# ADOMIAN AND BLOCK-BY-BLOCK METHODS TO SOLVE NONLINEAR TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATION

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**ABSTRACT**

In this paper, the existence of a unique solution of a nonlinear two-dimensional Volterra integral equation (**NT-DVIE**) with continuous kernel is discussed. Adomian Decomposition Method (**ADM**) and Block by block method (**B by BM**) are used to solve this type of **NT**–**DVIE.** Numerical examples are considered to illustrate the effectiveness of the proposed methods and the error is estimated.

**Keywords**: *Two-dimensional Volterra integral equation; Adomian method; Block-by-block method.*
**AMS**: 45B05**,** 45G10

## 1. INTRODUCTION

There are many well-written texts on the theory and applications of integral equations in different sciences. From 1960 to the present day, many new numerical methods have been developed for the solution of many types of integral equations, such as the Toeplitz matrix method, the product Nyström method, the Galerkin method; Runge-Kutta method and **B by BM** (see Linz [1], Baker et al. [2], and Delves and Mohamed [3]). More information for some numerical methods can be found especially in Delves and Mohamed [3], Atkinson [4, 5] and Golberg [6]. In the references [7-9], **ADM and B by BM** was used to solve the integral equation in one dimensional. In [10], the authors solved the **TD**-**NIE** of the second kind using degenerate kernel method. Guoqiang et al., in [11], obtained numerically the solution of two-dimensional nonlinear Volterra integral equation by collocation and iteration collocation methods. In [12], Guoqiang and Jiong analyzed the existence of asymptotic error expansion of the Nyström solution for two-dimensional nonlinear Fredholm integral equation of the second kind. In [13], Abdou obtained, using separation variables method, the solution of the linear **F-VIE** in one, two and three dimensional. In this paper, we use **B by BM** and **ADM** to discuss numerically the solution of the **NT-DVIE** of the second kind with continuous kernel of the form

$$\mu u(x,y) = f(x,y) + \lambda \int_0^x \int_0^y k(x,y,t,s)\gamma(t,s,u(t,s))dtds \quad . \tag{1}$$

Here, $u(t,s)$ is the unknown function, will be determined, the two analytic functions $f(x,y)$ and $k(x,y,t,s)$ are given and defined, respectively, on the following domains: $D=[0,X]\times[0,Y]$, and $E=\{(x,y,t,s)\gamma(t,s,u)$ : $:0\le t\le x\le X,\ 0\le s\le y\le Y,\ -\infty\le u\le\infty$. Also, $\gamma$(t, s, u(t, s)) is known function; $\mu$ and $\lambda$ are constants that have many physical meanings.

## 2. EXISTENCE OF A UNIQUE SOLUTION OF NT-DVIE

To prove the existence of a unique solution of Eq. (1) we assume the following conditions:

1- The given continuous function $f(x,y)$ in $0\le x\le X$ , $0\le y\le Y<\infty$, such that $\|f(x,y)\| = \max_{x,y\in D}|f(x,y)| \le A$

2- The kernel $k(x,y,t,s)$ satisfies the condition $|k(x, y, t, s)| \le N$, N is a constant.

3-The nonlinear term $\gamma(t,s,u(t,s))$, $0\le t\le X$ , $0\le s\le Y<\infty$, satisfies for the constants $B_1$ , $B_2$ the following conditions:

$$i - \ \|\gamma(t,s,u(t,s))\| \le B_1 \|u(x,y)\|$$

$$ii - \ |\gamma(x,y,u_1(x,y)) - \gamma(x,y,u_2(x,y))| \le B_2 |u_1(x,y) - u_2(x,y)| \ ,$$

Where $\|u(x,y)\| = \max_{x,y\in D} |u(x,y)|$ .

**Theorem 1:** The solution of the **NT-DVIE** (1) exist and unique, under the condition:

$$B|\lambda| < \frac{|\mu|}{N} \qquad \text{where, } B > B_1,\ B > B_2 \tag{2}$$

To proof the theorem, we must state the following lemmas:
**Lemma 1:** Beside the conditions (1-3), the infinite series $\sum_{i=0}^{\infty} \theta_i(x,y)$ is uniformly convergent to a continuous solution $u(x,y)$.
**Proof:** We construct the sequence of the function $u_n(x,y)$, as

$$\mu u_n(x,y) = f(x,y) + \lambda \int_0^x\int_0^y k(x,y,t,s)\gamma(t,s,u_{n-1}(t,s))dtds, \quad n = 1, 2, \ldots \tag{3}$$

$$u_0(x,y) = f(x,y) \tag{4}$$

It is convenient to introduce

$$\theta_n(x,y) = u_n(x,y) - u_{n-1}(x,y), \tag{5}$$

$$u_n(x,y) = \sum_{i=0}^{n} \theta_i(x,y), \quad \theta_0(x,y) = f(x,y) \tag{6}$$

Using the properties of the modulus, and condition (3-ii) the relation (5) takes the form:

$$|\theta_n(x,y)| \le \left|\frac{\lambda}{\mu}\right| B \int_0^x\int_0^y |k(x,y,t,s)||u_{n-1}(t,s) - u_{n-2}(t,s)| dtds. \tag{7}$$

Let $n=1$ in relation (7), and using the condition (1), we get $\|\theta_1\| \le \alpha A$, so we obtain

$$\|\theta_n\| \le \alpha^n A, \quad n = 0, 1, 2, \ldots \tag{8}$$

Finally, we get

$$\|\theta_n\| \le \alpha \|\theta_{n-1}\|, \quad \text{where } (\alpha = \frac{1}{|\mu|} N\{|\lambda| B\} < 1) \tag{9}$$

This bound makes the sequence $\{\theta_n\}$ converges, and hence we can write

$$u(x,y) = \sum_{i=0}^{\infty} \theta_i(x,y) \tag{10}$$

so, the infinite series (10) is uniformly convergent.

**Lemma 2:** A continuous function $u(x,y)$ of Eq. (10) represents a solution of (1).
**Proof:** For this we must prove that $u(x,y)$ satisfies Eq. (1), so we set

$$u(x,y) = u_n(x,y) + g_n(x,y), \quad \text{where } g_n(x,y) \to 0 \text{ as } n \to \infty.$$

Then, we get,

$$u(x,y) - g_n(x,y) = \frac{1}{\mu} f(x,y) + \frac{\lambda}{\mu} \int_0^x\int_0^y k(x,y,t,s)(\gamma(t,s,u(t,s) - g_{n-1}(t,s))dtds$$

Therefore, using the conditions (2) and (3-ii) we have

$$\left\| u(x,y) - \frac{1}{\mu} f(x,y) - \frac{\lambda}{\mu} \int_0^x\int_0^y k(x,y,t,s)\gamma(t,s,u(t,s))dtds \right\| \le \|g_n(x,y)\| - \alpha \|g_{n-1}(t,s)\| \tag{11}$$

If n is large enough, the **R.H.S**. of (11) can be made as small as desired. Thus, $u(x,y)$ satisfies Eq. (1). Now, to show that $u(x,y)$ is the only solution, we assume that $\bar{u}(x,y)$ is also a another continuous solution of Eq. (1), hence we have:

$$|u(x,y) - \bar{u}(x,y)| \le \left|\frac{\lambda}{\mu}\right| \int_0^x\int_0^y |k(x,y,t,s)||\gamma(t,s,u(t,s)) - \gamma(t,s,\bar{u}(t,s))| dtds \tag{12}$$

Using conditions (3-ii), Eq. (12) leads to

$$\|u(x,y) - \bar{u}(x,y)\| \le \alpha \|u(t,s) - \bar{u}(t,s)\|, \quad \text{where } \alpha = \frac{N}{|\mu|}\{|\lambda| B\} < 1. \tag{13}$$

Since $\alpha < 1$, then the inequality (13) is true only if $u(x,y) = \bar{u}(x,y)$, that is the solution of (1) is unique.

## 3. ADOMIAN DECOMPOSITION METHOD

In this section, **ADM** is used to solve **NT-DVIE** of the second kind of type (1), let $\mu=1$.
In this method, the unknown solution $u(x,y)$ is assumed to be an infinite series of the

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y) \tag{14}$$

And the nonlinear term $\gamma(t,s,u)$ in Eq. (1) is decomposed into an infinite series

$$\gamma(t,s,u)=\sum_{n=0}^{\infty}A_n \tag{15}$$

where $A_n$ is Adomian's polynomial, which is defined from the following equation:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}[\gamma(\sum_{i=0}^{\infty}\lambda^i u_i)]_{\lambda=0}, \qquad n=0,1,2,\ldots \tag{16}$$

Another formula of Adomian polynomial see [7] is given by:

$$A_n=\gamma(S_n)-\sum_{i=0}^{n-1}A_i \tag{17}$$

Substitute from Eqs. (15) and (16) into Eq. (1), we get

$$u_0(x,y)=f(x,y) \tag{18}$$

$$u_i(x,y)=\int_0^x\int_0^y k(x,y,t,s)A_{i-1}(u_0(t,s),\ldots,u_{i-1}(t,s))dtds \ , \ i\geq 1 \tag{19}$$

## 4. BLOCK BY BLOCK METHOD

**B by BM** is a generalized of the well-known implicit Runge-Kutta method for ordinary differential equations and one finds the latter term also used in connection with integral equations. We suppose that Eq. (1) has a unique solution. The idea behind **B by BM** is quite general, but is most easily understood by considering a specific case. Use the Simpson's rule as a numerical integration formula, we get

$$U_2(x,y)\approx u(x_2,y_2)=f(x_2,y_2)+\int_0^x\int_0^y k(x_2,y_2,t,s)\gamma(t,s,u_2(t,s))dtds. \tag{20}$$

Also, we can write the equation (20) in the form

$$U_2(x,y)\approx u(x_2,y_2)=f(x_2,y_2)+\int_0^x\int_0^y k(x_2,y_2,t,s,u(t,s))dtds. \tag{21}$$

Approximating the integrals by Simpson's rule, if we knew $U_1(x,y)$, then we could compute $U_2(x,y)$ by

$$U_2=f(x_2,y_2)+\frac{h}{3}\{k(x_2,y_2,x_0,y_0,U_0)+4k(x_2,y_2,x_1,y_1,U_1)+k(x_2,y_2,x_2,y_2,U_2)\} \tag{22}$$

where, $U_0=f(x_0,y_0)$. Now we have

$$U_1(x,y)\approx u(x_1,y_1)=f(x_1,y_1)+\int_0^x\int_0^y k(x_1,y_1,t,s,u_1(t,s))dtds \tag{23}$$

To evaluate the integrals on the right sides, we introduce another point $x_{1/2}=h/2$ and the corresponding value $U_{1/2}$ and use the Simpson's rule with step size $h/2$, then

$$U_1=f(x_1,y_1)+\frac{h}{6}\{k(x_1,y_1,x_0,y_0,U_0)+4k(x_1,y_1,x_{1/2},y_{1/2},U_{1/2})+k(x_1,y_1,x_1,y_1,U_1)\} \tag{24}$$

where $U_{1/2}$ have unknown values, that can be estimated by Lagrange interpolation points $x_0,y_0,x_1,y_1,x_2,y_2$.
Therefore we obtain:

$$U_{1/2} = \frac{3}{8}U_0 + \frac{3}{4}U_1 - \frac{1}{8}U_2 \tag{25}$$

Substituting from (25) into (24), we obtain:

$$U_1 = f(x_1, y_1) + \frac{h}{6}\{k(x_1, y_1, x_0, y_0, U_0) + 4k(x_1, y_1, x_{1/2}, y_{1/2}, \frac{3}{8}U_0 + \frac{3}{4}U_1 - \frac{1}{8}U_2) + k(x_1, y_1, x_1, y_1, U_1)\} \tag{26}$$

In Eq. (1) for x ϵ [0, a]; y ϵ [0, b], we divide $0 = x_0 < x_1 < \cdots < x_N = a$ , $0 = y_0 < y_1 < \cdots < y_N = b$ be a partition of $[0, a], [0, b]$ with the step size$h$ , such that $x_i = ih$ , $y_i = jh$ for $= 0,1, \ldots, N$ . Then we can construct **B by BM**, by setting $x = x_{2m+1}$ , $y = y_{2m+1}$ to get

$$U_{2m+1}(x, y) = f(x_{2m+1}, y_{2m+1}) + \int_0^{x_{2m}}\int_0^{y_{2m}} k(x_{2m+1}, y_{2m+1}, t, s, u)dtds + \int_{x_{2m}}^{x_{2m+1}}\int_{y_{2m}}^{y_{2m+1}} \mathrm{k}(\mathrm{x}_{2m+1}, y_{2m+1}, t, s, u)dtds \tag{27}$$

Now, integration over $[0, x_{2m}] \times [0, y_{2m}]$ can be accomplished by Simpson's rule and the integral over $[x_{2m}, x_{2m+1}] \times [y_{2m}, y_{2m+1}]$ is computed by using a quadratic interpolation, so if

$U_0 = f(x_0, y_0)$

We have,

$$\begin{aligned} U_{2m+1}(x, y) = {} & f(x_{2m+1}, y_{2m+1}) + \frac{h}{3}[k(x_{2m+1}, y_{2m+1}, x_0, y_0, U_0) + 4k(x_{2m+1}, y_{2m+1}, x_1, y_1, U_1) \\ & + \mathrm{k}(\mathrm{x}_{2m+1}, y_{2m+1}, x_{2m}, y_{2m}, U_{2m})] + \frac{h}{6}k(x_{2m+1}, y_{2m+1}, x_{2m}, y_{2m}, U_{2m}) \\ & + \mathrm{k}\frac{2h}{3}(\mathrm{x}_{2m+1}, y_{2m+1}, x_{2m+\frac{1}{2}}, y_{2m+\frac{1}{2}}, \frac{3}{8}U_{2m} + \frac{3}{4}U_{2m+1} - \frac{1}{8}U_{2m+2}) + \frac{h}{6}k(x_{2m+1}, y_{2m+1}, x_{2m+1}, y_{2m+1}, U_{2m+1}) \end{aligned}$$

Also, in a similar manner we have

$$U_{2m+2}(x, y) = f(x_{2m+2}, y_{2m+2}) + \int_0^{x_{2m+2}}\int_0^{y_{2m+2}} k(x_{2m+2}, y_{2m+2}, t, s, u(t, s))dtds, \tag{28}$$

$$\begin{aligned} U_{2m+2}(x, y) = {} & f(x_{2m+2}, y_{2m+2}) + \frac{h}{3}[k(x_{2m+2}, y_{2m+2}, x_0, y_0, U_0) + 4k(x_{2m+2}, y_{2m+2}, x_1, y_1, U_1) \\ & + \ldots + \mathrm{k}(\mathrm{x}_{2m+2}, y_{2m+2}, x_{2m+2}, y_{2m+2}, U_{2m+2})] \end{aligned}$$

## 5. NUMERICAL EXPERIMENTS AND DISCUSSIONS

**Exmaple1.** Consider the **NT-DVIE**

$$u(x, y) = f(x, y) + \int_0^x\int_0^y (xy)\, s^2\, (u(t, s))^K\, dtds \tag{29}$$

in which the exact solution is $u(x, y) = xy$ , if we *set K*=1, in (29), we have

$$u(x, y) = f(x, y) + \int_0^x\int_0^y (xy)s^2\, u(t, s)dtds \tag{30}$$

Which called the **LT-DVIE** of the second kind, and if we set $K \geq 2$ in (29) we obtain the nonlinear case. The errors for the linear and nonlinear cases are will be computed. Let the free term $f(x, y) = xy - 0.125x^5y^3$, for the linear case. While in the nonlinear case $f(x, y) = xy - 0.06666666667\, x^6y^4$ . We solve Eq. (29) using **ADM** and **B by-BM.** In the following Tables 1 and 2 we present the exact and the approximate numerical solutions and the corresponding errors for some points of $x, y,\ 0 \le x, y \le 1$.

i) Linear case: Numerical results by using **B by BM** and **ADM**.

*Table 1 (Linear case K=1)*

| ***x*** | ***y*** | $u_{Exact}$ | ***ADM*** | | ***B by BM*** | |
|---|---|---|---|---|---|---|
| | | | $u_{ADM}$ | $Error_{ADM}$ | $u_{BM}$ | $Error_{BM}$ |
| 0.0 | 0.0 | 0.0000000 | 0.00000000 | 0.00000E+00 | 0.00000000 | 0.00000E+00 |
| 0.1 | 0.1 | 0.0100000 | 0.01000000 | 3.01408E-21 | 0.01000000 | 1.95900E-08 |
| 0.2 | 0.2 | 0.0400000 | 0.04000000 | 4.93824E-17 | 0.04000448 | 4.48048E-06 |
| 0.3 | 0.3 | 0.0900000 | 0.09000000 | 1.44159E-14 | 0.09004059 | 4.05935E-05 |
| 0.4 | 0.4 | 0.1600000 | 0.16000000 | 8.09039E-13 | 0.16024690 | 2.46903E-04 |
| 0.5 | 0.5 | 0.2500000 | 0.25000000 | 1.83920E-11 | 0.25179211 | 1.79211E-03 |
| 0.6 | 0.6 | 0.3600000 | 0.36000000 | 2.35741E-10 | 0.46594983 | 1.05949E-01 |
| 0.7 | 0.7 | 0.4900000 | 0.49000000 | 2.01802E-09 | 0.50366165 | 1.36616E-02 |
| 0.8 | 0.8 | 0.6400000 | 0.64000001 | 1.23829E-08 | 0.67285243 | 3.28524E-02 |
| 0.9 | 0.9 | 0.8100000 | 0.81000005 | 4.99324E-08 | 0.85750594 | 4.75059E-02 |
| 1.0 | 1.0 | 1.0000000 | 1.00000000 | 4.06797E-09 | 1.08797645 | 8.79764E-02 |

ii) Nonlinear case: Numerical results by using **B by BM**, **ADM1** using formula (16) and **ADM2** using formula (17).

*Table 2 (Nonlinear case $K \geq 2$)*

| ***x*** | ***y*** | $u_{Exact}$ | ***ADM1*** | | ***ADM2*** | | ***B by BM*** | |
|---|---|---|---|---|---|---|---|---|
| | | | $u_{ADM1}$ | $Error_{ADM1}$ | $u_{ADM2}$ | $Error_{ADM2}$ | $u_{BM}$ | $Error_{BM}$ |
| 0.0 | 0.0 | 0.0000000 | 0.00000000 | 0.0000E+00 | 0.00000000 | 0.0000E+00 | 0.00000000 | 0.0000E+00 |
| 0.1 | 0.1 | 0.0100000 | 0.01000000 | 1.00000E-21 | 0.01000000 | 0.0000E+00 | 0.01000020 | 2.08300E-08 |
| 0.2 | 0.2 | 0.0400000 | 0.04000000 | 1.02400E-18 | 0.04000000 | 0.0000E+00 | 0.04000479 | 4.79368E-06 |
| 0.3 | 0.3 | 0.0900000 | 0.09000000 | 5.90497E-17 | 0.09000000 | 0.0000E+00 | 0.09004840 | 4.84010E-05 |
| 0.4 | 0.4 | 0.1600000 | 0.16000000 | 1.04871E-15 | 0.16000000 | 0.0000E+00 | 0.16032191 | 3.21912E-04 |
| 0.5 | 0.5 | 0.2500000 | 0.25000000 | 9.77325E-15 | 0.25000000 | 0.0000E+00 | 0.25221649 | 2.21649E-03 |
| 0.6 | 0.6 | 0.3600000 | 0.36000000 | 6.06693E-14 | 0.36000000 | 0.0000E+00 | 0.46767447 | 1.07674E-01 |
| 0.7 | 0.7 | 0.4900000 | 0.49000000 | 2.85734E-13 | 0.49000000 | 0.0000E+00 | 0.50907892 | 1.90789E-02 |
| 0.8 | 0.8 | 0.6400000 | 0.64000000 | 1.10986E-12 | 0.64000000 | 0.0000E+00 | 0.68732346 | 4.73234E-02 |
| 0.9 | 0.9 | 0.8100000 | 0.81000000 | 3.78891E-12 | 0.81000000 | 0.0000E+00 | 0.88997633 | 7.99763E-02 |
| 1.0 | 1.0 | 1.0000000 | 1.00000000 | 1.20300E-11 | 1.00000000 | 0.0000E+00 | 1.15301224 | 1.53012E-01 |

**Example2.** Consider the nonlinear two-dimensional **VIE**

$$u(x,y) = f(x,y) + \int_0^x \int_0^y (\text{xy})\,(u(t,s))^K\, dtds \tag{31}$$

For which the exact solution is $u(x,y) = (x\,y)/2$, if we *set K*=1, in (31), one has

$$u(x,y) = f(x,y) + \int_0^x \int_0^y (xy)\; u(t,s)dtds \tag{32}$$

In the linear, $f(x,y) = 0.5xy - 0.125x^3y^3$, and in nonlinear case, $f(x,y) = 0.5xy - 0.02777777778x^4y^4$

i) Linear case: Numerical results by using **B by BM** and **ADM**.

*Table 3 (Linear case K=1)*

| ***x*** | ***y*** | $u_{Exact}$ | ***ADM*** | | ***B by BM*** | |
|---|---|---|---|---|---|---|
| | | | $u_{ADM}$ | $Error_{ADM}$ | $u_{BM}$ | $Error_{BM}$ |
| 0.0 | 0.0 | 0.00000000 | 0.00000000 | 0.00000E+00 | 0.00000000 | 0.00000E+00 |
| 0.1 | 0.1 | 0.00500000 | 0.00500000 | 3.39051E-16 | 0.00500153 | 1.53667E-06 |
| 0.2 | 0.2 | 0.02000000 | 0.02000000 | 3.47110E-13 | 0.02007220 | 7.22007E-05 |
| 0.3 | 0.3 | 0.04500000 | 0.04500000 | 2.00113E-11 | 0.04537594 | 3.75945E-04 |
| 0.4 | 0.4 | 0.08000000 | 0.08000000 | 3.55208E-10 | 0.08120994 | 1.20994E-03 |
| 0.5 | 0.5 | 0.12500000 | 0.12500000 | 3.30049E-09 | 0.13198480 | 6.98480E-03 |
| 0.6 | 0.6 | 0.18000000 | 0.18000002 | 2.02177E-08 | 0.24004717 | 6.00471E-02 |
| 0.7 | 0.7 | 0.24500000 | 0.24500009 | 9.09427E-08 | 0.26548140 | 2.04814E-02 |
| 0.8 | 0.8 | 0.32000000 | 0.32000030 | 3.08155E-07 | 0.35516508 | 3.51650E-02 |
| 0.9 | 0.9 | 0.40500000 | 0.40500070 | 7.02844E-07 | 0.44732437 | 4.23243E-02 |
| 1.0 | 1.0 | 0.50000000 | 0.50000013 | 1.37769E-07 | 0.55943658 | 5.94365E-02 |

ii) Nonlinear case: Results using **B by BM** and **ADM1** using formulas (16) and **ADM2** using formulas( 17)

*Table 4 (Nonlinear case $K \geq 2$)*

| $x$ | $y$ | $u_{Exact}$ | ***ADM1*** | | ***ADM2*** | | ***B by BM*** | |
|---|---|---|---|---|---|---|---|---|
| | | | $u_{ADM1}$ | $Error_{ADM1}$ | $u_{ADM2}$ | $Error_{ADM2}$ | $u_{BM}$ | $Error_{BM}$ |
| 0.0 | 0.0 | 0.00000000 | 0.00000000 | 0.0000E+00 | 0.00000000 | 0.0000E+00 | 0.00000000 | 0.0000E+00 |
| 0.1 | 0.1 | 0.00500000 | 0.00500000 | 1.00000E-42 | 0.00500000 | 0.0000E+00 | 0.00500166 | 1.66081E-06 |
| 0.2 | 0.2 | 0.02000000 | 0.02000000 | 6.71092E-35 | 0.02000000 | 0.0000E+00 | 0.02008015 | 8.01514E-05 |
| 0.3 | 0.3 | 0.04500000 | 0.04500000 | 2.54205E-30 | 0.04500000 | 0.0000E+00 | 0.04546549 | 4.65490E-04 |
| 0.4 | 0.4 | 0.08000000 | 0.08000000 | 4.50544E-27 | 0.08000000 | 0.0000E+00 | 0.08170839 | 1.70839E-03 |
| 0.5 | 0.5 | 0.12500000 | 0.12500000 | 1.49244E-24 | 0.12500000 | 0.0000E+00 | 0.13387158 | 8.87158E-03 |
| 0.6 | 0.6 | 0.18000000 | 0.18000000 | 1.71377E-22 | 0.18000000 | 0.0000E+00 | 0.24574780 | 6.57478E-02 |
| 0.7 | 0.7 | 0.24500000 | 0.24500000 | 9.49792E-21 | 0.24500000 | 0.0000E+00 | 0.27936108 | 3.43610E-02 |
| 0.8 | 0.8 | 0.32000000 | 0.32000000 | 3.10154E-19 | 0.32000000 | 0.0000E+00 | 0.38621545 | 6.62154E-02 |
| 0.9 | 0.9 | 0.40500000 | 0.40500000 | 6.80401E-18 | 0.40500000 | 0.0000E+00 | 0.50825348 | 1.03253E-01 |
| 1.0 | 1.0 | 0.50000000 | 0.50000000 | 1.09454E-16 | 0.50000000 | 0.0000E+00 | 0.67317184 | 1.73171E-01 |

## 6. CONCLUSION

From the previous discussions we conclude the following:

1- In both linear and nonlinear cases, the **ADM** is more accurate for solving this type of integral equation**s**.

2- As $x$ and $y$ are increasing in interval [0,1], the errors due to **B by BM**, and **ADM** are also increasing.

3- The error when using **ADM** is smaller than the error when using **B by BM** method. So, the **ADM** is better than **B by BM** method for solving nonlinear two-dimensional **VIE** with **c**ontinuous kernel.

4- In the nonlinear case, the error resulted by using **ADM2** is smaller than the error resulted by using **ADM1**, for example at $x = 1, y = 1,$ in example1, the error value by **ADM1** is 1.20300E-11, while the error value by **ADM2** is 0.000000E+00.

## 7. REFRENCES


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