FREDHOLM-VOLTERRA INTEGRAL EQUATION WITH A GENERALIZED SINGULAR KERNEL AND ITS NUMERICAL SOLUTIONS

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ABSTRACT

In this paper, the existence and uniqueness of solution of the Fredholm-Volterra integral equation (**F-VIE**) with a generalized singular kernel, are discussed and proved in the space $L_2(\Omega)\times C(0,T)$. The Fredholm integral term (**FIT**) is considered in position while the Volterra integral term (**VIT**) is considered in time. Using a numerical technique we have a system of Fredholm integral equations (**SFIEs**). This system of integral equations can be reduced to a linear algebraic system (**LAS**) of equations by using two different methods. These methods are: Toeplitz matrix method and Product Nyström method. Numerical examples are considered when the generalized kernel takes the following forms: Carleman function, logarithmic form, Cauchy kernel, and Hilbert kernel.

Keywords: *Fredholm-Volterra integral equation; generalized singular kernel; Toeplitz matrix; product Nyström method; logarithmic form; Carleman functions and Cauchy kernel.*

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1. INTRODUCTION

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to singular integral equation, see [1-5]. The singular integral equations are considered to be of more interest than the others and a closed form of solution is generally not available. Therefore great attention must be considered for the numerical solution of these equations. The references Linz [6], Atkinson [7], Delves and Mohamed [8], Backer [9] contain many different methods for solving the integral equations numerically. The discussions of **F-VIE** started by Abdou [10], who discussed the solution of **F-VIE** of the first kind in one, two and three dimensions, using separation of variable method. Also, Abdou, in [11-13], used some different methods to obtain the solution of **F**–**VIE** of the first kind and of the second kind. Hendi and Albugami, in [14], obtained numerically, the solution of **F-VIE** of the second kind, using collocation and Galerkin methods. Abdou et al., in [15, 16], used two numerical methods to obtain the solution of **F–VIE** of the second kind when the kernel takes a logarithmic form and Hilbert kernel, respectively.

In all previous works the singular kernel is considered as $k|(x-y)|$.While, in this work, we considered the generalized form of the kernel $k|(g(x)-g(y)|)$. Consider the integral equation:

$$\mu\phi(x,t)-\lambda\int_{\Omega} k(|g(x)-g(y)|)\phi(y,t)dy-\lambda\int_0^t F(t,\tau)\phi(x,\tau)d\tau=f(x,t) \qquad . \tag{1}$$

The formula (1) is called the **F-VIE** of the second kind in the space $L_2(\Omega)\times C(0,T)$, $T<1$. Where, the **FIT** is considered in position with a generalized singular kernel $k(|g(x)-g(y)|)$; Ω is the domain of integration with respect to position. And, the **VIT** is considered in time with a positive and continuous kernel $F(t,\tau)$ for all $t,\ \tau\in[0,T]$,$T<1$. The free term $f(x,t)$ and the function $g(x)$ are known functions, while $\phi(x,t)$ is unknown function to be determined. The numerical coefficient λ is called the parameter of the integral equation, may be complex, and has physical meaning, while the parameter μ defines the kind of the integral equation. The kernel $k(|g(x)-g(y)|)$, where the function $g(x), x\in\Omega$, is continuous with its derivatives, contains most of the generalized forms of the singular kernels such as: logarithmic form, Carleman function, Cauchy kernel, Hilbert kernel and other singular forms.

In this paper, the existence and uniqueness of solution of **F-VIE** of the second kind, when the kernel has a generalized singular term, under certain conditions, are considered in the space $L_2(\Omega)\times C(0,T)$. Using a numerical technique, formula (1) can be reduced to **SFIEs**. By using the Toeplitz matrix method and product Nyström method, as the best two methods for solving numerically the singular integral equation, we obtain **LAS** of equations. Finally, some examples and numerical results are discussed and investigated, and the error is estimated.

2. THE EXSISTENCE OF A UNIQUE SOLUTION OF THE F-VIE

Now, to prove the existence of a unique solution of **F-VIE** of Eq. (1) using Banach Fixed Point Theorem we write Eq.(1) in the integral operator form:

$$\bar{Q}\phi(x,t)=\frac{1}{\mu}f(x,t)+Q\phi(x,t) \qquad (2)$$

$$Q\phi(x,t)=K\phi(x,t)+F\phi(x,t) \qquad (3)$$

where,

$$K\phi=\frac{\lambda}{\mu}\int_{\Omega}k(|g(x)-g(y)|)\phi(y,t)dy\ ,\ F\phi=\frac{\lambda}{\mu}\int_0^t F(t,\tau)\phi(x,\tau)d\tau \qquad (4)$$

Then, we assume the conditions:

a- The generalized kernel of the **FIT** $k(|g(x)-g(y)|)$ satisfies the discontinuity condition:

$$[\int_{\Omega}\int_{\Omega}k^2(|g(x)-g(y)|)dxdy]^{\frac{1}{2}}=B \qquad (B \text{ is a constant})$$

b-The kernel of the **VIT** $F(t,\tau)\in C[0,T]$; $T<1$, with respect to the time satisfies

$|F(t,\tau)|\le C,\ \forall\ t,\ \tau\in[0,T],\ (C \text{ is a constant})$

c- The given function $f(x,t)$ with its partial derivatives with respect to x and t are continuous in $L_2(\Omega)\times C(0,T)$ where

$$\|f(x,t)\|=\max\int_0^t[\int_{\Omega}|f(x,\tau)|^2dx]^{\frac{1}{2}}d\tau=D,\quad (D \text{ is a constant})$$

d- The unknown function $\phi(x,t)\in L_2(\Omega)\times C(0,T)$ is called the potential function and in this space it behaves as the know function $f(x,t)$.

Theorem 1

The integral equation(1) has an exact and a unique solution in the space $L_2(\Omega)\times C(0,T)$ under the condition:

$$|\mu|>|\lambda|(B+CKT) \qquad (5)$$

The proof of this theorem depends on the following two lemmas:

Lemma 1

Under the conditions (a)-(d), the integral operator $\bar{Q}$ of (2), maps the space $L_2(\Omega)\times C(0,T)$ into itself.

Proof

In view of the formulas (3) and (4); after taking the norm of the formula (2), we found

$$\|\bar{Q}\phi(x,t)\|\le\left|\frac{1}{\mu}\right|\|f(x,t)\|+\left|\frac{\lambda}{\mu}\right|\left(\left\|\int_{\Omega}k(|g(x)-g(y)|)\phi(y,t)dy\right\|+\left\|\int_0^t|F(t,\tau)|\phi(x,\tau)d\tau\right\|\right) \qquad (6)$$

By using Cauchy – Schwarz inequality, we have

$$\|F\phi\|\le\left|\frac{\lambda}{\mu}\right|\left\|\int_0^t|F(t,\tau)|\phi(x,\tau)d\tau\right\|\le\left|\frac{\lambda}{\mu}\right|CKT\ \|\phi(x,t)\| \qquad (7)$$

Also, for the integral operator $K\phi$, we get

$$\|K\phi\|\le\left|\frac{\lambda}{\mu}\right|B\ \|\phi\| \qquad (8)$$

Hence, using (7) and (8) in the formula (6), we obtain

$$\|\bar{Q}\phi(x,t)\|\le\sigma\|\phi(x,t)\|+\left|\frac{D}{\mu}\right|,\ (\ \sigma=\left|\frac{\lambda}{\mu}\right|(B+CKT)). \qquad (9)$$

Where $T=\max_{0\le t\le T}t,\ K=(\int_{\Omega}dx)^{\frac{1}{2}}$

The last inequality (9) shows that, the operator $\bar{Q}$ maps the ball $R_p\subset L_2(\Omega)\times C(0,T)$ into itself, where

$$\rho=\frac{D}{[|\mu|-|\lambda|(B+CKT)]}=\frac{D}{|\mu|}(1-\sigma)^{-1}.$$

Since $\rho > 0,\ D > 0,$ therefore we have $\sigma < 1.$ Furthermore, the inequality (9) yields the boundedness of the operator Q defined by (3), where $\|Q\phi(x,t)\| \le \sigma \|\phi(x,t)\|.$

Also, the above inequality and (9) define the boundedness of the operator $\bar{Q}$.

Lemma 2

The integral operator (2), under the condition (5), is a continuous and contraction operator.

Proof

To prove the continuity of the integral operator $\bar{Q}$, we consider two functions $\phi_1(x,t),\ \phi_2(x,t) \in L_2(\Omega)\times C[0,T]$, then the formula (2) yields:

$$\|(\bar{Q}\phi_1 - \bar{Q}\phi_2)(x,t)\| = \|Q(\phi_1 - \phi_2)(x,t)\| \tag{10}$$

Using the formula (4), then applying Cauchy-Schwarz inequality and using conditions (a) and (b), we obtain

$$\|\bar{Q}\phi_1(x,t) - \bar{Q}\phi_2(x,t)\| \le \left[\left|\frac{\lambda}{\mu}\right|(B + CKT)\right]\|\phi_1(x,t) - \phi_2(x,t)\| \tag{11}$$

Hence, $\bar{Q}$ is a continuous operator in the space $L_2(\Omega)\times C(0,T)$, and under the condition (5), $\bar{Q}$ is a contraction operator.

So, from lemmas (1) and (2) and Banach Fixed Point Theorem we can decide that the operator $\bar{Q}$ has a unique fixed point which is the unique solution of integral equation (1), and theorem 1 is completely proved.

3. THE SYSTEM OF FREDHOLM INTEGRAL EQUATIONS

To represent (1) as SFIEs we divide the interval [0, *T*] as

$0 = t_0 < t_1 < \ldots < t_N = T$, and let $t = t_n,\ n = 0,1,2,\ldots,N$.

Therefore Eq. (1) reduces to **SFIEs** of the second kind, in the form:

$$\eta_n \phi_n(x) = \psi_n(x) + \lambda \int_{-a}^{a} k(|g(x) - g(y)|)\phi_n(y)dy \tag{12}$$

where

$$\eta_n = (\mu - \lambda w_n F_{n,n})$$

$$\psi_n(x) = f_n(x) + \lambda \sum_{j=0}^{n-1} w_j F_{n,j}\phi_j(x), \qquad n = 0, 1, 2, \ldots, N .$$

Equation (12) can be solved using recurrence relations.

4. THE GENERALIZE TOEPLITZ MATRIX METHOD

In this section, we present the Toeplitz matrix method to obtain numerically the solution of **SFIEs** of the second kind with a generalized kernel. The idea of this method is to obtain system of $2N+1$ **LAEs**, where $2N+1$ are the number of discretion points used. The coefficients are expressed as a sum of two matrices one of them is the Toeplitz matrix and the other is a matrix with zero elements except the first and the last rows (columns).

To discuss the solution of Eq. (12) numerically, using Toeplitz matrix method, we write the integral term as:

$$\int_{-a}^{a} k(|g(x) - g(y)|)\phi(y)dy = \sum_{n=-N}^{N} \int_{nh}^{nh+h} k(|g(x) - g(y)|)\phi(y)dy, \qquad (h = \frac{a}{N}). \tag{13}$$

Then, we approximate the integral term in the right hand side by

$$\int_{nh}^{nh+h} k(|g(x) - g(y)|)\phi(y)dy = A_n(g(x))\phi(nh) + B_n(g(x))\phi(nh+h) + R \tag{14}$$

where $A_n(g(x)),\ B_n(g(x))$ are two arbitrary functions to be determined and *R* is the error estimate *y*.

As the principle idea of the Toeplitz matrix to obtain the values of the functions $A_n(g(x)),\ B_n(g(x))$, we assume $\phi(y) = g'(y)$ and $g'(y)g(y)$ respectively, in Eq. (13), where $g'(x)$ is a monotonic increasing function . This yields a set of two equations in terms of two unknown functions where, in this case, the error is vanishing. Solving the results we have:

$$A_n(g(x)) = \frac{1}{g'(nh)(g(nh+h)-g(nh))}[g(nh+h)I(g(x)) - J(g(x))] ,$$

$$B_n(g(x)) = \frac{1}{g'(nh+h)(g(nh+h)-g(nh)}[J(g(x)) - g(nh)I(g(x))] , \tag{15}$$

where

$$I(g(x)) = \int_{nh}^{nh+h} k(|g(x)-g(y)|).g'(y)dy \quad , J(g(x)) = \int_{nh}^{nh+h} k(|g(x)-g(y)|).g'(y).g(y)dy \tag{16}$$

Using (13) and the following notations, for $x = mh$,

$$\phi(x) = \phi(mh) = \phi_m \quad , D_n(g(x)) = D_n(g(mh)) \; = D_{nm} \quad , \quad \psi(x) = \psi(mh) = \psi_m , \tag{17}$$

We get the following system of **LAEs**

$$\eta\phi_m - \lambda \sum_{\text{n=-N}}^{\text{N}} D_{\text{n,m}}\phi_n = \psi_m \; , \tag{18}$$

where

$$D_{n,m} = \begin{cases} A_{-N}(g(mh)) , & n = \text{-}N \\ \text{A}_n(g(mh)) + B_{n-1}(g(mh)), & \text{-}N < n < N \\ \text{B}_{N\text{-}1}(g(mh)), & n = N \quad . \end{cases} \tag{19}$$

The matrix $D_{n,m}$ may be written as $D_{n,m} = G_{n,m} - E_{n,m}$, where

$$G_{n,m} = A_n(g(mh)) + B_{n-1}(g(mh)), \qquad \text{-}N \le m, n \le N \tag{20}$$

is a Toeplitz matrix of order $2N+1$ and

$$E_{mn} = \begin{cases} B_{-N-1}(g(mh)) , & n = \text{-}N \\ 0 \; , & \text{-}N < n < N \\ \text{A}_N(g(mh)) , & n = N \end{cases} \tag{21}$$

represents a matrix of order $2N+1$ whose elements are zeros except the first and the last columns (rows).
The solution of the system (19) can be obtained in the form

$$\phi(mh) = [\eta\, I - \lambda(G_{mn} - E_{mn})]^{-1}\psi(mh), \qquad |\eta I \text{-} \lambda(G_{mn} - E_{mn})| \ne 0 \; . \tag{22}$$

The error term R is determined from equation (14) by letting $\phi(y) = g'g^2$ to get,

$$R = \left| \int_{nh}^{nh+h} g'g^2 k(|g(x)-g(y)|)dy - A_n(g(x))g'(nh)g^2(nh) - B_n(g(x))g'(nh+h)g^2(nh+h) \right| . \tag{23}$$

Remarks:

(1) The method is said to be convergent of order r in [-a,a] if and only if for N sufficiently large there exists a constant D^* > 0, independent of N, such that

$$\|\phi(x) - \phi_N(x)\| \le D^* N^{-r} .$$

(2) The linear algebraic system (9) has a unique solution, under the convergence condition

$$\sup_N \left\| \sum_{n=-N}^{N} D_{n,m} \right\| \le c' \; . \qquad (\; c' \text{ is a constant })\tag{24}$$

5. THE GENERALIZED PRODUCT NYSTROM METHOD

We discuss the solution of **FIE** using the product Nyström method. For this consider [7]

$$\eta\phi(x) - \lambda\int_a^b p(g(x), g(y))\bar{k}(|g(x)-g(y)|)\phi(y)dy \; = \psi(x) \tag{25}$$

When the kernel $k(|g(x)-g(y)|)$ is singular within the range of integration, we can often factor out the singularity in $k(|g(x)-g(y)|)$ by writing

$$k(|g(x)-g(y)|) = p(g(x), g(y)\bar{k}(|g(x)-g(y)|), \tag{26}$$

Where p and $\bar{k}$ are badly behaved and well behaved functions of their arguments, respectively,$\phi(x)$is the unknown function, while $f(x)$ is a given function. Equation (25) can be written in the form.

$$\eta\phi_n(x_i)-\lambda\sum_{j=0}^{N}w_{ij}\bar{k}(|g(x_i)-g(y_j)|)\phi_n(y_j)=\psi_n(x_i), \quad i=0,1,\dots,N \tag{27}$$

where $x_i = y_i = a+ih$, $i = 0, 1, \dots, N$ with $h = (b-a)/N$, N even and w_{ij} are weights .
Following the same way of Delves and Mohamed [6], we have

$$w_{i,0}=\beta_1(y_i), \qquad w_{i,2j+1}=2\gamma_{j+1}(y_i)$$
$$w_{i,2j}=\alpha_j(y_i)+\beta_{j+1}(y_i), \qquad w_{i,N}=\alpha_{N/2}(y_i) \tag{28}$$

where

$$\alpha_j(y_i)=\frac{(g(h))^2}{g(2h)}\int_0^2\zeta(\zeta-1)p(g(y_{2j-2})+\zeta g(h),g(y_i))d\zeta,$$

$$\beta_j(y_i)=\frac{(g(h))^2}{g(2h)}\int_0^2(\zeta-1)(\zeta-2)p(g(y_{2j-2})+\zeta g(h),g(y_i))d\zeta,$$

$$\gamma_j(y_i)=\frac{(g(h))^2}{g(2h)}\int_0^2\zeta(2-\zeta)p(g(y_{2j-2})+\zeta g(h),g(y_i))d\zeta. \tag{29}$$

Here, in (29), we introduce the variable $g(y)=g(y_{2j-2})+\zeta g(h),\ 0\le\zeta\le 2$. Therefore, the system (27) has a solution:

$$\Phi=[\eta I-\lambda W]^{-1}\Psi \tag{30}$$

where I is the identity matrix, and $|\eta I-\lambda W|\neq 0$.

6. NUMERICAL RESULTS

We apply the two previous methods to solve Eq. (1) numerically with Carleman function, logarithmic form, Cauchy kernel, and Hilbert kernel using **Maple10**. The approximate solution and the error, in each case, is obtained and computed.

6.1 Application for a Generalized Carleman Kernel

Example 1: Consider the integral equation

$$\phi(x,t)=f(x,t)+\lambda\int_{-1}^{1}\left|x^4-y^4\right|^{-\nu}\phi(y,t)dy+\lambda\int_0^t\tau^2\phi(x,\tau)d\tau,\ (0\le t\le T;\ |x|\le 1)$$

at the three times $T = 0.006$, 0.03 and $T = 0.9$ with $\lambda =$ 0.31579 and 0.6666666667. Divide the position interval by $N =$ 41 units and take $0 < \upsilon < ½$; υ is called Poisson ratio, where the ratio of lateral strain and axial strain is defined as *Poisson's ratio.* In the theory of elasticity the relation between the coefficients λ, υ, E and between μ, v, E are given by

$$\lambda=\frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)},\ \mu=\frac{E}{2(1+\upsilon)}\Rightarrow E=2\mu(1+\upsilon)$$

Where E is called Young modulus and λ, μ are called Lame constants . The exact solution is $\phi(x,t)=x^5t^6$

Case1: $\lambda = 0.31579$, $\nu = 0.12$:

Table 1.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-1.00	-4.66560E-14	-4.66560E-14	1.31000E-20	-4.66568E-14	8.00000E-19
	-0.60	-3.62797E-15	-3.62797E-15	1.00400E-21	-3.62858E-15	6.10000E-19
	-0.20	-1.49299E-17	-1.49299E-17	4.32000E-24	-1.52189E-17	2.89000E-19
	0.20	1.49299E-17	1.49299E-17	4.12000E-24	1.48085E-17	1.21400E-19
	0.60	3.62797E-15	3.62797E-15	1.00500E-21	3.62794E-15	3.00000E-20
	1.00	4.66560E-14	4.66560E-14	1.29200E-20	4.66560E-14	0.0000E+00
0.03	-1.00	-7.29000E-10	-7.29004E-10	4.50500E-15	-7.29014E-10	1.40000E-14
	-0.60	-5.66870E-11	-5.66873E-11	3.50060E-16	-5.66967E-11	9.70000E-15
	-0.20	-2.33280E-13	-2.33281E-13	1.44100E-18	-2.37796E-13	4.51600E-15
	0.20	2.33280E-13	2.33281E-13	1.44100E-18	2.31384E-13	1.89600E-15
	0.60	5.66870E-11	5.66873E-11	3.49990E-16	5.77867E-11	3.00000E-16
	1.00	7.29000E-10	7.29004E-10	4.50130E-15	7.29002E-10	2.00000E-15
0.9	-1.00	-5.31441E-01	-5.21420E-01	1.00208E-02	-5.49481E-01	1.80400E-02
	-0.60	-4.13248E-02	-4.05456E-02	7.79217E-04	-4.27345E-02	1.40970E-03
	-0.20	-1.70061E-04	-1.66854E-04	3.20665E-06	-1.79442E-04	9.38100E-06
	0.20	1.70061E-04	1.66854E-04	3.20665E-06	1.74314E-04	4.25300E-06
	0.60	4.13248E-02	4.05456E-02	7.79217E-04	4.27265E-02	1.40170E-03
	1.00	5.31441E-01	5.21420E-01	1.00208E-02	5.49471E-02	1.80300E-02

Case2: λ = 0.6666666667 , ν = 0.2 :

Table 2.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-1.00	-4.66560E-14	-4.66560E-14	2.72000E-20	-4.66665E-14	1.05000E-17
	-0.60	-3.62797E-15	-3.62797E-15	2.13500E-21	-3.63295E-15	4.98000E-18
	-0.20	-1.49299E-17	-1.49299E-17	8.80000E-24	-1.66030E-17	1.67310E-18
	0.20	1.49299E-17	1.49299E-17	8.80000E-24	1.44457E-17	4.84200E-19
	0.60	3.62797E-15	3.62797E-15	2.12100E-21	3.62792E-15	5.00000E-20
	1.00	4.66560E-14	4.66560E-14	2.72600E-20	4.66565E-14	5.00000E-19
0.03	-1.00	-7.29000E-10	-7.29009E-10	9.50200E-15	-7.29166E-10	1.66000E-13
	-0.60	-5.66870E-11	-5.66877E-11	7.38720E-16	-5.67650E-11	7.80000E-14
	-0.20	-2.33280E-13	-2.33283E-13	3.03800E-18	-2.59424E-13	2.61440E-14
	0.20	2.33280E-13	2.33283E-13	3.04700E-18	2.25714E-13	7.56600E-15
	0.60	5.66870E-11	5.66877E-11	7.38900E-16	5.66865E-11	5.00000E-16
	1.00	7.29000E-10	7.29009E-10	9.50200E-15	7.29010E-10	1.00000E-14
0.9	-1.00	-5.31441E-01	-5.10132E-01	2.13082E-02	-5.72421E-01	4.09800E-02
	-0.60	-4.13248E-02	-3.96679E-02	1.65693E-03	-4.45696E-02	3.24480E-03
	-0.20	-1.70061E-0	-1.63242E-04	6.81865E-06	-2.06491E-04	3.64300E-05
	0.20	1.70061E-04	1.63242E-04	6.81865E-06	1.76365E-04	6.30400E-06
	0.60	4.13248E-02	3.96679E-02	1.65693E-03	4.44994E-02	3.17460E-03
	1.00	5.31441E-01	5.10132E-01	2.13082E-02	5.72281E-02	4.08400E-02

Example 2:

Consider the integral equation:

$$\phi(x,t) = f\,(x,t) + \lambda\int_{-1}^{1}\left|\sin(x) - \sin(y)\right|^{-\upsilon}\phi(y,t)dy\ + \lambda\int_{0}^{t}\tau^2\phi(x,\tau)d\tau$$

at T= 0.006 , 0.03 and 0.9 with λ = 0.13636 and 0.19048 , N = 41, and 0.5 < υ < 1 ,

exact solution $= t\ \sin(x)$

Here, the material is solid.

Case1: λ = 0.13636 , υ = 0.6 :

Table 3.

T	X	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	-5.04882E-03	-4.90956E-03	1.39259E-04	-4.98583E-03	6.29924E-05
	-0.60	-3.38785E-03	-3.38843E-03	5.75298E-07	-3.33112E-03	5.67273E-05
	-0.20	-1.19201E-03	-1.19259E-03	5.82948E-07	-1.17922E-03	1.27932E-05
	0.20	1.19201E-03	1.19259E-03	5.82949E-07	1.17927E-03	1.27381E-04
	0.60	3.38785E-03	3.38843E-03	5.75300E-07	3.33656E-03	5.12864E-05
	1.00	5.04882E-03	4.90956E-03	1.39259E-04	4.99343E-03	5.53948E-05
0.03	1.00-	-2.52441E-02	-2.45478E-02	6.96242E-04	-2.49291E-02	3.14962E-04
	-0.60	-1.69392E-02	-1.69421E-02	2.91543E-06	-1.66556E-02	2.83636E-04
	-0.20	-5.96007E-03	-5.96300E-03	2.92847E-06	-5.89611E-03	6.39664E-05
	0.20	5.96007E-03	5.96300E-03	2.92847E-06	5.89689E-03	6.36906E-05
	0.60	1.69392E-02	1.69421E-02	2.91543E-06	1.66828E-02	2.56432E-04
	1.00	2.52441E-02	2.45478E-02	6.96242E-04	2.49671E-02	2.76974E-04
0.9	1.00-	-7.57323E-01	-7.24387E-01	3.29365E-02	-7.47625E-01	9.69861E-03
	-0.60	-5.08178E-01	-5.00032E-01	8.14570E-03	-4.99443E-01	8.73518E-03
	-0.20	-1.78802E-01	-1.75989E-01	2.81309E-03	-1.76825E-01	1.97737E-03
	0.20	1.78802E-01	1.75989E-01	2.81309E-03	1.76852E-01	1.95034E-03
	0.60	5.08178E-01	5.00032E-01	8.14570E-03	5.00298E-01	7.87973E-03
	1.00	7.57323E-01	7.24387E-01	3.29365E-02	7.48812E-01	8.51173E-03

Case2: λ = 0.19048 , υ = 0.8 :

Table 4.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	-5.04882E-03	-4.78351E-03	2.65310E-04	-4.93225E-03	1.16569E-04
	-0.60	-3.38785E-03	-3.38870E-03	8.46200E-07	-3.28403E-03	1.03815E-04
	-0.20	-1.19201E-03	-1.19305E-03	1.03534E-06	-1.17458E-03	1.74281E-05
	0.20	1.19201E-03	1.19305E-03	1.03534E-06	1.16307E-03	2.89456E-05
	0.60	3.38785E-03	3.38870E-03	8.46199E-07	3.28510E-03	1.02748E-04
	1.00	5.04882E-03	4.78515E-03	2.65310E-04	4.93835E-03	1.10473E-04
0.03	1.00-	-2.52441E-02	-2.39176E-02	1.32647E-03	-2.46612E-02	5.82846E-04
	-0.60	-1.69392E-02	-1.69435E-02	4.28619E-06	-1.64201E-02	5.19079E-04
	-0.20	-5.96007E-03	-5.96527E-03	5.19623E-06	-5.87293E-03	8.71410E-05
	0.20	5.96007E-03	5.96527E-03	5.19622E-06	5.81535E-03	1.44728E-04
	0.60	1.69392E-02	1.69435E-02	4.28619E-06	1.64255E-02	5.13743E-04
	1.00	2.52441E-02	2.39176E-02	1.32647E-03	2.46917E-02	5.52367E-04
0.9	1.00-	-7.57323E-01	-7.00710E-01	5.66138E-02	-7.39076E-01	1.82476E-02
	-0.60	-5.08178E-01	-4.96613E-01	1.15649E-02	-4.91910E-01	1.62675E-02
	-0.20	-1.78802E-01	-1.74833E-01	3.96934E-03	-1.75956E-01	2.84548E-03
	0.20	1.78802E-01	1.74833E-01	3.96934E-03	1.74438E-01	4.36369E-03
	0.60	5.08178E-01	4.96613E-01	1.15649E-02	4.92347E-01	1.58310E-02
	1.00	7.57323E-01	7.00710E-01	5.66138E-02	7.40292E-01	1.70316E-02

6.2 Application for a Generalized Logarithmic Kernel

Example 1: For the integral equation:

$$\phi(x,t) = f\ (x,t) + \lambda \int_{-1}^{1} \ln\left|x^4 - y^4\right| \phi(y,t)dy\ + \lambda \int_{0}^{t} \tau^2 \phi(x,\tau)d\tau, \qquad (0 \le t \le T;\quad |x| \le 1)$$

With λ = 0.6666666667, 1.083333; T= 0.006, 0.03 and 0.9 and N = 41.

Exact solution $\phi(x,t) = x^5 t^6$.

Case1: λ = 0.6666666667 :

Table 5.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	-4.66560E-14	-4.66560E-14	2.72600E-20	-4.66549E-14	1.00884E-18
	-0.60	-3.62797E-15	-3.62797E-15	2.14000E-21	-3.62706E-15	9.05741E-19
	-0.20	-1.49299E-17	-1.49299E-17	9.40000E-24	-1.43729E-17	5.57014E-19
	0.20	1.49299E-17	1.49299E-17	9.00000E-24	1.52237E-17	2.93832E-19
	0.60	3.62797E-15	3.62797E-15	2.14000E-21	3.62805E-15	8.65480E-20
	1.00	4.66560E-14	4.66560E-14	2.72600E-20	4.66558E-14	1.12920E-19
0.03	1.00-	-7.29000E-10	-7.29009E-10	9.50280E-15	-7.28986E-10	1.39550E-14
	-0.60	-5.66870E-20	-5.66877E-11	7.39000E-16	-5.66730E-11	1.40117E-14
	-0.20	-2.33280E-13	-2.33283E-13	3.04900E-18	-2.24577E-13	8.70283E-15
	0.20	2.33280E-13	2.33283E-13	3.03000E-18	2.37871E-13	4.59174E-15
	0.60	5.66870E-11	5.66877E-11	7.38600E-16	5.66885E-11	1.49290E-15
	1.00	7.29000E-10	7.29009E-10	9.50280E-15	7.29000E-10	4.36000E-17
0.9	1.00-	-5.31441E-01	-5.10132E-01	2.13083E-02	-5.72260E-01	4.08194E-02
	-0.60	-4.13248E-02	-3.96679E-02	1.65693E-03	-4.44874E-02	3.16256E-03
	-0.20	-1.70061E-04	-1.63242E-04	6.81866E-06	-1.75350E-04	5.28962E-06
	0.20	1.70061E-04	1.63242E-04	6.81867E-06	1.87230E-04	1.71692E-05
	0.60	4.13248E-02	3.96679E-02	1.65693E-03	4.45012E-02	3.17642E-03
	1.00	5.31441E-01	5.10132E-01	2.13083E-02	5.72272E-01	4.08319E-02

Case2: λ = 1.083333 :

Table 6.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	-4.66560E-14	-4.66560E-14	4.42900E-20	-4.66543E-14	1.63927E-18
	-0.60	-3.62797E-15	-3.62797E-15	3.29000E-21	-3.62649E-15	1.47180E-18
	-0.20	-1.49299E-17	-1.49299E-17	7.00000E-24	-1.40248E-17	9.05115E-19
	0.20	1.49299E-17	1.49299E-17	8.00000E-24	1.54073E-17	4.77449E-19
	0.60	3.62797E-15	3.62797E-15	3.36000E-21	3.62811E-15	1.40618E-19
	1.00	4.66560E-14	4.66560E-14	4.42900E-20	4.66558E-14	1.83490E-19
0.03	1.00-	-7.29000E-10	-7.29015E-10	1.54419E-14	-7.28977E-10	2.26762E-14
	-0.60	-5.66870E-20	-5.66882E-11	1.19800E-15	-5.66642E-11	2.27687E-14
	-0.20	-2.33280E-13	-2.33284E-13	4.88000E-18	-2.19138E-13	1.41416E-14
	0.20	2.33280E-13	2.33284E-13	4.87000E-18	2.40741E-13	7.46116E-15
	0.60	5.66870E-11	5.66882E-11	1.19980E-15	5.66894E-11	2.42562E-15
	1.00	7.29000E-10	7.29015E-10	1.54422E-14	7.29000E-10	7.08000E-17
0.9	1.00-	-5.31441E-01	-4.96514E-01	3.49263E-02	-6.04047E-01	7.26063E-02
	-0.60	-4.13248E-02	-3.86089E-02	2.71587E-03	-4.69490E-02	5.62418E-03
	-0.20	-1.70061E-04	-1.58884E-04	1.11764E-05	-1.78702E-04	8.64101E-06
	0.20	1.70061E-04	1.58884E-04	1.11764E-05	2.01005E-04	3.09447E-05
	0.60	4.13248E-02	3.86089E-02	2.71587E-03	4.69750E-02	5.65019E-03
	1.00	5.31441E-01	4.96514E-01	3.49263E-02	6.04070E-01	7.26298E-02

Example 2:

For the integral equation:

$$\phi(x,t) = f(x,t) + \lambda\int_{-1}^{1} \ln\left|e^{x^2} - e^{y^2}\right|\phi(y,t)dy + \lambda\int_{0}^{t} \tau^2\phi(x,\tau)d\tau, \qquad (0 \le t \le T)$$

With λ = 0.001 , 0.01,and the time T=0.006 , 0.03 and 0.9 for N = 41.

exact solution is $\phi(x,t)= e^{x^5}t^3$.

Case1: λ = 0.001 :

Table 7.

T	X	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	7.94619E-08	4.62457E-08	3.32161E-08	4.62457E-08	3.32162E-08
	-0.60	1.99840E-07	2.06637E-07	6.79699E-09	2.06637E-07	6.79700E-09
	-0.20	2.15930E-07	2.19450E-07	3.51941E-09	2.19450E-07	3.52000E-09
	0.20	2.16069E-07	2.17678E-07	1.60962E-09	2.17678E-07	1.60900E-09
	0.60	2.33466E-07	2.34863E-07	1.39742E-09	2.34863E-07	1.39700E-09
	1.00	5.87148E-07	5.71855E-07	1.52928E-08	5.71855E-07	1.52930E-08
0.03	1.00-	9.93274E-06	5.78072E-06	4.15202E-06	5.78072E-06	4.15201E-06
	-0.60	2.49800E-05	2.58296E-05	8.49624E-07	2.58296E-05	8.49600E-07
	-0.20	2.69913E-05	2.74312E-05	4.39927E-07	2.74312E-05	4.39900E-07
	0.20	2.70086E-05	2.72098E-05	2.01203E-07	2.72098E-05	2.01200E-07
	0.60	2.91833E-05	2.93579E-05	1.74678E-07	2.93579E-05	1.74600E-07
	1.00	7.33936E-05	7.14820E-05	1.91160E-06	7.14819E-05	1.91160E-06
0.9	1.00-	7.94619E-02	4.62443E-02	3.32176E-02	2.42643E-02	5.51976E-02
	-0.60	1.99840E-01	2.06632E-01	6.79220E-03	2.04953E-01	5.11330E-03
	-0.20	2.15930E-01	2.19445E-01	3.51429E-03	2.19458E-01	3.52842E-03
	0.20	2.16069E-01	2.17673E-01	1.60452E-03	2.17690E-01	1.62184E-03
	0.60	2.33466E-01	2.34858E-01	1.39190E-03	2.36573E-01	3.10710E-03
	1.00	5.87148E-01	5.71842E-01	1.53065E-03	5.93855E-01	6.70700E-03

Case2: λ =0.01 :

Table 8.

T	X	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	7.94195E-08	1.28261E-09	7.81793E-08	1.28261E-09	7.81368E-08
	-0.60	1.99840E-07	2.70596E-07	7.07566E-08	2.70596E-07	7.07560E-08
	-0.20	2.15930E-07	2.55163E-07	3.92327E-08	2.55163E-07	3.92330E-08
	0.20	2.16069E-07	2.33440E-07	1.73710E-08	2.33440E-07	1.73710E-08
	0.60	2.33466E-07	2.46392E-07	1.29258E-08	2.46392E-07	1.29260E-08
	1.00	5.87148E-07	5.50372E-07	3.67759E-08	5.50372E-07	3.67760E-08
0.03	1.00-	9.93274E-06	1.60324E-07	9.77242E-06	1.60324E-07	9.77241E-06
	-0.60	2.49800E-05	3.38246E-05	8.84458E-06	3.38246E-05	8.84460E-06
	-0.20	2.69913E-05	3.18954E-05	4.90409E-06	3.18954E-05	4.90410E-06
	0.20	2.70086E-05	2.91800E-05	2.17138E-06	2.91800E-05	2.17140E-06
	0.60	2.91833E-05	3.07990E-05	1.61573E-06	3.07990E-05	1.61570E-06
	1.00	7.33936E-05	6.87966E-05	4.59697E-06	6.87965E-05	4.59700E-06
0.9	1.00-	7.94619E-02	1.26764E-03	7.81943E-02	6.07623E-02	1.86995E-02
	-0.60	1.99840E-01	2.70542E-01	7.07026E-02	2.65811E-01	6.59710E-02
	-0.20	2.15930E-01	2.55110E-01	3.91798E-02	2.55152E-01	3.92223E-02
	0.20	2.16069E-01	2.33389E-01	1.73200E-02	2.33441E-01	1.73726E-02
	0.60	2.33466E-01	2.46336E-01	1.28700E-02	2.51177E-01	1.77118E-02
	1.00	5.87148E-01	5.50249E-01	3.68991E-02	6.12379E-01	2.52310E-02

6.3 Application for a Generalized Cauchy Kernel

Example 1: For the integral equation:

$$\phi(x,t) = f(x,t) + \lambda\int_{-1}^{1}\frac{1}{(x^2-y^2)}\phi(y,t)dy + \lambda\int_{0}^{t}\tau^2\phi(x,\tau)d\tau, \quad (0 \le t \le T; \quad |x| \le 1)$$

λ = 1.5, 1.941176, and the time T= 0.006, 0.03 and 0.9 for N = 41.

exact solution $\phi(x,t) = x^5 t^6$

Case1: λ =1.5 :

Table 9.

T	X	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-1.00	-4.66560E-14	-4.66560E-14	6.29700E-20	-4.66568E-14	8.00000E-19
	-0.60	-3.62797E-15	-3.62798E-15	1.38400E-20	-3.62859E-15	6.20000E-19
	-0.20	-1.49299E-17	-1.49850E-17	5.50800E-20	-1.52740E-17	3.44100E-19
	0.20	1.49299E-17	1.49860E-17	5.60800E-20	1.48645E-17	6.53200E-20
	0.60	3.62797E-15	3.62798E-15	1.41400E-20	3.62795E-15	2.00000E-20
	1.00	4.66560E-14	4.66560E-14	6.31600E-20	4.66560E-14	0.0000E+00
0.03	-1.00	-7.29000E-10	-7.29021E-10	2.14073E-14	-7.29031E-10	3.10000E-14
	-0.60	-5.66870E-20	-5.66888E-11	1.80400E-15	-5.66982E-11	5.66981E-11
	-0.20	-2.33280E-13	-2.34130E-13	8.50000E-16	-2.38645E-13	5.36500E-15
	0.20	2.33280E-13	2.34160E-13	8.80000E-16	2.32263E-13	1.01700E-15
	0.60	5.66870E-11	5.66888E-11	1.80800E-15	5.66882E-11	1.20000E-16
	1.00	7.29000E-10	7.29021E-10	2.14087E-14	7.29019E-10	1.90000E-14
0.9	-1.00	-5.31441E-01	-4.82657E-01	4.87830E-02	-5.10718E-01	2.07230E-02
	-0.60	-4.13248E-02	-3.75314E-02	3.79336E-03	-3.97235E-02	1.60450E-03
	-0.20	-1.70061E-04	-1.54440E-04	1.56211E-05	-1.67028E-04	3.03300E-06
	0.20	1.70061E-04	1.54450E-04	1.56111E-05	1.61910E-04	8.15100E-06
	0.60	4.13248E-02	3.75314E-02	3.79336E-03	3.97123E-02	1.61250E-04
	1.00	5.31441E-01	4.82657E-01	4.87830E-02	5.10708E-01	2.07330E-02

Case2: λ =1.941176 :

Table 10.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-1.00	-4.66560E-14	-4.66560E-14	8.04100E-20	-4.66665E-14	1.05000E-17
	-0.60	-3.62797E-15	-3.62798E-15	1.14400E-20	-3.63296E-14	4.99000E-18
	-0.20	-1.49299E-17	-1.49624E-17	3.24800E-20	-1.66355E-17	1.70560E-18
	0.20	1.49299E-17	1.49626E-17	3.26800E-20	1.44784E-17	4.51500E-19
	0.60	3.62797E-15	3.62798E-15	1.16400E-20	3.62793E-15	4.00000E-20
	1.00	4.66560E-14	4.66560E-14	8.04900E-20	4.66565E-14	5.00000E-19
0.03	-1.00	-7.29000E-10	-7.29027E-10	2.76958E-14	-2.03656E-13	1.57000E-13
	-0.60	-5.66870E-20	-5.66893E-11	2.28400E-15	-8.09279E-14	7.73000E-14
	-0.20	-2.33280E-13	-2.34090E-13	8.10000E-16	-2.61559E-14	2.61410E-14
	0.20	2.33280E-13	2.34110E-13	8.30000E-16	-7.55403E-15	7.56896E-14
	0.60	5.66870E-11	5.66893E-11	2.28600E-15	2.42789E-15	1.19999E-15
	1.00	7.29000E-10	7.29027E-10	2.76965E-14	4.76560E-14	1.00000E-15
0.9	-1.00	-5.31441E-01	-4.67719E-01	6.37214E-02	-3.60372E-01	1.71069E-01
	-0.60	-4.13248E-02	-3.63698E-02	4.95497E-03	-2.81959E-02	1.31289E-02
	-0.20	-1.70061E-04	-1.49731E-04	2.03301E-05	-2.32025E-04	6.19648E-05
	0.20	1.70061E-04	1.49746E-04	2.03151E-05	5.37920E-05	1.16269E-04
	0.60	4.13248E-02	3.63698E-02	4.95497E-03	2.79879E-02	1.33369E-02
	1.00	5.31441E-01	4.67719E-01	6.37214E-02	3.60184E-01	1.71257E-01

Example 2: Solve the integral equation:

$$\phi(x,t) = f(x,t) + \lambda\int_{-1}^{1}\frac{1}{(e^x - e^y)}\phi(y,t)dy + \lambda\int_0^t \tau^2\phi(x,\tau)d\tau$$

the values of $\mu = 1$, at the times $t\in[0 , 0.006]$, $t\in[0 , 0.03]$, $t\in[0 , 0.6]$ with $\lambda =$ 0.001 , 0.004, and we divided the position interval by $N = 41$.

Exact solution $\phi(x,t) = e^x t^3$

case1: λ = 0.001 :

Table 11.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-1.00	7.94619E-08	4.66819E-08	3.27800E-08	4.66819E-08	3.27800E-08
	-0.60	1.18543E-07	1.14672E-07	3.87127E-09	1.14672E-07	3.87100E-09
	-0.20	1.76845E-07	1.72844E-07	4.00183E-09	1.72844E-07	4.00100E-09
	0.20	2.63822E-07	2.58041E-07	5.78116E-09	2.58041E-07	5.78100E-09
	0.60	3.93577E-07	3.81497E-07	1.20800E-08	3.81497E-07	1.20800E-08
	1.00	5.87148E-07	1.05022E-06	4.63072E-07	1.05022E-06	4.63072E-07
0.03	-1.00	9.93274E-06	5.83524E-06	4.09750E-06	5.83523E-06	4.09750E-06
	-0.60	1.48179E-05	1.43340E-05	4.83908E-07	1.43339E-05	4.83900E-07
	-0.20	2.21057E-05	2.16055E-05	5.00229E-07	2.16055E-05	5.00200E-07
	0.20	3.29778E-05	3.22552E-05	7.22645E-07	3.22552E-05	7.22600E-07
	0.60	4.91972E-05	4.76872E-05	1.51000E-06	4.76872E-05	1.51000E-06
	1.00	7.33946E-05	1.31277E-04	5.78841E-05	1.31277E-04	5.78824E-05
0.6	-1.00	7.94619E-02	4.66804E-02	3.27814E-02	2.46574E-02	5.48045E-02
	-0.60	1.18543E-01	1.14669E-01	3.87413E-03	1.12952E-01	5.59050E-03
	-0.20	1.76845E-01	1.72839E-01	4.00607E-03	1.72829E-01	4.01527E-03
	0.20	2.63822E-01	2.58035E-01	5.78743E-03	2.58041E-01	5.78094E-03
	0.60	3.93577E-01	3.81488E-01	1.20893E-02	3.83199E-01	1.03772E-02
	1.00	5.87148E-01	1.05020E+00	4.63054E-01	1.07221E+00	4.85069E-01

Case2: $\lambda = 0.004$:

Table 12.

T	x	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	1.00-	7.94619E-08	7.26675E-09	7.21952E-08	7.26674E-09	7.21951E-08
	-0.60	1.18543E-07	1.01139E-07	1.74035E-08	1.01139E-07	1.74040E-08
	-0.20	1.76845E-07	1.59947E-07	1.68981E-08	1.59947E-07	1.68980E-08
	0.20	2.63822E-07	2.41259E-07	2.25632E-08	2.41259E-07	2.25630E-08
	0.60	3.93577E-07	3.89513E-07	4.06402E-09	3.89513E-07	4.06400E-09
	1.00	5.87148E-07	2.94642E-06	2.35928E-06	2.94642E-06	2.35927E-06
0.03	1.00-	9.93274E-06	9.08344E-07	9.02440E-06	9.08343E-07	9.02439E-06
	-0.60	1.48179E-05	1.26424E-05	2.17543E-06	1.26423E-05	2.17550E-06
	-0.20	2.21057E-05	1.99934E-05	2.11226E-06	1.99934E-05	2.11230E-06
	0.20	3.29778E-05	3.01574E-05	2.82040E-06	3.01574E-05	2.82040E-06
	0.60	4.91972E-05	4.86892E-05	5.07997E-07	4.86892E-05	5.08000E-07
	1.00	7.33936E-05	3.68303E-04	2.94910E-04	3.68303E-04	2.94909E-04
0.6	1.00-	7.94619E-02	7.26494E-03	7.21970E-02	2.07960E-02	5.86658E-02
	-0.60	1.18543E-01	1.01127E-01	1.74157E-02	9.89349E-02	1.96081E-02
	-0.20	1.76845E-01	1.59929E-01	1.69158E-02	1.59916E-01	1.69285E-02
	0.20	2.63822E-01	2.41234E-01	2.25888E-02	2.41241E-01	2.25805E-02
	0.60	3.93577E-01	3.89483E-01	4.09384E-03	3.91663E-01	1.91310E-03
	1.00	5.87148E-01	2.94627E+00	2.35912E+00	2.97432E+00	2.38717E+00

6.4 Application for a Generalized Hilbert Kernel

Example: For the integral equation:

$$\phi(x,t) = f(x,t) + \lambda\int_{-\pi}^{\pi} \cot\left(\frac{y^3 - x^3}{2}\right)\phi(y,t)\,dy + \lambda\int_0^t \tau^2\phi(x,\tau)\,d\tau.$$

λ =0.001, 0.006 and T= 0.006, 0.03 and 0.9 for N = 21. exact solution $\phi(x,t) = t^6(\sin(x))^5$

case1: $\lambda = 0.001$

Table 13.

T	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-3.27341E-15	-3.81547E-15	5.42060E-16	-3.81227E-15	5.38863E-16
	-3.63027E-14	-3.61399E-14	1.62736E-16	-3.61378E-14	1.64888E-16
	0.00000E+00	-1.73115E-16	1.73115E-16	-1.71872E-16	1.71872E-16
	3.63027E-14	3.54468E-14	8.55862E-16	3.54474E-14	8.55220E-16
	3.27341E-15	2.64486E-15	6.28550E-16	2.64476E-15	6.28644E-16
0.03	-5.11470E-11	-5.95598E-11	8.41280E-12	-5.95668E-11	8.41974E-12
	-5.67229E-10	-5.64648E-10	2.58007E-12	-5.64653E-10	2.57638E-12
	0.00000E+00	-2.68358E-12	2.68358E-12	-2.68551E-12	2.68551E-12
	5.67229E-10	5.53867E-10	1.33612E-11	5.53867E-10	1.33628E-11
	5.11470E-11	4.13245E-11	9.82250E-12	4.13245E-11	9.82257E-12
0.9	-3.72862E-02	-3.62098E-02	1.07640E-03	-4.34298E-02	6.14360E-03
	-4.13510E-01	-4.11444E-01	2.06566E-03	-4.11673E-01	1.83666E-03
	0.00000E+00	-1.95714E-03	1.95714E-03	-1.95866E-03	1.95866E-03
	4.13510E-01	4.03582E-01	9.92787E-03	4.03807E-01	9.70284E-03
	3.72862E-02	2.29117E-02	1.43745E-02	3.01267E-02	7.15948E-03

Case2: $\lambda = 0.006$:

Table 14.

T	$\phi(x)$	ϕ^T	E^T	ϕ^N	E^N
0.006	-3.27341E-15	-9.46393E-16	2.32701E-15	-9.45593E-16	2.382E-15
	-3.63027E-14	-1.87550E-14	1.75476E-14	-1.87546E-14	1.780E-14
	0.00000E+00	1.69611E-14	1.69611E-14	1.69613E-14	1.613E-14
	3.63027E-14	4.99713E-14	1.36686E-14	4.99714E-14	1.387E-14
	3.27341E-15	3.08061E-14	2.75326E-14	3.08060E-14	2.726E-14
0.03	-5.11470E-11	-1.47858E-11	3.63612E-11	-1.47748E-11	3.622E-11
	-5.67229E-10	-2.93047E-10	2.74181E-10	-2.93041E-10	2.788E-10
	0.00000E+00	2.65018E-10	2.65018E-10	2.65021E-10	2.621E-10
	5.67229E-10	7.80831E-10	2.13602E-10	7.80833E-10	2.173E-10
	5.11470E-11	4.81343E-10	4.30196E-10	4.81344E-10	4.397E-10
0.9	-3.72862E-02	-1.96937E-02	1.75925E-02	-1.04917E-02	2.644E-02
	-4.13510E-01	-2.14092E-01	1.99417E-02	-2.13800E-01	1.909E-02
	0.00000E+00	1.92723E-01	1.92723E-02	1.92726E-01	1.926E02-
	4.13510E-01	5.62802E-01	1.55270E-02	5.68494E-01	1.543E-02
	3.72862E-02	3.58731E-01	3.21444E-02	3.49538E-01	3.152E-02

7. CONCLUSION

1- For fixed values of λ and υ, the error is increasing when the time T is increasing.

2- For fixed value of T, the error is increasing when the values of λ and υ are increasing.

3-In the Carleman function, for fixed values of N and λ, the error increases when value of υ is increasing (In the theory of elasticity υ is called Poisson's ratio).

4-The error using the Toeplitz matrix method, is less than the error using the product Nyström method.

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