} + \Big\{m\Big(2p + (1+p)(1+m^2p)\big(p + (1-2p)\delta(p)\big)\Big)\zeta^{p+1} - \zeta^{2p+2} - m^2p^2\Big\}e^{-i\alpha}\Big]. \tag{2.24}$$

Where $\varepsilon(p) = p + (1-2p)\delta(p)$.

To get the thermo-elastic stresses on the boundary, we have from (2.9) that

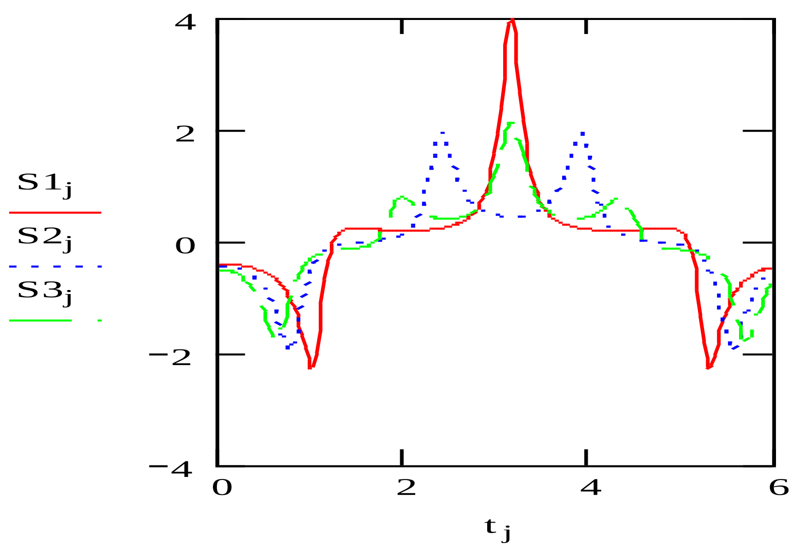
$$\sigma_\rho = \tau_{\rho\theta} = 0, \sigma_\theta = 4Re\{\Phi_1(\sigma)\} \quad at\ \rho = 1 ,$$

i.e., $$S = \frac{\sigma_\theta|_{\rho=1}}{cqE\,\alpha_T} = \frac{1}{U(\theta)}\left[m\{p+\varepsilon(p)\}\cos(p\theta+\alpha) - \{m^2 p\varepsilon(p)+1\}\cos(\theta-\alpha)\right]. \qquad (2.25)$$



p=3,m=.2 α=0 , π/4 , π/2

``Figure 5" Distribution of tangential thermo-elastic shear stresses on the square hole



p=2, 3, 4 m=-2/[p(p+1)] , α=0

``Figure 6" Distribution of tangential thermo-elastic shear stresses on the hypitrochoidal holes

where $U(\theta) = 1 + m^2p^2 - 2mp\cos(\mathrm{p}+1)\,\theta$.

The formula(2.23)-(2.25) agreeing with the formulae (VII.80)-(VII.82)for p=3 ,m=$-\frac{1}{6}$ and formulae (VII.83)-(VII.84)for p=2, m=$\frac{1}{3}$,formulae (VII.85)and (VII.87) for p=1 ,p.537 of Savin [6].

2.2. Second Mapping Function

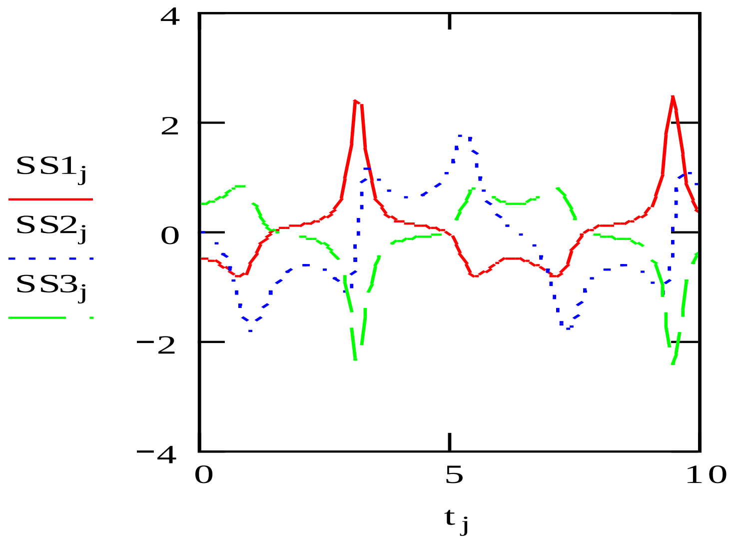
We now insert the results concerned with the case of an infinite plate with a hole of a curvilinear sides by using the mapping function (1.2).Applying the procedure of the previous problem of the mapping function (1.1),the complex potentials and the tangential component of the thermo-elastic stresses on the boundary are given by

$$\Phi(\zeta) = -\frac{cqE\ \alpha_T}{4V(\zeta)}\Phi_0(\zeta)\ , \qquad (2.26)$$

$$\Psi(\zeta) = \frac{cqE\alpha_T\zeta}{4V^3(\zeta)}\left[\zeta^4 V(\zeta)V(\zeta^{-1})\Phi_0(\zeta) + V^2(\zeta)\{\zeta^4\overline{\Phi}_0(\zeta^{-1}) - P(\zeta)\} + \zeta^2(1 + m\zeta^2 + n\zeta^3)\{V'(\zeta)\Phi_0(\zeta) - V(\zeta)\Phi_0'(\zeta)\}\right] \quad (2.27)$$

and

$$\sigma_\theta|_{\rho=1} = \frac{cq\alpha_T E}{U(\theta)}\{(2n^2 - m^2 - 1)\cos(\theta - \alpha) + 2m\cos(\theta + \alpha) + mn[\cos\alpha - 2\cos(2\theta - \alpha)]\} \quad (2.28)$$



m=.2 n=-.2 ; α=0, π/4, π/2

``Figure 7'' Distribution of tangential thermo-elastic shear stresses on the curvilinear holes.

where $V(\zeta) = \zeta^3 - m\zeta - 2n$,

$\Phi_0(\zeta) = e^{i\alpha}(m + 2n\zeta^{-1}) + e^{-i\alpha}\zeta(3n - \zeta^3)$,

$P(\zeta) = e^{i\alpha}(3 - m\zeta^2 - 3n\zeta^3) + e^{-i\alpha}[(1 - 3m)\zeta^2 - 5n\zeta^5]$,

and $U(\theta) = 1 + m^2 + 4n^2 + 4mn\cos\theta - 2m\cos 2\theta - 4n\cos 3\theta$.

Formulae (2.26)_(2.28) for m=0 and n=0 of the second mapping function agreeing with the formulae (2.23)-(2.25)for p=2 and p=1 of the first mapping function and also agreeing with the formulae (VII.85) and (VII.87) p.537 of [8] respectively on noting the difference in notations .

3. REFERENCES

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