

# NUMERICAL ANALYSIS FOR SOME AUTONOMOUS STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

**Hamdy M.Ahmed**

Higher Institute of Engineering, El-Shorouk Academy, Cairo, Egypt.

Email: hamdy\_17eg@yahoo.com

## ABSTRACT

Numerical solution of stochastic delay differential equations is studied by using explicit one-step methods. One-step method is asymptotically zero-stable in the quadratic mean square sense.

**Keywords:** *Stochastic delay differential equations, Explicit one-step methods, Conditional expectation, Lipschitz condition and linear growth condition.*

**AMS Subject Classifications:** 65C30, 60H20, 60H10 .

## 1. INTRODUCTION

In Many areas of science (such as medicine, physics, biology, economics) there has been an increasing interest in the investigation of stochastic delay differential equations. The important of stochastic delay equations derives from the fact that many of the phenomena witnessed around us do not have an immediate effect from the moment of their occurrence. Of course the stochastic delay differential equations can also be regarded as a generalization of stochastic differential equations (see [1], [2], [3]).

The numerical solutions of stochastic delay differential equations or SDDEs are studied in many papers. C.T.H.Baker and E.Buckwar [4] studied the numerical analysis of explicit one-step method for stochastic delay differential equations, C.T.H.Baker and N.Ford [5] studied the Convergence of linear multistep methods for a class of delay-integro-differential equations, E.Buckwar and T.Shardlow [6] studied the weak approximation of stochastic differential delay equations, Y.Hu,S.E-A.Mohammed and F.Yan [7] studied the discrete-time approximations of stochastic delay equations, X.Mao and S.Sabanis [8] studied the numerical solutions of stochastic differential delay equations under local Lipschitz condition, U.Kuchler and E.Platen [9] studied the Strong discrete time approximation of stochastic differential equations with time delay, A.Bellen and M.Zennaro [10] studied the numerical methods for delay differential equations, G.A.Bocharov and F.A.Rihan [11] studied the numerical modeling in biosciences using delay differential equations and E.Buckwar and R.Winkler [12] studied the multi-step Maruyama methods for stochastic delay differential equations. In this paper we study the numerical analysis for autonomous stochastic delay differential equations by using a stochastic explicit one-step method.

Let  $(\Omega, F, \{F_t\}_{t \geq 0}, P)$  be a complete probability space with the filtration  $\{F_t\}_{t \geq 0}$  satisfying the usual conditions ( that is increasing and right continuous, and  $\{F_t\}, t \geq 0$ , contains all P- null sets in F ).Throughout this paper let  $|\cdot|$  denote the Euclidean vector norm and we shall use the notation

$$\sup_x |u(x,t)| = P u(.,t)P \text{ and } \sup_x |v(x,t)| = P v(.,t)P.$$

The  $L^p$  – norm of a vector-valued  $L^p$  – integrable random variable  $Y \in L^p(\Omega, F, P)$  will be denoted by

$$PYP_p = (EPYP^p)^{1/p}, \quad 1 \leq p < \infty,$$

where E is the expectation with respect to P .

In this paper let W(t) be a n-dimensional Wiener process given on the filtered probability space  $(\Omega, F, P)$  .

We consider the stochastic delay differential equation  $(0 = t_0 < T < \infty)$  :

$$du(x,t) = f(x,t,u(x,t),u(x,t-\tau))dt + g(x,t,u(x,t),u(x,t-\tau))dW(t), \quad t \in [0,T]$$

$$u(x,t) = \Psi(x,t), \quad t \in [-\tau,0] \quad (1.1)$$

with one fixed lag, where  $x \in R^v$  ,(  $R^v$  is the  $v$  -dimensional Euclidean space) and  $\Psi(x,t)$  is an  $F_{t_0}$  -measurable  $C(R^v \times [-\tau,0], R^n)$  valued random variable such that

$$EP\Psi P < \infty.$$

$(C(R^v \times [-\tau, 0], R^n))$  is the Banach space of all continuous paths from  $R^v \times [-\tau, 0] \rightarrow R^n$  equipped with the supremum norm

$$P\eta P_C = \sup_{t \in [-\tau, 0]} P\eta(\cdot, t)P$$

where

$$P\eta(\cdot, t)P = \sup_x |\eta(x, t)|.$$

We have  $f : R^{n+v+1} \rightarrow R^n$ ,  $g : R^{n+v+1} \rightarrow R^{n \times n}$  and  $\Psi : R^v \times [-\tau, 0] \rightarrow R^n$

If the functions  $f$  and  $g$  do not explicitly depend on  $x$  and  $t$  the equation is called autonomous, and we consider this case for simplicity. Equation (1.1) can then be formulated equivalently as

$$u(x, t) = u_0(x) + \int_0^t f(u(x, s), u(x, s - \tau)) ds + \int_0^t g(u(x, s), u(x, s - \tau)) dW(s), \quad (1.2)$$

for  $t \in [0, T]$  and with

$$u(x, t) = \Psi(x, t), \text{ for } t \in [-\tau, 0].$$

In this paper we need the following conditions:

(A1) The functions  $f$  and  $g$  are continuous

(A2) The functions  $f$  and  $g$  satisfy a uniform Lipschitz condition; that is, there exists positive constants

$L_1, L_2, L_3$  and  $L_4$  such that  $\phi_1, \phi_2, \psi_1, \psi_2 \in R^n$  and  $t \in [0, T]$ ,

$$Pf(\phi_1, \psi_1) - f(\phi_2, \psi_2)P \leq L_1 P\phi_1 - \phi_2 P + L_2 P\psi_1 - \psi_2 P, \quad (1.3)$$

and

$$Pg(\phi_1, \psi_1) - g(\phi_2, \psi_2)P \leq L_3 P\phi_1 - \phi_2 P + L_4 P\psi_1 - \psi_2 P. \quad (1.4)$$

(A3) The function  $\Psi$  is Holder continuous with exponent  $\gamma$ ; that is, such that

for  $t, s \in [-\tau, 0]$

$$EP\Psi(\cdot, t) - \Psi(\cdot, s)P^p \leq L_5 P|t - s|^{p\gamma}, \quad p = 1, 2 \quad (1.5)$$

(A4) The functions  $f$  and  $g$  satisfy a linear growth condition; that is, there exist positive constants  $K_1$  and  $K_2$  such

that for all  $\phi, \phi_1, \psi, \psi_1 \in R^n$  and  $t \in [0, T]$ ,

$$Pf(\phi, \phi_1)P^2 \leq K_1(1 + P\phi P^2 + P\phi_1 P^2), \quad (1.6)$$

$$Pg(\psi, \psi_1)P^2 \leq K_2(1 + P\psi P^2 + P\psi_1 P^2). \quad (1.7)$$

## 2. NUMERICAL ANALYSIS FOR AN AUTONOMOUS SDDES

For simplicity we shall in the sequel consider equation (1.1) in the autonomous form; that is, we shall work with

$$du(x, t) = f(u(x, t), u(x, t - \tau))dt + g(u(x, t), u(x, t - \tau))dW(t), \quad t \in [0, T]$$

$$u(x, t) = \Psi(x, t), \quad t \in [-\tau, 0]. \quad (2.1)$$

We define a mesh with a uniform step on the interval  $[0, T]$ ,  $h = T/N$ ,  $t_n = n \cdot h$ , where  $n = 0, \dots, N$ , and where we assume that for the given  $h$  there is a corresponding integer  $N_\tau$  such that the lag can be expressed in terms of the step size as  $\tau = N_\tau \cdot h$ . We consider strong approximations  $v_n(x)$  of the solution to equation (2.1), using a stochastic explicit one-step method of the form

$$v_{n+1}(x) = v_n(x) + \phi(h, v_n(x), v_{n-N_\tau}(x), I_\phi), \quad n = 0, \dots, N - 1, \quad (2.2)$$

where the initial values are given by

$$v_{n-N_\tau}(x) = \Psi(x, t_n - \tau) \text{ for } n - N_\tau \leq 0.$$

The increment function  $\phi(h, \dots, I_\phi) : R^n \times R^n \rightarrow R^n$  incorporates a finite number of multiple Ito-integrals (see

[13]) of the form

$$I_{(j_1, \dots, j_l), h} = \int_t^{t+h} \int_t^{s_l} \dots \int_t^{s_2} dW^{j_1}(s_1) \dots dW^{j_{l-1}}(s_{l-1}) dW^{j_l}(s_l),$$

where  $j_i \in \{0, 1\}$  and  $dW^0(t) = dt$  with  $t = t_n$  in the case (2.1). We denote by  $I_\phi$  the collection of Ito-integrals required to compute the increment function  $\phi$ . The increment function  $\phi$  is assumed to generate approximations  $v_n(x)$  which are  $F_{t_n}^-$ -measurable.

We suppose there exist positive constants  $C_1, C_2, C_3$  such that  $\zeta, \zeta, \eta, \eta \in R^n$ ,

$$PE(\phi(h, \zeta, \eta, I_\phi) - \phi(h, \zeta, \eta, I_\phi))P \leq C_1 h (P\zeta - \zeta P + P\eta - \eta P), \quad (2.3)$$

$$E(P\phi(h, \zeta, \eta, I_\phi) - \phi(h, \zeta, \eta, I_\phi)P^2) \leq C_2 h (P\zeta - \zeta P^2 + P\eta - \eta P^2), \quad (2.4)$$

and

$$E(P\phi(h, \zeta, \eta, I_\phi)P^2) \leq C_3 h (1 + P\zeta P^2 + P\eta P^2). \quad (2.5)$$

**Lemma 2.1.** If the increment function  $\phi$  in equation (2.2) satisfies condition (2.5), then  $EPv_n(\cdot)P^2 \leq \infty$  for all  $n \leq N$ .

**Proof.** We have

$$\begin{aligned} E(Pv_n(\cdot)P^2 | F_{t_0}) &= E(Pv_{n-1}(\cdot) + \phi(h, v_{n-1}(\cdot), v_{n-1-N_\tau}(\cdot), I_\phi)P^2 | F_{t_0}) \\ &\leq 2E(Pv_{n-1}(\cdot)P^2 | F_{t_0}) + 2E(P\phi(h, v_{n-1}(\cdot), v_{n-1-N_\tau}(\cdot), I_\phi)P^2 | F_{t_0}) \\ &\leq 2E(Pv_{n-1}(\cdot)P^2 | F_{t_0}) + 2C_3 h E(1 + Pv_{n-1}(\cdot)P^2 + Pv_{n-1-N_\tau}(\cdot)P^2 | F_{t_0}) \\ &= 2(1 + C_3 h)E(Pv_{n-1}(\cdot)P^2 | F_{t_0}) + 2C_3 h E(Pv_{n-1-N_\tau}(\cdot)P^2 | F_{t_0}) + 2C_3 h. \end{aligned}$$

The lemma follows from this result. To display the argument in detail, we define

$$\begin{aligned} \rho_0 &= \max_{-N_\tau \leq r \leq 0} E(Pv_r(\cdot)P^2 | F_{t_0}), \quad \rho_n = \max_{0 \leq r \leq n} E(Pv_r(\cdot)P^2 | F_{t_0}); \\ \tilde{\rho}_0 &= \rho_0, \quad \tilde{\rho}_n = \max_{-N_\tau \leq r \leq n} E(Pv_r(\cdot)P^2 | F_{t_0}) = \max(\rho_0, \rho_n). \end{aligned}$$

Note that the sequences  $\{\rho_n\}_{n \geq 1}$  and  $\{\tilde{\rho}_n\}_{n \geq 0}$  are monotonically non-decreasing. Thus, we obtain

$$\tilde{\rho}_n \leq 2(1 + C_3 h)\tilde{\rho}_{n-1} + 2C_3 h \tilde{\rho}_0 + 2C_3 h, \text{ for } 0 < n \leq N_\tau,$$

$$\tilde{\rho}_n \leq 2(1 + C_3 h)\tilde{\rho}_{n-1} + 2C_3 h, \text{ for } n > N_\tau,$$

$$\text{hence } \tilde{\rho}_n \leq 2(1 + C_3 h)\tilde{\rho}_{n-1} + 2C_3 h \tilde{\rho}_0 + 2C_3 h, \text{ for } n > 0.$$

By induction, when  $\tilde{\rho}_n \leq \alpha \tilde{\rho}_{n-1} + \zeta$  for  $n > 0$ , we find that

$$\tilde{\rho}_n \leq \alpha^n \zeta + (1 + \alpha + \dots + \alpha^n) \tilde{\rho}_0;$$

setting  $\alpha = 2(1 + C_3 h)$  and  $\zeta = 2C_3 h \tilde{\rho}_0 + 2C_3 h$ , and using the assumptions on the initial function  $\Psi$  to bound  $\tilde{\rho}_0$ , we deduce the desired result.

**Notation 2.1.** We denote by  $u(x, t_{n+1})$  the value of the exact solution of equation (2.1) at the meshpoint  $t_{n+1}$ , by  $v_{n+1}(x)$  the value of the approximate solution using equation (2.2), and by  $v(x, t_{n+1})$  the value obtained after just one step of equation (2.2); that is,

$$v(x, t_{n+1}) = v(x, t_n) + \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi).$$

**Definition 2.1.** The error of the above approximation  $v_n(x)$  on the meshpoints is the sequence of random variables

$$\varepsilon_n := u(x, t_n) - v_n(x), \quad n = 1, \dots, N. \quad (2.6)$$

Note that  $\varepsilon_n$  is  $F_{t_n}$ -measurable since both  $u(x, t_n)$  and  $v_n(x)$  are  $F_{t_n}$ -measurable random variables, and that  $(E\mathbf{P}\varepsilon_n\mathbf{P}^2)^{1/2}$  is the  $L^2$ -norm of (2.6).

**Definition 2.2.** Let

$$\delta_{n+1}(x) = u(x, t_{n+1}) - v(x, t_{n+1}), \quad n = 0, \dots, N-1. \quad (2.7)$$

The method (2.2) is said to be consistent with order  $p_1$  in the mean and with order  $p_2$  in the mean-square sense if, with

$$p_2 \geq \frac{1}{2} \text{ and } p_1 \geq p_2 + \frac{1}{2}, \quad (2.8)$$

the estimates

$$\max_{0 \leq n \leq N-1} PE(\delta_{n+1}(\cdot))\mathbf{P} \leq Ch^{p_1} \text{ as } h \rightarrow 0, \quad (2.9)$$

and

$$\max_{0 \leq n \leq N-1} (E\mathbf{P}\delta_{n+1}(\cdot)\mathbf{P}^2)^{\frac{1}{2}} \leq Ch^{p_2} \text{ as } h \rightarrow 0, \quad (2.10)$$

hold, where the (generic) constant  $C$  does not depend on  $h$ , but may depend on  $T$ , and on the initial data. We now state the main theorem of this paper

**Theorem 2.1.** We assume that the functions  $f$  and  $g$  satisfy the conditions from A1 to A4 . Suppose that the method defined by equation (2.2) is consistent with order  $p_1$  in the mean and order  $p_2$  in the mean-square sense, with  $p_1$  and  $p_2$  satisfying inequality (2.8), and the increment function  $\phi$  in equation (2.2) satisfies the estimates (2.3) and (2.4). Then the approximation (2.2) for equation (2.1) is convergent in  $L^2$  (as  $h \rightarrow 0$  with  $\tau/h \in N$ ) with order  $p = p_2 - \frac{1}{2}$ . That is, convergence is in the mean-square sense, and

$$\max_{1 \leq n \leq N} (E\mathbf{P}\varepsilon_n\mathbf{P}^2)^{\frac{1}{2}} \leq ch^p \text{ as } h \rightarrow 0. \quad (2.11)$$

**Proof.** Using Notation 2.1, adding and subtracting  $u(x, t_n)$  and

$\phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi)$ , and rearranging, we obtain

$$\begin{aligned} \varepsilon_{n+1} &= u(x, t_{n+1}) - v_{n+1}(x) \\ &= u(x, t_n) - v_n(x) + u(x, t_{n+1}) - u(x, t_n) - \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi) \\ &\quad + \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi) - \phi(h, v_n(x), v_{n-N_\tau}(x), I_\phi) \\ &= \varepsilon_n + \delta_{n+1} + z_n, \end{aligned}$$

where

$$\varepsilon_n = u(x, t_n) - v_n(x),$$

$$\delta_{n+1} = u(x, t_{n+1}) - u(x, t_n) - \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi)$$

and

$$z_n = \phi(h, u(x, t_n), u(x, t_n - \tau), I_\phi) - \phi(h, v_n(x), v_{n-N_\tau}(x), I_\phi). \quad (2.12)$$

Thus, squaring, employing the conditional means with respect to the  $\sigma$ -algebra  $F_{t_0}$  and taking the norm, we obtain

$$E(\mathbf{P}\varepsilon_{n+1}\mathbf{P}^2 | F_{t_0}) \leq E(\mathbf{P}\varepsilon_n\mathbf{P}^2 | F_{t_0}) + E(\mathbf{P}\delta_{n+1}\mathbf{P}^2 | F_{t_0}) + E(\mathbf{P}z_n\mathbf{P}^2 | F_{t_0})$$

$$+ 2PE(\delta_{n+1} \cdot \varepsilon_n | F_{t_0})P + 2PE(\delta_{n+1} \cdot z_n | F_{t_0})P + 2PE(\varepsilon_n \cdot z_n | F_{t_0})P, \quad (2.13)$$

which holds almost surely.

We shall now estimate the separate terms in inequality (2.13) individually and in sequence; all the estimates hold almost surely. We shall frequently use the Holder inequality, the inequality  $2ab \leq a^2 + b^2$  and properties of conditional expectation, which can be found in [14]. In the sequel we shall use  $c$  to denote an unspecified constant, which depends only on the constants  $L_1, L_2, L_3, L_4, k_1, k_2, c_1$  and  $c_2$ , and on  $T$  and the initial data .

Due to the assumed consistency in the mean-square sense of the method, we have

$$E(P\delta_{n+1}P^2 | F_{t_0}) = E(E(P\delta_{n+1}P^2 | F_{t_n}) | F_{t_0}) \leq ch^{2p_2}.$$

Due to property (2.4) of the increment function, we have

$$E(Pz_nP^2 | F_{t_0}) \leq chE(P\varepsilon_nP^2 | F_{t_0}) + chE(P\varepsilon_{n-N_\tau}P^2 | F_{t_0}).$$

We have, due to the consistency condition,

$$\begin{aligned} 2PE(\delta_{n+1} \cdot \varepsilon_n | F_{t_0})P &\leq 2PE(E(\delta_{n+1} | F_{t_n})\varepsilon_n | F_{t_0})P \leq 2(EPE(\delta_{n+1} | F_{t_n})P^2)^{\frac{1}{2}} \cdot (E(P\varepsilon_nP^2 | F_{t_0}))^{\frac{1}{2}} \\ &= 2(E(ch^{2p_1-1}))^{\frac{1}{2}} \cdot (hE(P\varepsilon_nP^2 | F_{t_0}))^{\frac{1}{2}} \leq ch^{2p_1-1} + hE(P\varepsilon_nP^2 | F_{t_0}). \end{aligned}$$

By employing the consistency condition and property (2.4) of the increment function  $\phi$ , we have

$$\begin{aligned} 2PE(\delta_{n+1} \cdot z_n | F_{t_0})P &\leq 2(E(P\delta_{n+1}P^2 | F_{t_0}))^{\frac{1}{2}} (E(Pz_nP^2 | F_{t_0}))^{\frac{1}{2}} \\ &\leq E(E(P\delta_{n+1}P^2 | F_{t_n}) | F_{t_0}) + E(Pz_nP^2 | F_{t_0}) \\ &\leq ch^{2p_2} + chE(\varepsilon_n^2 | F_{t_0}) + chE(\varepsilon_{n-N_\tau}^2 | F_{t_0}). \end{aligned}$$

Using definition (2.12) and property (2.3) of the increment function  $\phi$ , we have

$$\begin{aligned} 2P(z_n \cdot \varepsilon_n | F_{t_0})P &\leq 2E(PE(z_n | F_{t_n})P\varepsilon_nP | F_{t_0}) \leq chE(P\varepsilon_nP^2 | F_{t_0}) + 2chE(P\varepsilon_nPP\varepsilon_{n-N_\tau}P | F_{t_0}) \\ &\leq chE(P\varepsilon_nP^2 | F_{t_0}) + ch[2(E(P\varepsilon_nP^2 | F_{t_0}))^{\frac{1}{2}} \cdot (E(P\varepsilon_{n-N_\tau}P^2 | F_{t_0}))^{\frac{1}{2}}] \\ &\leq chE(P\varepsilon_nP^2 | F_{t_0}) + chE(P\varepsilon_nP^2 | F_{t_0}) + chE(P\varepsilon_{n-N_\tau}P^2 | F_{t_0}) \\ &\leq chE(P\varepsilon_nP^2 | F_{t_0}) + chE(P\varepsilon_{n-N_\tau}P^2 | F_{t_0}). \end{aligned}$$

Combining these results with  $2p_2 \leq 2p_1 - 1$ , we obtain

$$E(\varepsilon_{n+1}^2 | F_{t_0}) \leq (1 + ch)E(\varepsilon_n^2 | F_{t_0}) + ch^{2p_2} + chE(P\varepsilon_{n-N_\tau}P^2 | F_{t_0}).$$

Now we shall prove the assertion by an induction argument over consecutive intervals of length  $\tau$  up to the end of the interval  $[0, T]$ . Since we have exact initial values, we set

$$\varepsilon_n = 0 \text{ for } n = -N_\tau, \dots, 0.$$

**Step1 .** Suppose that  $t_n \in [0, \tau]$ ; that is,  $n = 1, \dots, N_\tau$  and  $\varepsilon_{n-N_\tau} = 0$ .

$$\begin{aligned}
 E(\varepsilon_{n+1}^2 | F_{t_0}) &\leq (1+ch)E(\varepsilon_n^2 | F_{t_0}) + ch^{2p_2} \leq ch^{2p_2} \sum_{k=0}^n (1+ch)^k \\
 &= ch^{2p_2} \frac{(1+ch)^{n+1} - 1}{(1+ch) - 1} \leq ch^{2p_2-1} ((e^{ch})^{n+1} - 1) \leq ch^{2p_2-1} (e^{cT} - 1).
 \end{aligned}$$

**Step2.** Suppose that  $t_n \in [k\tau, (k+1)\tau]$ , and make the assumption that

$$E(\mathbf{P}\varepsilon_{n-N_\tau} \mathbf{P}^2 | F_{t_0}) \leq ch^{2p_2-1}.$$

Then

$$\begin{aligned}
 E(\varepsilon_{n+1}^2 | F_{t_0}) &\leq (1+ch)E(\varepsilon_n^2 | F_{t_0}) + ch^{2p_2} + chE(\mathbf{P}\varepsilon_{n-N_\tau} \mathbf{P}^2 | F_{t_0}) \leq (1+ch)E(\varepsilon_n^2 | F_{t_0}) + ch^{2p_2} + hch^{2p_2-1} \\
 &= (1+ch)E(\varepsilon_n^2 | F_{t_0}) + ch^{2p_2} \leq ch^{2p_2-1} (e^{cT} - 1),
 \end{aligned}$$

by the same arguments as above. This, implies, almost surely, that

$$(E(\varepsilon_{n+1}^2 | F_{t_0}))^{\frac{1}{2}} \leq ch^{2p_2-\frac{1}{2}},$$

which proves the theorem.

**Definition 2.3.** The stochastic one-step size (2.2) is zero stable in the quadratic mean square sense if, given  $\varepsilon > 0$ , there is a  $\delta = \delta(\varepsilon, h_0) > 0$  such that for all  $0 < h < h_0$  and positive integers  $n \leq T/h$ ,

$$\rho_0 = \max_{-N_t \leq r \leq 0} E\mathbf{P}v_r - \tilde{v}_r \mathbf{P}^2 \leq \delta \Rightarrow \rho_n = E\mathbf{P}v_n - \tilde{v}_n \mathbf{P}^2 \leq \varepsilon$$

holds, where  $\tilde{v}_n$  denotes the sequence defined by the method (2.2) with the initial values  $v_r$  for  $r = -N_t, \dots, 0$  replaced by  $\tilde{v}_r$  for  $r = -N_t, \dots, 0$ . If the method is stable and if  $\rho_n \rightarrow 0$  whenever  $\rho_0$  is sufficiently small, the method is asymptotically zero-stable in the quadratic mean square sense.

**Theorem 2.2.** If the increment function  $\phi$  of the approximation method (2.2) satisfies the estimate (2.3) and (2.4), then the one-step method (2.2) is zero-stable in the quadratic mean-square sense.

**Proof.** We have, for  $0 < n \leq N = T/h$ ,

$$\begin{aligned}
 (v_n - \tilde{v}_n)^2 &\leq (v_{n-1} - \tilde{v}_{n-1})^2 + 2(v_{n-1} - \tilde{v}_{n-1}) \cdot (\phi(h, v_{n-1}, v_{n-1-N_\tau}, I_\phi) - \phi(h, \tilde{v}_{n-1}, \tilde{v}_{n-1-N_\tau}, I_\phi)) \\
 &\quad + (\phi(h, v_{n-1}, v_{n-1-N_\tau}, I_\phi) - \phi(h, \tilde{v}_{n-1}, \tilde{v}_{n-1-N_\tau}, I_\phi))^2.
 \end{aligned}$$

By using the properties of conditional expectation and the estimates (2.3) and (2.4), we obtain

$$\begin{aligned}
 E(\mathbf{P}v_n - \tilde{v}_n \mathbf{P}^2 | F_{t_0}) &\leq E(\mathbf{P}v_{n-1} - \tilde{v}_{n-1} \mathbf{P}^2 | F_{t_0}) \\
 &\quad + 2\mathbf{P}((v_{n-1} - \tilde{v}_{n-1}) \cdot (\phi(h, v_{n-1}, v_{n-1-N_\tau}, I_\phi) - \phi(h, \tilde{v}_{n-1}, \tilde{v}_{n-1-N_\tau}, I_\phi)) | F_{t_0}) \mathbf{P} \\
 &\quad + E(\mathbf{P}(\phi(h, v_{n-1}, v_{n-1-N_\tau}, I_\phi) - \phi(h, \tilde{v}_{n-1}, \tilde{v}_{n-1-N_\tau}, I_\phi))^2 | F_{t_0}) \\
 &\leq E(\mathbf{P}v_{n-1} - \tilde{v}_{n-1} \mathbf{P}^2 | F_{t_0}) + chE(\mathbf{P}v_{n-1} - \tilde{v}_{n-1} \mathbf{P}^2 | F_{t_0}) + chE(\mathbf{P}v_{n-1-N_\tau} - \tilde{v}_{n-1-N_\tau} \mathbf{P}^2 | F_{t_0}) \\
 &\quad + chE(\mathbf{P}v_{n-1} - \tilde{v}_{n-1} \mathbf{P}^2 | F_{t_0}) + chE(\mathbf{P}v_{n-1-N_\tau} - \tilde{v}_{n-1-N_\tau} \mathbf{P}^2 | F_{t_0})
 \end{aligned}$$

$$= (1+ch)E(\mathbf{P}v_{n-1} - \tilde{v}_{n-1}\mathbf{P}^2 | F_{t_0}) + chE(\mathbf{P}v_{n-1-N_\tau} - \tilde{v}_{n-1-N_\tau}\mathbf{P}^2 | F_{t_0}),$$

where  $c$  denotes a generic positive constant.

We define the quantities

$$R_0 = \max_{-N_\tau \leq r \leq 0} E(\mathbf{P}v_r - \tilde{v}_r\mathbf{P}^2 | F_{t_0}), \quad R_n = \max_{0 \leq r \leq n} E(\mathbf{P}v_r - \tilde{v}_r\mathbf{P}^2 | F_{t_0});$$

$$\tilde{R}_0 = R_0, \quad R_n = \max_{-N_\tau \leq r \leq n} E(\mathbf{P}v_r - \tilde{v}_r\mathbf{P}^2 | F_{t_0}) = \max(R_0, R_n)$$

Note that  $\{R_n\}_{n>0}$  and  $\{\tilde{R}_n\}_{n \geq 0}$  are monotonically non-decreasing.

We obtain,  $\tilde{R}_n \leq (1+ch)\tilde{R}_{n-1} + ch\tilde{R}_0$  for  $0 < n \leq N_\tau$  while for  $n > N_\tau$  we have  $\tilde{R}_n \leq (1+ch)\tilde{R}_{n-1} + ch\tilde{R}_{j(n)}$  for some  $j(n) < n$ . Thus  $\tilde{R}_n \leq (1+ch)\tilde{R}_{n-1}$  for  $n > 0$ .

By induction and using the property that  $1+2ch \leq e^{2ch}$ , we find that  $\tilde{R}_n \leq e^{(2cT)}\tilde{R}_0$ .

We deduce that, given  $\varepsilon > 0$ , we have  $\tilde{R}_n \leq \varepsilon$  if  $R_0 \leq \delta = \varepsilon e^{-2cT}$ , when  $n \leq N$ , which proves the theorem.

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