INVERSE CAUCHY PROBLEMS FOR NONLINEAR FRACTIONAL PARABOLIC EQUATIONS IN HILBERT SPACE

Mahmoud M. El-Borai

Faculty of Science, Alexandria University, Alexandria, Egypt
Email : m_m_elborai @ yahoo.com

ABSTRACT
This note is devoted to study an inverse Cauchy problem in a Hilbert space H for fractional abstract differential equations of the form:

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + f(t)\, g(t) + F(t, W(t)),$$

with the initial condition $u(0) = u_0 \in H$ and the overdetermination condition:

$$(u(t), v) = w(t),$$

where (.,.) is the inner product in H, f is a real unknown function w is a given real function, u_0, v are given elements in H, g is a given abstract function with values in H, $0 < \alpha \le 1$, u is unknown, and A is a linear closed operator defined on a dense subset of H, $W(t) = (B_1(t)u(t), ..., B_r(t)u(t)), \{B_i(t) : i = 1, ..., r, t \in J\}$ is a family of linear closed operators defined on dense sets $S_1, ..., S_r \supset S$ respectively in H into H, F is a given abstract function on $J \times H^r$ into H.
It is supposed that A generates a semigroup. An application is given to study an inverse problem in a suitable Sobolev space for general nonlinear fractional parabolic partial differential equations with unknown source functions.

Keywords and phrases: *Nonlinear fractional evolution equation,inverse Cauchy problem.*
2000 Mathematics Subject Classifications: 35A 05, 77D 09, 47 G 10.

1. INTRODUCTION

Successful utilization of any fractional differential equation as a modeling tool requires results about existence, uniqueness and regularity properties of the solution under sufficiently general assumptions.
The general form of the equation is known and the details must be determined by reconciling the model with the observation of the process. In other words an inverse problem must be solved to find, on the basis of the observation, the coefficients, free term, the right-hand side, and sometimes, initial and boundary conditions.
Several authors [1-5] studied the unique solvability of inverse problems for various parabolic equations with unknown source functions under an integral overdetermination condition.
Cannon and Duchateu considered the identification of an unknown state-dependent source term in the heat equation[6].
In this note, the following nonlinear model is considered:

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + f(t)g(t) + F(t, W), (1.1)$$

$$u(0) = u_0, (1.2)$$

where A is a linear closed operator defined on a dense set S in a real Hilbert space H into H, $W(t) = (B_1(t)u(t), ..., B_r(t)u(t)), \{B_i(t) : i = 1, ..., r, t \in J\}$ is a family of linear closed operators defined on dense sets $S_1, ..., S_r \supset S$ respectively in H into H, F is a given abstract function on $J \times H^r$ into H, u_0 is a given element in S, g is a given abstract function defined on $J = [0, T], (T > 0)$ with values in H, f is an unknown real function and $0 < \alpha \le 1$.
It is assumed that A generates an analytic semigroup $Q(t)$. This condition implies $\| Q(t) \| \le c$ for all $t \ge 0$, where $\| . \|^2 = (.,.)$, (.,.) is the inner product in H and c is a positive constant.
It is supposed also that

(C_1) There is a number $\gamma \in (0,1)$ such that

$$\| B_i(t_2)\, Q(t_1) h \| \leq \frac{K_1}{t_1^{\gamma}} \| h \|,$$

where $t_1 \in (0,T], t_2 \in J, h \in H$ and K_1 is a positive constant, $i = 1,...,r,$

(C_2) The functions $B_1(t)h,..., B_r(t)h$ are uniformly Hölder continuous in $t \in J$ for every element h in $\cap_i S_i$,

(C_3) g is continuous in t on J with respect to the norm in H ,

(C_4) F is continuous on $J \times H^r$ with respect to the norm in H .

In section 2, the inverse Cauchy problem is studied under the overdetermination condition:

$$(u(t), v) = w(t) \quad ,(1.3)$$

where v is a given element in H and w is a given real function defined on J .

We shall suppose that the adjoint operator A^* of the closed operator A exists and that if

$$\frac{d^{\alpha}\phi(t)}{dt^{\alpha}} = \psi(t),$$

then

$$\phi(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds,$$

where $\Gamma(\alpha)$ is the gamma function, $0 < \alpha \leq 1, \phi, \psi$ are abstract functions of t with values in H and the integral is taken in Bochner`s sense [7].

We shal also assume that :

(i) F is uniformly Hölder continuous in t, that is there exist constants $c_1 > 0$, $\beta \in (0,1]$ such that

$$\| f(t_2, W) - f(t_1, W) \| \leq c_1 \,| t_2 - t_1 |^{\beta},$$

for all $t_1, t_2, \in J$ and all $W \in H^r$,

(ii)The Lipschitz condition

$$\| F(t,W) - F(t,W^*) \| \leq c_2 \sum_{i=1}^{r} \| w_i - w_i^* |,$$

is satisfied for all $W, W^* \in H^r$ and all $t \in J$, where $c_2 > 0$ is a constant.

We shall assume the following conditions;

A_1**:** $u_0, v \in S, g(t) \in S$ for all $t \in J$,

A_2**:** $| g_1(t) | \geq g_0, t \in J$, where $g_1(t) = (g(t), v)$ and g_0 is a positive constant,

A_3**:** The abstract functions g and Ag are continuous on J with respect to the norm in H ,

A_4**:** $\dfrac{dw}{dt} \in C(J)$.

In section 3 an application is given to the inverse Cauchy problem for nonlinear fractional differential equations of parabolic type.

2. AN INVERSE CAUCHY PROBLEM

A pair of functions $\{u, f\}$ is said to be a strictly solution of the inverse problem (1.1)-(1.3) if

$$u \in S, \frac{d^{\alpha}u(t)}{dt^{\alpha}} \in H$$

for each $t \in (0,T], f \in C(J)$ and the relations (1.1)-(1.3) are satisfied. In this case we say that the inverse problem (1.1)-(1.3) is solvable.

Let us consider the following equation:

$$f = h + P, (2.1)$$

where

$$h(t) = \frac{1}{g_1(t)} \frac{d^{\alpha}w(t)}{dt^{\alpha}},$$

and P is defined on $C(J)$ by:

$$P(t) = -\frac{1}{g_1(t)}\ (Au(t),v) - \frac{1}{g_1(t)}\ (F(t,W(t)),v).(2.2)$$

Theorem 2.1. Suppose that the conditions $(A_1 - A_4)$ are satisfied. Then the following assertions are valid :

(I) If the inverse problem (1.1) is solvable, then so equation (2.1) has a solution $f \in C(J)$,

(II) If equation (2.1) has a solution $f \in C(J)$ and the compatibility condition

$$(u_o, v) = w(0), (2.3)$$

holds, then the inverse problem (1.1) - (1.3) is solvable.

Proof. Assume that the inverse problem (1.1) - (1.3) is solvable. Multiplying both sides of (1.1) by v scalarly in H, we obtain the relation

$$\frac{d^\alpha}{dt^\alpha}(u(t),v) = (Au(t),v) + f(t)g_1(t) + (F(t,W(t),v).(2.4)$$

To prove assertion (II), we notice that by the assumption, equation (2.1) has a solution $f \in C(J)$. When inserting this function in (1.1), the resulting problem (1.1), (1.2) can be treated as a direct problem . Using previous results [8], this solution is given by

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0\, d\theta$$

$$+\alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta)[f(s)g(s) + F(s,W(s)]d\theta\, ds, (2.5)$$

Using previous results [8,9], we can see that the solution $u(t)$ of equation (2.5)exists and unique in S. Let us prove now that u satisfies the overdetermination condition (1.3). In this case u and f are known, consequently (2.4) will represent an identity,

$$f(t)\, g_1(t) = \frac{d^\alpha w(t)}{dt^\alpha} - (Au,v) - (F(t,W),v).(2.6)$$

Subtracting equation (2.4) from (2.6), one gets

$$\frac{d^\alpha w(t)}{dt^\alpha} = \frac{d^\alpha}{dt^\alpha}\ (u(t),v)).$$

applying the fractional integral of order α and taking into account the compatibility condition (2.3), we find out that u satisfies the overdetermination condition (1.3) and that the pair $\{u, f\}$ is a strictly solution of the inverse problem (1.1) - (1.3). This completes the proof of the theorem, [10-16].

Theorem 2.2. Let the conditions $(A_1 - A_4)$ and the compatibility condition (2.3) hold, then there exists a unique strictly solution of the inverse problem (1.1) - (1.3).

Proof. Substituting from (2.1) into (1.1), one gets:

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + F(t,W(t)) + g(t)h(t) - \frac{1}{g_1(t)}\ g(t)(u, A^* v) - \frac{1}{g_1(t)} g(t)(F(t,W(t),v)).(2.7)$$

Using similar techniques as in [8] , we deduce that the direct Cauchy problem (1.2) , (2.7) has a unique strong solution u. To prove the uniqueness of u and f, we assume to the contrary that there were two different solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$ of the inverse problem (1.1) - (1.3). We claim that in this case $f_1 \neq f_2$ for all points of J. In fact if $f_1 = f_2$ on J then applying the uniqueness theorem to the corresponding direct problem (1.2), (2.7) , we would have $u_1 = u_2$. Since both pairs satisfy identity (2.4), the functions f_1 and f_2 give two different solutions of equation (2.7). But this contradicts the uniqueness of solutions to equation (2.1). This completes the proof of the theorem.

3. APPLICATIONS

Consider the nonlinear integro-partial differential equation of fractional order;

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \sum_{|q|=2m} a_q(x)\, D_x^q\, u(x,t) = F(x,t,W) + f(t)\, g(x,t),\ \ 0 < t \le T(3.1)$$

with the initial condition

$$u(x,0) = u_o(x)\,, (3.2)$$

where $x \in R^n$, R^n is the n-dimensional Euclidem space, $D_x^q = D_{x_1}^{g_1} ... D_{x_n}^{q_n}, D_{x_i} = \frac{\partial}{\partial x_i}, q = (q_1,...,q_n)$ is an n-dimensional multi - index,

$$|q| = q_1 + ... + q_n \,, W = (w_1,..., w_r)$$

and

$$w_j(x,t) = \sum_{|q| \le 2m-1} b_{qj}(x,t)\, D_x^q\, u(x,t) + \int_Y \sum_{|q| \le 2m-1} c_{qj}\,(x,y,t)\, D_y^q\, u(y,t)\, dy\,,$$

where Y is an open subset of R^n.

Let $L_2(R^n)$ be the set of all square integrable functions on R^n. We denote by $C^m(R^n)$ the set of all continuous real valued functions defined on R^n, which have continuous partial derivatives of order less than or equal to m. By $C_0^m(R^n)$ we denote the set of all functions $f \in C^m(R^n)$ with compact supports. Let $H^m(R^n)$ be the completion of $C_0^m(R^n)$ with respect to the norm

$$\| f \|_m = [\sum_{|q| \le m} \int_{R^n} |\, D_x^q\, f(x)\,|^2\, dx]^{\frac{1}{2}}.$$

It is supposed that the operator

$$A = \sum_{|q| = 2m} a_q\,(x)\, D_x^q$$

is uniformly elliptic. In other words it is supposed that all the coefficients $a_q, |q| = 2m$, are continuous and bounded on R^n and that there is a positive number c such that

$$(-1)^{m+1} \sum_{|q| = 2m} a_q\,(x)\, \zeta^q \ge c\,|\,\zeta\,|^{2m},$$

is satisfied for all $x \in R^n$ and all $\zeta \ne (0,...,0)$, $(\zeta^q = \zeta_1^{q_1} ... \zeta_n^{q_n}$, $|\,\zeta\,|^2 = \zeta_1^2 + ... + \zeta_n^2)$. It is supposed also that all the coefficients $a_q = 2m$, satisfy a Hölder condition on R_n.

Under these conditions the operator A with the domain of definition $S = H^{2m}(R^n)$ generates an analytic semigroup $Q(t)$ [17-21].

It is well known that $H^{2m}(R^n)$ is dense in $L_2(R^n)$.

If $g \in H^{2m}(R^n)$, then $Q(t)g = \varphi$ can be written in the form

$$\varphi(x,t) = \int_{R^n} G(x,y,t) g(y)\, dy\,,$$

where G is the fundamental solution of the Cauchy problem

$$\frac{\partial u}{\partial t} = Au, \quad u(x,0) = g(x).$$

It can be proved that

$$\| D_x^q\, Q(t) g \| \le \frac{K}{t^\gamma} \| g \|,$$

where $0 < \gamma < 1, K$ is a positive constant, $|q| \le 2m - 1,\ t > 0$.

and $\| g \|^2 = \int_{R^n} g^2(x) dx$.

The properties of the fundamental solution G can be found in [20,21].

To solve the Cauchy problem (3.1), (3.2), we suppose the following conditions;

(a) All the coefficients $b_{qj}, |q| \le 2m-1,\ j = 1,...,r$ are bounded continuous on $R^n \times J$ and satisfy the Hölder condition

$$\sup_{R^n} |\, b_{qj}(x,t_2) - b_{qj}(x,t_1)\,| \le K(t_2 - t_1)^\beta,$$

where K is a positive constant and $0 < \beta < 1$.

(b) The integrals $\int_{R^n}\int_Y c_{qj}^2(x,y,t)\,dy\,dx$ exit for all $t \in J$, $|q| \le 2m-1,\ j=1,...,r.$

(c) All the coefficients $c_{qj}, |q| \le 2m-1$, $j=1,...,r$ satisfy the Hölder condition $\int_{R^n}\int_Y [c_{qj}(x,y,t_2) - c_{qj}(x,y,t_1)]^2\,dy\,dx \le K(t_2-t_1)^{2\beta}$, where K is a positive constant and $0<\beta<1$.

The Cauchy problem (3.1), (3.2) can be written in the abstract form (1.1), (1.2), where A is the operator $\sum_{|q|=2m} a_q(x)\,D^q$ with domain of definition $S = H^{2m}(R^n)$. The operators $B_j(t)$ are defined by $B_j(t)u = u_j,\ j=1,...,r,$

where
$$u_j(x,t) = \sum_{|q|\le 2m-1} b_{qj}(x,t)\,D_x^q\,u(x,t) + \int_Y \sum_{|q|\le 2m-1} c_{qj}(x,y,t)\,D_y^q\,u(y,t)\,dy$$

The domain of definition of the operators $B_1(t),...,B_r(t)$ can be taken $S_1 = ... = S_r = H^{2m-1}(R^n)$, [14],[15].

We suppose that F satisfies the conditions (1.3) and (1.4) with respect to the norm in $L_2(R^n)$.

Now it is clear that we can apply theorems (2.1), (2.2) and (2.3) to the Cauchy problem (3.1), (3.2).

In other words the Cauchy problem (3.1), (3.2) has a unique solution in the space $H^{2m}(R^n)$. Also the considered problem is correctly formulated. We suppose that u satisfies the integral overdetermination condition:

$$\int_{R^n} u(x,t)v(x)dx = w(t), (3.3)$$

The functions u_o, v, w and g are known and satisfy the conditions of theorem 2 in the space $L_2(R^n)$. The function f is unknown. Applying theorems (2.1) and (2.2) we can see that the inverse problem (3.1)- (3.3) is uniquely solvable in the considered Sobolev space.

4. REFERENCES

[1]. I.A. Vasin, V.L. Kamysin. On the asymptotic behavior of solutions of inverse problems for parabolic equations. Siberian Math. J. 38 (1997) 647-662.

[2]. V.L. Kamysin, I.A. Vasin, Asymptotic behavior of solutions of inverse problems for parabolic equations with irregular coefficients, Soborink Math. 18 (1997) 371-387.

[3]. A.I. Prileplo. D.G. Orlovsky, L.A. Vasin, Methods for solving inverse problems in mathematical physics, Marcel Dekker Inc. 1999.

[4]. A.F. Guvenilir and V.K. Kalantrov, The asymptotic behavior of solutions to an inverse problem for differential operator equations, Mathematical and computer Modeling, 37(2003), 907-914.

[5]. A.I. Prilepko and D.S. Tkachenko, Properties of solutions of parabolic equation and the uniqueness of the solution of the inverse source problem with integral overdetermination, compt. Math.Math. phys. 43(4) (2003), 537-546.

[6]. J.R. cannon, P .Duchateav, Structural identification of an term in a heat equation, Inverse probl. 14 (1998) 535 - 551.

[7]. Mahmoud M. El-Borai, Some probability densities and fundamental solution of fractional evolution equations , Chaos, Soliton and Fractals 14 (2002) 433-440.

[8]. Mahmoud M. El-Borai, Semigroup and some nonlinear fractional differential equations, Appl. Math. and Computations, 149(2004) 823-831.

[9]. Mahmoud M. El-Borai, On some fractional evolution equations with nonlocal conditions, Int. J. Pure Appl. Math. 24(3)(2005) 405-415.

[10]. Mahmoud M. El-Borai , The fundamental solutions for fractional evolution equations of parabolic type, J. of Appl.Math. Stochastic Analysis (JAMSA) (2004) 199-211.

[11]. Mahmoud M. El-Borai , On some fractional evolution equations With non local conditions, International J. of Pure and Applied Mathematics, Vol. 24, No. 3, (2005) 405-413

[12]. Mahmoud M. El-Borai, On some fractional differential equations in the Hilbert-space, Journal of Discrete and Continuous Dynamical Systems, Series A, (2005) 233-241.

[13]. Mahmoud M. El-Borai, On some stochastic integro-differential equations, Advances in Dynamical Systems and Applications, Volume 1, Number 1 ,(2006) 49- 57.

[14]. Mahmoud M. El-Borai, Khairia El-Said El-Nadi, Hoda A. Fouad, On some fractional stochastic delay differential equations, Computers and Mathematics With Applications, 59(2010) 1165-1170.

[15]. Mahmoud M. El-Borai, Khairia El-Said El-Nadi, Eman G. El-Akabawy, On some fractional evolution equations, Computers and Mathematics With Applications, 59(2010) 1352-1355.

[16]. Mahmoud M. El-Borai, On the solvability of an inverse fractional abstract Cauchy problem, IJRRAS 4(4) September (2010) 411-415.

[17]. Khairia El-Said El-Nadi, On some stochastic parabolic differential equations in a Hilbert space, J. of Applied Mathematics and Stochastic Analysis, Vol.2 (2005) 167-179.

[18]. Khairia El-Said El-Nadi, On some stochastic differential equations and fractional Brownian motion. Int. J. of Pure And Applied Math., Vol.24, No.3. (2005) 415- 425.

[19]. Khairia El-Said El-Nadi, Asymptotic methods and difference fractional differential equations, Int. J. Contemp. Math. Sci., Vol1, No.1 (2006) 3-14.

[20]. S.D. Eidelman, Bounds for solutions of parabolic systems and some applications, Math.Sb. 1953, 33, 387-91.

[21]. S.D. Eidelman, On the fundamental solution of parabolic systems, Math. Sb.(1956), 38, 51-92.