ON STABILITY OF POPULATION SYSTEM

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ABSTRACT
By using the spectral properties of population operator, we investigated the existence and asymptotic behavior of the population system.
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1. INTRODUCTION

Population evolution processes can be described by age structure models. Recently population modeling has been developed rapidly. The stability problem of the population system is of great significance both theoretically and practically. In [1, 3, 4, 5, 6, 7] the analytical expression of the critical fertility rate was obtained for first time and the necessary and sufficient condition for the stability of the population system was given. In this paper, by applying the spectral theory of linear operators and the semigroup theory in functional analysis [2], the population operator and the process of the population growth are studied. As the substance of the critical fertility rate, it is proved that the spectral set of the population operator consists of isolated eigenvalues with finite multiplicity, involving a unique real value in it. It is also proved that the population operator is an infinitesimal generator of a C_0 -semigroup, for which the asymptotic behavior is discussed. Besides, the steady solution of the population equation is obtained in the critical case of the fertility rate.

In this paper we adopt the continuous model of the population process used in [7].

Let $F(r,t)$ be the total number of the people of age less than *r* at the time *t* and $p(r,t)=\frac{\partial F}{\partial r}$ be the population density function of *t* .

The steady and continuous model of the process of population growth is described as follows:

$$\begin{aligned} &\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=\mu(r)p(r,t), \qquad 0<r<r_m \\ &p(r,0)=p_0(r) \qquad\qquad (1) \\ &p(0,t)=\beta\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr \end{aligned}$$

where r_m is the highest age attained by the individuals in the population, $[r_1,r_2]$ is the fecundity period of the females, $\mu(r)$ is the death rate, $k(r)$ is the female sex ratio function and $h(r)$ is fertility pattern of the females subject to the following normalization condition

$$\int_{r_1}^{r_2} h(r)dr=1.$$

It is assumed that the function $k(r)$ and $h(r)$ are continuous or piecewisely continuous on the interval $[0,r_m]$.

The system (1) is a distributed parameter system with the boundary condition of positive feadback. As state space we take the ordinary Hilbert space $L^2(0,r_m)$ where the inner product and norm are defined as follows

$$\langle\varphi,\psi\rangle=\int_0^{r_m}\phi(r)\overline{\psi(r)}dr, \qquad \|\phi\|=\sqrt{\langle\varphi,\psi\rangle}.$$

The population system, $p(r,t)$, can be regarded as an abstract function defined on $[0,\infty]$ and valued in $L^2(0,r_m)$. Then the equation (1) may be written as

$$\frac{dp(t)}{dt}=A\ p(t)\ ,\quad p(0)=p_0(r) \qquad (2)$$

where A is a linear operator in $L^2(0,r_m)$:

$$A=-\frac{d}{dr}-\mu(r)$$

$$D(A)=\{p\in L^2(0,r_m)\,|-\frac{dp}{dr}-\mu p\in L^2(0,r_m) \tag{3}$$

$$p(0)=\beta\int_{r_1}^{r_2} k(r)b(r)p(r)dr\}$$

A is called population operator.

2. MAIN RESULTS

For any complex number λ , set

$$F(\lambda)=\beta\int_{r_1}^{r_2} e^{-\int_0^r \mu(\tau)d\tau} k(r)h(r)dr \tag{4}$$

and let

$\sigma(A)$ denotes the spectral set of A

$p(A)$ denotes the resolvent set of A

$R(\lambda;A)$ denotes resolvent of A

$r(A)$ denotes the spectral radius of A

Theorem 1. (i) If $F(\lambda)\neq 1$, then $\lambda\in p(A)$ and the corresponding resolvent $R(\lambda;A)$ can be written as

$$R(\lambda;A)=\frac{\beta\int_{r_1}^{r_2} k(r)h(r)(V_\lambda g)(r)dr}{1-F(\lambda)}e^{-\lambda\tau-\int_0^r \mu(\tau)d\tau}+(V_\lambda g)(r)\,,$$

where

$$V_\lambda g=\int_0^r g(s)\exp\left[-\lambda(r-s)-\int_s^r \mu(\tau)d\tau\right]ds \tag{5}$$

furthermore, $R(\lambda;A)$ is a compact operator.

(ii) $\lambda\in\sigma(A)$ if and only if $F(\lambda)=1$, for any $\lambda\in\sigma(A)$, its geometrical multiplicity equal 1 and the corresponding eigenfunction $\phi(r)$ is $\exp(-\lambda r-\int_0^r \mu(\tau)d\tau)$.

Proof. The formula (5) is easily obtained. It also easy to verify that this first term on the right sides of (5) is an one-dimensional operator and the second term is a Volterra integral operator. So $R(\lambda;A)$ is compact, i.e. A has the compact resolvent.

Now let $\lambda\in\sigma(A)$ with a corresponding eigenfunction $\phi(r)$, then

$$\phi'(r)+\mu(r)\phi(r)+\lambda\phi(r)=0 \tag{6}$$

$$\phi(0)=\beta\int_{r_1}^{r_2} k(r)h(r)\phi(r)dr \tag{7}$$

From (6) it follows that $\phi(r)=Ce^{-\lambda r-\int_0^r \mu(s)ds}$, where C is nonzero constant. Then using (7) we obtain $F(\lambda)=1$. Besides, from this it can be also seen that the geometrical multiplicity of λ is equal to 1. Inversely, if $F(\lambda)=1$, then it is easy to verify directly that ϕ_λ belongs to $D(A)$ and satisfies

$$A\ \phi_\lambda = \lambda\ \phi_\lambda$$

That is to say, $-\lambda \in \sigma(A)$. This completes the proof.

Theorem 2. (i) The operator *A* has the unique real eigenvalue γ_0 with the algebraic multiplicity 1. Moreover

$$\begin{aligned} \gamma_0 &= 0 \quad \text{if} \quad F(0) = 1 \\ \gamma_0 &< 0 \quad \text{if} \quad F(0) < 1 \\ \gamma_0 &> 0 \quad \text{if} \quad F(0) > 1 \end{aligned} \tag{8}$$

(ii) The real parts of the other eigenvalues of *A* are less than γ_0.

(iii) There exists at most finitely many eigenvalues of *A* in any strip region $\{\lambda \mid -\infty < a_1 \le \operatorname{Re}\lambda < a_2 < \infty\}$ of the complex plane.

Proof. For any $r \in [0, r_m]$, we have

$$\beta k(r)h(r)e^{-\int_0^r \mu(s)ds} \ge 0$$

$$\beta r k(r)h(r)e^{-\int_0^r \mu(s)ds} \le 0\ .$$

By hypotheses, neither of the left sides of the above two inequalities is identically vanishing. It is easy to see that:

(1) $F(x)$ is strictly decreasing function of $x, -\infty < x < \infty$.

(2) $F(x)$ has one only real root γ_0 with (8).

Besides, it may shown that the eigenfunction of A^* corresponding to γ_0 is

$$\psi_{\gamma_0} = \left[\phi_{\gamma_0}(r)\right]^{-1} \int_r^{r_m} k(\tau)h(\tau)e^{-\lambda\tau - \int_0^r \mu(s)ds} d\tau\ ,$$

then we have

$$\left\langle \varphi_{\gamma_0}, \psi_{\gamma_0} \right\rangle = \int_0^{r_m} dr \int_r^{r_m} k(s)h(s)e^{-\lambda s - \int_0^s \mu(\tau)d\tau} ds$$

$$= \int_r^{r_m} rk(r)h(r)e^{-\lambda r - \int_0^r \mu(\tau)d\tau} dr \ne 0$$

From here it follows that the algebraic multiplicity of γ_0 is 1. Hence the statement (i) holds .

Let γ be any non-real eigenvalue of $A, \gamma = \alpha + i\tau\ ,\ \tau \ne 0$.

Than

$$\beta \int_{r_1}^{r_2} k(r)h(r)\exp\left[-(\alpha + i\tau)r - \int_0^r \mu(s)ds\right]dr$$

$$= \beta \int_{r_1}^{r_2} k(r)h(r)\exp\left[-(\gamma_0 r + \int_0^r \mu(s)ds\right]dr = 1$$

Therefore

$$\int_{r_1}^{r_2} k(r)h(r)\exp\left[-\alpha r - \int_0^r \mu(s)ds\right]dr =$$

$$= \int_{r_1}^{r_2} k(r)h(r)\exp\left[-\gamma_0 r - \int_0^r \mu(s)ds\right]dr$$

From here it follows

$$\int_{r_1}^{r_2} k(r)h(r)\exp\left[-\gamma_0 r - \int_0^r \mu(s)ds\right]dr \le$$

$$\le \int_{r_1}^{r_2} k(r)h(r)\exp\left[-\alpha r - \int_0^r \mu(s)ds\right]dr$$

Thus $\alpha \le \gamma_0$. If $\alpha = \gamma_0$, $\tau \ne 0$, then

$$\int_{r_1}^{r_2} k(r)h(r)\exp\left(-\gamma_0 r - \int_0^r \mu(s)ds\right)(1-\cos\tau r)dr = 0$$

But by hypotheses, there is an interval $[r_1', r_2'] \subset [r_1, r_2]$ such that $k(r)h(r) > 0, r \in [r_1', r_2']$, then we have $1-\cos\tau r = 0, r \in [r_1', r_2']$. Hence $\tau = 0$. This contradicts the assumption $\tau \ne 0$. The part (ii) is thus proved. Now we prove (iii). If the assertion (iii) is not true, then there must be infinitly many eigenvalues of *A*,

$$\{\lambda_n \mid a_n + i\tau_n = \lambda_n\}_{n=1}^{\infty} \ , \ a_1 \le a_n \le a_2$$

Then the subsequence $\{\lambda_k \mid \lambda_k = \alpha_k + i\tau_k\}_{k=1}^{v}$ can be selected such that $\alpha_n \to \alpha_0, |\tau_k| \to \infty$ and $F(\lambda_k) = 1$, i.e

$$\beta\int_{r_1}^{r_2} k(r)h(r)\exp\left[-(\alpha_k + i\tau_k)r - \int_0^r \mu(s)ds\right]dr = 1$$

Therefore

$$1 = \beta\int_{r_1}^{r_2} k(r)h(r)e^{-a_k r - \int_0^r \mu(s)ds}\cos\tau_k r\ dr =$$

$$= \beta\int_{r_1}^{r_2} k(r)h(r)e^{-\int_0^r \mu(s)ds}\left[e^{-a_k r} - e^{-a_0 r}\right]\cos\tau_k r\ dr$$

$$+\beta\int_{r_1}^{r_2} k(r)h(r)e^{-a_0 r - \int_0^r \mu(s)ds}\cos|\tau_k| r\ dr + \beta\int_{r_1}^{r_2} k(r)h(r)\left|e^{-a_k r} - e^{-a_0 r}\right|dr .$$

Since $|\tau_k| \to \infty$, the first term on the right side converges to zero, and since $a_k \to a_0$, the second term also converges to zero. It is impossible. Hence the theorem is proved.

For the sake of convenience, we denote by *K* the cone of nonnegative function in $L^2(0, r_m)$

$K \square \{u \mid u \in L^2(0, r_m)\ ,\ u \ge 0\ ,\ a.e.\}$.

Definition: A linear operator *T* is called positive if $TK \subset K$. The inequality $T \ge T_2$ means that $T \ge T_2$ means that $T_1 - T_2$ to a positive operator.

Theorem 3: (i) of $\lambda \ge \gamma_0$, then $R(\lambda; A)$ is a positive operator and the spectral radius of $R(\lambda, A)$, $r(R(\lambda, A)) > 0$

(ii) of $\tau > \gamma_0$, then the spectral radius of $R(\lambda, A)$ is an eigenvalue of $R(\lambda, A)$ and there is an eigenfunction $\tilde{\varphi}(r) > 0$, $r \in [0, r_m]$ in the corresponding eigenspace.

Proof: By Theorem 2, $F(\lambda)$ is strictly decreasing function on $(-\infty, \infty)$, and $F(\gamma_0) = 1$. So $1 - F(\lambda) > 0$ if $\lambda > \gamma_0$. Hence from (5) it follows that $R(\lambda, A)g \ge 0$ if $g \ge 0$. This means that $R(\lambda, A)$ is positive. It follows that $r(\lambda)=r(R(\lambda,A))>0$ for $\lambda>\gamma_0$ and $r(\lambda)$ is an eigenvalue of $R(\lambda,A)$, and there exists an eigenfunction $\tilde{\varphi}\in K$.

By spectral mapping Theorem [2], for the eigenvalues and eigenfunctions of *A* and $R(\lambda;A)$, we have

$$A\phi=\gamma_i\phi \quad , \quad R(\lambda;A)\varphi=\frac{1}{\lambda-\gamma_i}\phi \tag{9}$$

So the real number $\gamma_0=\lambda-\dfrac{1}{r(\lambda)}$ is an eigenvalue of *A*. By (ii) of Theorem 1, the corresponding eigenfunctions is $\exp(-\gamma_0 r-\int \mu(s)ds)$. Then the conclusion (ii) follows from formula (B).

Theorem 4: The operator *A* is the infinitesimal generator of some C^0-semigroup $E(t)$ in $L^2(0,r_m)$.

Proof: By Theorem 1, *A* is closed linear operator, we now prove that $D(A)$ is dense in $L^2(0,r_m)$. For any $g\in L^2(0,r_m)$, $\varepsilon>0$ there is b_m, where $r_2<b_m<r_m$ such that

$$\int_{b_m}^{r_m} |g(r)|^2 dr<\frac{\varepsilon^2}{16}$$

Moreover, a function $\phi_1\in C^2[0,r_m]$ can be found such that

$$\phi_1(r)=0 \ , \quad b_m<r\le r_m \ , \ \|\phi_1-g\|^2\le\varepsilon^2/16$$

Let

$$M=2\max\left\{\begin{matrix}\max\phi_1(r)\\ 0\le r\le r_m\end{matrix},\beta\int_{r_1}^{r_2} k(r)h(r)\phi_1(r)dr\right\}$$

and take r_0 with, $0<r_0<r_1,\ 4r_0M^2\le\varepsilon^2/16$. Then we define the function $\phi(r)$ as follows

$$\phi(r)=\varphi_1(r) \ , \quad r_0<r\le r_m$$

$$\phi(0)=\beta\int_{r_1}^{r_2} k(r)h(r)\phi(r)dr$$

$\phi(r)$ is continuous by differentiable, $\max\limits_{0\le r\le r_0}|\varphi(r)|\le M \ , \ \ \phi'(r_0+0)=\phi'(r_0-0)$

Obviously, $\phi\in D(A)$ and

$$\|\phi-g\|\le\|\varphi-\varphi_1\|+\|\varphi_1-g\|=\left(\int_0^r |\varphi(r)-\varphi_1(r)|^2 dr\right)^{1/2}$$

$$+\|\varphi_1-g\|\le\frac{\varepsilon}{4}+\frac{\varepsilon}{4}<\varepsilon$$

This shows that $D(A)$ is dense in $L^2(0,r_m)$. Therefore, *A* is an operator with dense domain. For any $\phi\in D(A)$, we have

$$R_e(A\varphi,\varphi)=\lim_{a\to r_m-0}\left\{-\frac{1}{2}\int_0^a \left[\phi'(r)\overline{\phi(r)}+\phi(r)\overline{\phi'(r)}\right]dr\right.$$

$$\left.-\int_0^a \mu(r)|\varphi(r)|^2 dr\right\}\le\lim_{a\to r_m-0}\left\{-\frac{1}{2}\int_0^a \frac{d}{dr}|\varphi(r)|^2 dr\right\}$$

$$=\lim_{a\to r_m-0}\left\{-\frac{1}{2}|\varphi(r)|^2+\frac{1}{2}|\varphi(0)|^2\right\}$$

$$\leq \frac{1}{2}|\varphi(0)|^2 = \frac{1}{2}\beta^2\left(\int_{r_1}^{r_2} k(r)h(r)\phi(r)dr\right)^2 \leq a\|\phi\|^2$$

where $a = \frac{1}{2}\beta^2\int_{r_1}^{r_2} k^2h^2dr$. Then, by [4], A is an infinitesimal generator of some C_0-semigroup $E(t)$ in $L^2(0,r_m)$.

Theorem 5: $E(t)$ is a positive operator for any $t \geq 0$. If the initial distribution of population $P_0(r) \geq 0$, then there exists the unique solution of the problem (2) and it is a nonnegative function.

Proof: The positivity of the operator $E(t)$ is direct, so $E(t)P_0 \geq 0$ if $P_0(r) \geq 0$. The existence and uniqueness of the solution for (2) is direct consequence of Theorem 4.

Since the population density function must be nonnegative, Theorem 5 is quite natural. In a sense this shows the reasonableness of the above mathematical mode; for the population process.

Theorem 5: Let $m_{\gamma_0}(A)$ be the root subspace corresponding to this eigenvalue γ_0 of A and Q_{γ_0} be the projection operator from $L^2(0,r_m)$ onto $m_{\gamma_0}(A)$. Then for any $f \in L^2(0,r_m)$.
we have

$$Q_{\gamma_0}f = \frac{\beta}{-F'(\gamma_0)}\left\{\int_{r_1}^{r_2} k(\tau)h(\tau)\left[\int_0^{\tau} f(s)\exp\left(\gamma_0 s + \int_0^{s}\mu(\xi)d\xi\right)ds\right] \times \right.$$
$$\left. \times \exp\left(-\gamma_0\tau - \int_0^{\tau}\mu(\xi)d\xi\right)d\tau\right\}e^{-\gamma_0 r - \int_0^{\tau}\mu(\tau)d\tau} \qquad (10)$$

Proof: By Theorem 2, γ_0 is an eigenvalue of A with the algebraic multiplicity 1 and by [2] we have

$$Q_{\gamma_0}f = \lim_{\lambda\to\gamma_0}(\lambda-\gamma_0)R(\lambda,A)f \ .$$

Then by (5), the formula (10) can be verified by direct computations.

According to Theorem 2, there exists $\sigma > 0$ such that holds the inequality $R_e\lambda < \gamma_0 - \varepsilon$, $0 < \varepsilon \leq \xi$ for any eigenvalue $\lambda \neq \gamma_0$ of A.

As the above, we denote by Q_{γ_0} the projection operator corresponding to γ_0.

Theorem 7: For any ε, $0 < \varepsilon < \sigma$ there must be constants $c(\varepsilon)$ and $T_0(\varepsilon)$ such that

$$\left\|E(t) - E(t)Q_{\gamma_0}\right\| \leq c(\varepsilon)\exp[(\gamma_0-\varepsilon)t] \qquad \varepsilon \geq T_0(\varepsilon) \qquad (11)$$

Proof: From

$$E(t) - E(t)Q_{\gamma_0} = E(t)(I - Q_{\gamma_0}) = (I - Q_{\gamma_0})E(t)$$

and

$$\sigma(E(t)(I - Q_{\gamma_0})) = \left\{e^{\lambda t} \mid \lambda \in \sigma(A)/\gamma_0\right\}u\{0\},$$

we may obtain

$$\lim_{n\to\infty}\left\|E(n)(I - Q_{\gamma_0})\right\|^{1/n} = \lim_{n\to\infty}\left\|E(I)(I - Q_{\gamma_0})^n\right\|^{1/n} \leq \exp[(\gamma_0 - \varepsilon)]$$

Therefore there exists a constants $T_0'(\varepsilon)$ such that

$$\left\|E(n)(I - Q_{\gamma_0})\right\| \leq \exp[n(\gamma_0-\varepsilon)], \quad n \geq T_0'(\varepsilon)$$

Let $t > 1 + T_0'(\varepsilon)$ and take n with $n \leq t \leq n+1$, then

$$\left\|E(t)(I - Q_{\gamma_0})\right\| = \left\|E(t-n)E(n)(I - Q_{\gamma_0})\right\|$$
$$\leq \left\|E(t-n)\right\|\left\|E(n)(I - Q_{\gamma_0})\right\| \leq \left\|E(t-n)\right\|\exp[n(\gamma_0-\varepsilon)]$$
$$\leq \left\|E(t-n)\right\|\exp[(n-t)(\gamma_0-\varepsilon)]\exp t(\gamma_0-\varepsilon)]$$

Put

$$c(\varepsilon)=\max_{0\le t\le 1}\left\{\left\|E(t)\right\|e^{-t(\gamma_0-\varepsilon)}\right\} ,$$

Then we have

$$\left\|E(t)(I-Q_{\gamma_0})\right\|\le c(\varepsilon)e^{t(\gamma_0-t)} \quad , \quad \text{for} \quad t\ge T_0(\varepsilon)=1+T_0(\varepsilon)$$

and the proof follows.
The following theorem is the immediate consequence of Theorems.
Theorem 8: The solution $p(r,t)$ of the equation (2) has the following behavior .

$$p(r,t)=E(t)p_0=e^{\gamma_0 t}(Q_{\gamma_0},p_0)(r)+O(e^{(\gamma_0-\varepsilon)}) \qquad (12)$$

where $Q_{\gamma_0}\phi$ is defined as (9) .

By Theorem 8 we may obtain the stability of the population system (2). In fact, let

$$\beta_{cr}=\left[\int_{r_1}^{r_2} k(r)h(r)\exp(-\int_0^T \mu(s)ds)dr\right]^{-1} \qquad (13)$$

This β_{cr} is just critical fertility of women, when $\beta>\beta_{cr}$, the population growth increases exponentially, when $\beta<\beta_{cr}$, the size of population decreases to zero exponentially and when $\beta=\beta_{cr}$, the population process converges asymptotically to

$$p(r)=c_0\exp\left(-\int_0^r \mu(s)ds\right) \qquad (14)$$

where

$$c_0=\frac{\int_{r_1}^{r_2} k(r)h(r)\left[\int_0^r p_0(s)\exp\left(\int_0^r \mu(\tau)d\tau\right)\exp\left(-\int_0^r \mu(\tau)d\tau\right)ds\right]dr}{\int_{r_1}^{r_2} k(\tau)h(\tau)\tau\,\mathrm{ex}\left(\int_0^\tau \mu(s)ds\right)d\tau}$$

From (14) we may also obtain an interesting result, i.e.

$$\int_{r_1}^{r_2} c_0 p(r)\beta_{cr}k(r)h(r)dr=c_0F(0)=c_0 .$$

This means c_0 is just the absolute rate of infants in the society.

Integrating $p(r)$, we obtain the asymptotic value of the total number of people

$$N=\int_0^{r_m} p(r)d(r)=c_0\int_0^{r_m} e^{-\int_0^r \mu(\tau)d\tau} dr=c_0s_0$$

where $s_0=\int_0^{r_m} \exp\left[-\int_0^r \mu(\tau)d\tau\right]dr$

Is the average life-span of the population. Therefore for the steady population states the total number of people in the society is equal to the product of the life expectancy with absolute birth rate of infants.

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