AN INTEGRAL METHOD TO DETERMINE THE STRESS COMPONENTS OF STRETCHED INFINITE PLATE WEAKENED BY A CURVILINEAR HOLE

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ABSTRACT
An integral method, complex variable method, is used to obtain exact and closed expressions for Goursat functions for the stretched infinite plate weakened by a hole having arbitrary shape. The inner of the infinite plate is free from stresses. The plates considered are conformally mapped on the area of the right half – plane.
The interesting cases of an infinite plate weakened by a crescent like hole or by a cut having the shape of a circular arc , also when the hole takes the form of hypotrochoidal with four round corners, are included as special cases.

1. INTRODUCTION
The boundary value problems for isotropic homogenous perforated infinite plates have been discussed by several authors, see Muskhelishvili [1], England [2], Abdou [3], and Abdou and Asseri [4,5] .
Muskhelishvili [1] proved that the stresses, for perforated infinite plate, stretched at infinite by the application of a uniform tensile stress of intensity P making an angle θ with the x – axis can be written in the form

$$\overline{xx}+\overline{yy}=4\,\mathrm{Re}\{\phi'(z)\}$$

$$\overline{yy}-\overline{xx}+2i\ \overline{xy}=4[\bar{z}\phi''(z)+\psi'(z)]. \qquad (1.1)$$

Here, the functions $\phi(z)$ and $\psi(z)$ take the form

$$\phi(z)=\frac{P}{4}z+\phi_0(z), \qquad \psi(z)=\frac{P}{2}e^{-2i\theta}z+\psi_0(z), \qquad (1.2)$$

Also, $\phi_0(z)$ and $\psi_0(z)$ are single valued analytic functions within the region of the plate and bounded at infinity. When the edge of the hole is free from stresses, the boundary conditions can be written in the form

$$\phi_0(t)+t\overline{\phi_0'(t)}+\overline{\psi_0(t)}=-\frac{1}{2}P(t-e^{2i\theta}t) \qquad (1.3)$$

Where, t is an arbitrary point on the inner boundary.
El-Sirafy and Abdou [6] used the complex variable method and rational function to obtain the Goursat functions for a stretched infinite plate, weakened by inner curvilinear hole, using the transformation (1.4), see Figs. (1, 2, 3, 4)

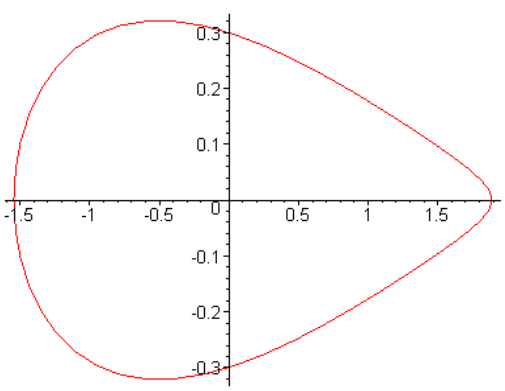

Fig. (1): n=0.1, m=0.7

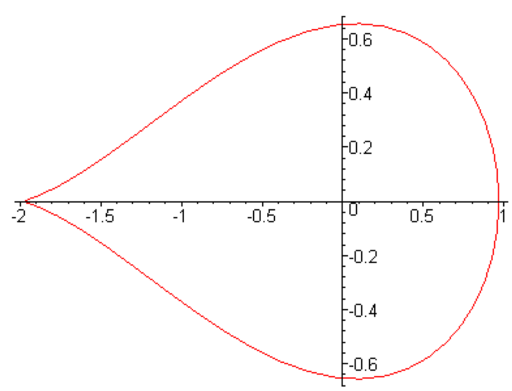

Fig. (2): n=-0.342, m=0.298

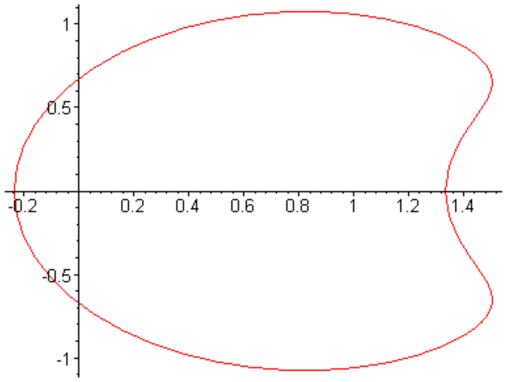

Fig. (3): n=0.7, m=-0.6

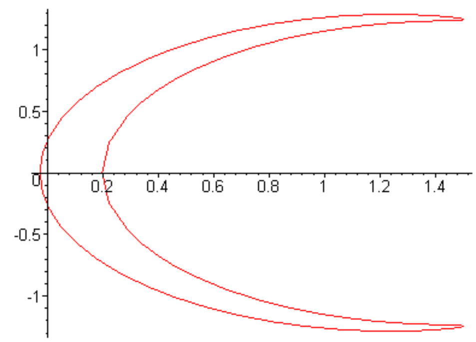

Fig. (4): n=0.7512, m=-0.951

$$z = c\frac{\xi + m\xi^{-1}}{1 - n\xi^{-1}} \quad \xi = \frac{s+1}{s-1}, c > 0, |n| < 1 \ , s = \sigma + i\tau \tag{1.4}$$

In the present paper, we use Cauchy method of the singular integral equation to obtain the solution of the first fundamental problem. For this aim complex variable methods and the following rational mapping function

$$\frac{z}{c} = \frac{w(s)}{c} = \frac{(s+1)^4 + m(s^2-1)^2 + l(s-1)^4}{(s+1)(s^2-1)[s+1-n(s-1)]}, \quad c > 0, |n| < 1, s = \sigma + i\tau \tag{1.5}$$

where m ,n , l are real parameters subject to the condition that $z(\infty)$ is bounded and $z'(s)$ does not vanish within the right half – plane , are used to obtain exact and closed expression for two analytic functions. The two analytic functions, **Goursat functions**, will be determined for the stretched infinite plate weakened by an arbitrary curvilinear hole. The edge of the hole is free from stresses. The interesting cases when the hole takes the form of an ellipse, a crescent or a cut having the shape of a circular arc, and the hypotrochoidal hole with four rounds are considered as special cases and the functions $\phi(z)$ and $\psi(z)$ are obtained in a closed form. The work of several authors is considered as special cases of this work.

2. METHOD OF SOLUTION

We consider an infinite plate weakened by a curvilinear hole, conformally mapped on the right half –plane $\mathrm{Re}\, s \geq 0$ by (1.5).

The expression $\frac{\overline{w(i\tau)}}{w'(i\tau)}$ can be written in the form

$$\frac{\overline{w(i\tau)}}{w(i\tau)} = \overline{\alpha(i\tau)} + \beta(i\tau) \tag{2.1}$$

where

$$\alpha(i\tau) = \frac{K}{a + i\tau}, \quad a = \frac{1+n}{1-n} \ , K = 4a^2(n^4 + n^2 m + 1)J_0^{-1},$$

$$J_0 = (21n^6 - 31n^4 - mn - 2n^2 + 1) \tag{2.2}$$

and

$$\beta(s) = \frac{1}{s-a}\left[\frac{H(s)}{E(s)} + K\right]$$

$$H(s) = (1-n)(1+s)^2(a+s)^2\left[1(s+1)^4 + m(s+1)^2(s-1)^2 + (s-1)^4\right],$$

$$E(s) = 2[-(1+s)^5 + 2n(s+1)^4(s-1) + m(s+1)^3(s-1)^2$$
$$+ 31(s+1)(s-1)^4 + 21\ (s-1)^4], \tag{2.3}$$

$\beta(s)$ is a regular function within the right half- plane except at infinity .

The boundary condition (1.3) takes the form

$$\phi(i\tau)+\alpha(\tau)\overline{\phi'(i\tau)}+\overline{\psi(i\tau)}=f(\tau), \qquad (2.4)$$

where

$$f(\tau)=-\frac{P}{2}[w(i\tau)-e^{2i\theta}\overline{w(i\tau)}]+(1-e^{2i\theta}),$$

$$\psi(s)=\psi_0(s)+\beta(s)\ \phi'(s) \qquad (2.5)$$

$$\phi(s)=\phi_0(w(s)).$$

and we assume that $\phi(\infty)=\psi(\infty)=0$.

Multiplying both sides of (2.4) by $\dfrac{1}{2\pi(s-i\tau)}$ and integrating with respect to τ from $-\infty$ *to* ∞ , one has

$$\phi(s)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha(\tau)\overline{\phi'(i\tau)}}{s-i\tau}d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{s-i\tau}d\tau \qquad (2.6)$$

Using (2.2) and (2.5), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha(\tau)\overline{\phi'(i\tau)}}{s-i\tau}d\tau=\frac{KcPb}{s+a}, \qquad (2.7)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{s-i\tau}d\tau=\frac{cP}{(1-n)^2}\left[\frac{m+n^2}{s+a}+\frac{l[(3-2n)s^2+2ns+1]}{(s+1)^2(s+a)}-\frac{(1-n)^2}{s+1}e^{2i\theta}\right] \qquad (2.8)$$

Where, b is a complex constant will be determined.

Substituting from (2.7) and (2.8) in (2.6) , one obtains

$$\phi(s)=\frac{cP}{(1-n)^2}\left[\frac{m+n^2}{s+a}+\frac{l[(3-2n)s^2+2ns+1]}{(s+1)^2(s+a)}-\frac{(1-n)^2}{s+1}e^{2i\theta}-\frac{Kb(1-n)^2}{s+a}\right] \qquad (2.9)$$

Inserting $\overline{\phi'(i\tau)}$ from (2.9) in (2.7), the complex constant *b* will be determined

$$b=(1+n)^2\left[\frac{\cos2\theta}{4a^2-K}-\frac{i\sin2\theta}{4a^2+K}\right]-\frac{m+n^2}{(1-n)^2(4a^2-K)}+\frac{l\ n^2(2n^2-3)}{(1-n)^2(4a^2-K)}, \qquad (2.10)$$

Substituting this value in (2.9), we obtain the function $\phi(s)$ in the form

$$\phi(s)=\frac{cP}{(1-n)^2}\left[\frac{(n^4+mn^2+1)(J_1+iJ_2)+J_3}{s+a}+\frac{l[(3-2n)s^2+2ns+1]}{(s+1)^2(s+a)}-\frac{(1-n)^2}{s+1}e^{2i\theta}\right] \qquad (2.11)$$

where

$$J_1=\frac{m+2+(1-n^2)^2\cos2\theta}{n^4+mn^2+1-J_0},\quad J_2=\frac{(1-n^2)^2\sin2\theta}{n^4+mn^2+1-J_0},$$

and

$$J_3=\frac{l^2n^2(2n^2-3)-1(m+2)-(m+n^2)}{n^4+mn^2+1-J_0} \qquad (2.12)$$

From the boundary condition (1.3) the final closed expressions for $\psi(s)$ takes the form

$$\psi(s)=\frac{cPK}{4a^2(1-n)^4(s+a)^2}\left[l(s-3)+(1-n)^2(s+3a)L_1+\frac{l\sum_{i=0}^{4}A_is^i}{(1+s)^3}\right] \qquad (2.13)$$

$$-\frac{cPK(1-n)^2(s+a+2)e^{2i\theta}}{4(1+s)^2}+\frac{cP}{1+s}\left[1-\left\{sL_2+L_3+\frac{4(1-n)l}{1+s}\right\}\frac{e^{-2i\theta}}{(s+a)(1-n)^2}\right]$$

where

$$A_0 = -2n^6 + 5n^4 - 8n^3 - 15n^2 + 8n + 15,$$
$$A_1 = -12n^5 + 12n^4 + 50n^3 - 10n^2 - 40n + 8,$$
$$A_2 = 4n^6 - 4n^5 - 38n^4 + 14n^3 + 52n^2 - 16n - 6,$$
$$A_3 = 12n^5 + 12n^4 - 18n^3 + 18n^2, \quad A_4 = -2n^6 + 4n^5 + n^4 - 6n^3 + 3n^2 - 1,$$
$$L_1 = (n^4 + mn^2 + 1)(J_1 + iJ_2) + J_3 \quad , \quad L_2 = n^2 + m + (3-2n)l \qquad (2.14)$$
$$L_3 = L_2 + 6(n-1)l$$

3. SPECIAL CASES

We are now in a position to consider several interesting special cases:

(i) For n=0, m=0, we get the mapping function, see Figs. (5, 6)

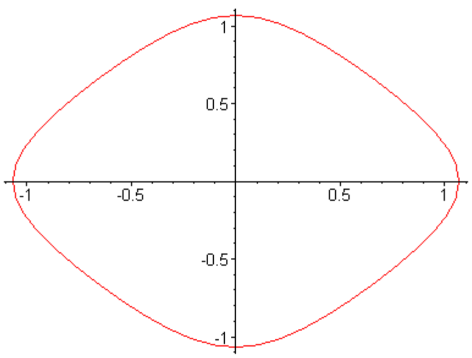

Fig. (5): l=0.6754

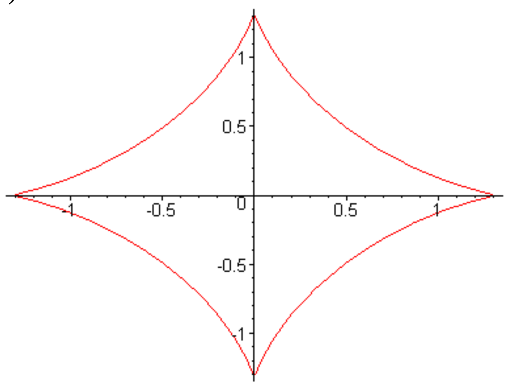

Fig. (6): l=0.3

$$z = c\left[\frac{s+1}{s-1} + l\left(\frac{s-1}{s+1}\right)^3\right], \quad c>0, \quad |l| \le \frac{1}{3}$$

The corresponding formulas for $\phi(z)$ and $\psi(z)$ become

$$\phi(s) = \frac{cP}{1+s}\left[3l - \gamma - \frac{6l}{1+s} + \frac{4l}{(1+s)^2}\right], \qquad (3.1)$$

$$\psi(s) = \frac{cP}{1+s}\left[1 - l\gamma - 3le^{-2i\theta} + 2l\frac{3e^{-2i\theta} - \gamma}{s+1} - 41\frac{3l + e^{-2i\theta}}{(1+s)^2} + \frac{24l^2}{(1+s)^3}\right], \qquad (3.2)$$

where

$$\gamma = \frac{\cos 2\theta}{1-l} + i\frac{\sin 2\theta}{1+l}$$

Also, for *m=n=0*, *l=1/3*, we get the mapping function

$$z = c\left[\frac{s+1}{s-1} + \frac{1}{3}\left(\frac{s-1}{s+1}\right)^3\right], \quad c>0. \qquad (3.3)$$

Then, the corresponding formulae for $\phi(z)$ and $\psi(z)$ become

$$\phi(s) = \frac{cP}{1+s}\left[1 - \gamma_1 - \frac{2}{1+s} + \frac{4}{3(1+s)^2}\right], \qquad (3.4)$$

$$\psi(s) = \frac{cP}{1+s}\left[1 - \frac{1}{3}\gamma_1 - e^{-2i\theta} + \frac{2[3e^{-2i\theta} - \gamma_1]}{3(s+1)} - \frac{4(1+e^{2i\theta})}{3(1+s)^2} + \frac{8}{3(1+s)^3}\right], \qquad (3.5)$$

where

$$\gamma_1 = \frac{3}{4}(2\cos 2\theta + i\sin 2\theta)$$

The two formulas (3.4) and (3.5) represent the Goursat functions for an infinite Plate when the hole takes a hypotrchoidal with four cusps. .

(ii) For *n=m=l=0*, we have the case of an infinite plate weakened by a circular hole $|z| = c$, and (2.11), (2.13) reduces to

$$\phi(s) = -\frac{cP}{1+s}e^{2i\theta}, \quad \psi(s) = \frac{cP}{1+s} \tag{3.6}$$

(iii) For n=0, we have the mapping function

$$\frac{z}{c} = \left(\frac{s+1}{s-1}\right) + m\left(\frac{s-1}{s+1}\right) + 1\left(\frac{s-1}{s+1}\right)^3 \tag{3.7}$$

The Goursat functions reduce to

$$\phi(s) = \frac{cP}{1+s}\left[B + \frac{1+31s^2}{(s+1)^2} - e^{2i\theta}\right] \tag{3.8}$$

$$\psi(s) = \frac{cP1}{(1+s)^2}\left[1(s-3)(s+3)B + \frac{1\sum_{s=0}^{4} D_i s^i}{(1+s)^3}\right] + \frac{cP}{(1+s)} - \frac{cP1(s+3)}{(1+s)^2}e^{2i\theta} \tag{3.9}$$

$$-\frac{cPe^{-2i\theta}}{(1+s)^2}\left[m(s+1) + 31(s-1) + \frac{41}{1+s}\right]$$

$$B = \frac{1sos 2\theta - m}{1-1} + i\frac{1\sin 2\theta}{1+1}, (1 \neq \pm 1)$$

$$D_0 = 15,\ D_1 = 8,\ D_2 = -6, D_3 = 0,\ D_4 = -1$$

The interested of this map when $\xi = \frac{s+1}{s-1}$ is mentioned in Abdou [3].

(iv) For *l=0* the rational function takes the form

$$z = c\frac{(s+1)^2 + m(s-1)^2}{s^2 - 1 - n(s-1)^2}, \quad c > 0, \quad |n| < 1$$

The corresponding formulas for $\phi(z)$ and $\psi(z)$ become

$$\phi(s) = \frac{cP}{(1-s)^2}\left[\frac{(m+n^2)J}{s+a} - \frac{(1-n)^2 e^{2i\theta}}{(1+s)}\right], \tag{3.10}$$

$$\psi(s) = cP\left[\frac{1}{s+1} - \frac{(m+n^2)e^{-2i\theta}}{(1-n)^2(s+a)}\right] + cPK^*\left[\frac{(m+n^2)(s+3a)}{4(1+n)^2(s+a)^2}J - \frac{(s+a+2)e^{2i\theta}}{(1+a)(s+1)^2}\right], \tag{3.11}$$

$$J = \frac{(m+2)n^2 - 1 + n^2(n^2-1)^2\cos 2\theta}{n^4 - 1 + 2n^2(m+1)} + in^2\sin 2\theta;$$

$$K^* = \frac{4n^2a^2(m+n^2)}{1-(m+2)n^2}$$

The previous result agree with (17) and (18) of El-Sirafy and Abdou [6]. Also, when *l=0*, $s = \frac{\xi+1}{\xi-1}$, and excluding the constant term , we have the rational function $z = \frac{\xi + m\xi^{-1}}{1-n\xi^{-1}}$ and the two complex functions will take the form

$$\phi(\xi) = \frac{1}{2}cP\left[e^{2i\theta}\xi^{-1} + (m+n)^2(1/2 - E)(\xi - n)^{-1}\right] \tag{3.12}$$

$$\psi(\xi) = -\frac{cP}{4\xi} + \frac{cPw(\xi^{-1})}{2w'(\xi)}\left[e^{2i\theta}\xi^{-2} + (m+n^2)(1/2-E)(\xi-n)^{-2} - 1/2\right] \tag{3.13}$$
$$-\frac{cPn^2h}{2(1-n\xi)}\left[e^{2i\theta} + (m+n^2)(1/2-E)(1-n^2)^{-2} - \frac{1}{2n^2}\right]$$
$$E = J + in^2\sin^2\theta\ ,\ h = \frac{(m+n^2)()1-n^2)^2}{1-(m+2)n^2}$$

(v) For $l=0, m=-n^2$, we have the case of an infinite plate weakened by a circular hole $|z-cn| = c$ and the two complex functions reduce to

$$\phi(s) = -\frac{cP}{1+s}e^{2i\theta}, \quad \psi(s) = \frac{cP}{1+s}, \tag{3.14}$$

(vi) For $l=0, m=-1$, we have the mapping function

$$z = c\frac{\xi + m\xi^{-1}}{1-n\xi^{-1}}\ ,\ \xi = \frac{s+1}{s-1},$$

In this case, $\phi(s)$ and $\psi(s)$ reduce to

$$\phi(s) = -cP\left[\frac{e^{2i\theta}}{1+s} + \frac{(1+n)F}{(1-n)s+1+n}\right] \tag{3.15}$$
$$\psi(s) = cP\left[\frac{1}{1+s} + \frac{e^{-2i\theta}}{s+a} + \frac{n^2a(s+3a)F}{(s+a)^2} + \frac{n^2(n+1)^2(s+a+2)e^{2i\theta}}{(s+1)^2}\right], \tag{3.16}$$
$$F = \frac{1+n^2(n^2-1)\cos 2\theta}{1+n^2} + in^2\sin 2\theta,$$

The above case represents an infinite plate weakened by a circular cut. For values of m near edge of the hole resembles the shape of crescent.

(vii) For $n=1=0$, $0 \le m < 1$, we get the mapping function

$$z = c(\xi + m\xi^{-1})\ ,\ \xi = \frac{s+1}{s-1},$$

The corresponding formulas for $\phi(s)$ and $\psi(s)$ become

$$\phi(s) = cP\frac{m-e^{2i\theta}}{1+s}, \quad \psi(s) = cP\frac{1-me^{-2i\theta}}{1+s}.$$

This case represents and infinite plate weakened by a hole in the form of ellipse.

(viii) For $l=m=0$, $s = \frac{\xi+1}{\xi-1}$, we have the mapping function $z = \frac{c\xi}{1-n\xi^{-1}}$, the inner edge of the infinite plate is the inners of an elliptic limacon, and the two Goursat functions can be determined from (2.12) and (2.14) .

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