A NOTE ON HADAMARD INEQUALITIES FOR THE PRODUCT OF THE CONVEX FUNCTIONS

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ABSTRACT

The main aim of the present note is to prove new Hadamard like integral inequalities for the product of the convex functions.

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1. INTRODUCTION

Let f be a real valued convex function defined on the interval $[a,b]$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known as Hadamard's inequality for the convex function. [1]
If $f = uv$ and u, v are convex functions then we have

$$u\left(\frac{a+b}{2}\right)v\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b u(x)v(x)dx \le \frac{u(a)v(a)+u(b)v(b)}{2} \tag{2}$$

by Hadamard inequality.
Therefore by Cauchy-Scwartz inequality

$$\frac{u(a)v(a)+u(b)v(b)}{2} \le \frac{\sqrt{u^2(a)+u^2(b)}\sqrt{v^2(a)+v^2(b)}}{2} \tag{3}$$

the inequality

$$\frac{1}{b-a}\int_a^b u(x)v(x)dx \le \frac{\sqrt{u^2(a)+u^2(b)}\sqrt{v^2(a)+v^2(b)}}{2} \tag{4}$$

holds.
Note that if u, v are convex functions, then the function $f = uv$ may not be convex function.

Example1. $u(x) = x^2, v(x) = (1-x)^2$ are convex functions defined on $[0,1]$,but the function $f(x) = u(x)v(x) = x^2(1-x)^2 = (x^2-x)^2$ is not convex function on this interval.

In fact, the first and second derivatives $f'(x)$, $f''(x)$ of $f(x)$ can be calculated by the formulas.

$$f'(x) = 2(x^2-x)(2x-1)$$

$$f''(x) = 2(2x-1)^2 + 4(x^2-x)$$

Therefore we have $f''\left(\frac{1}{2}\right) = -1$.

It means that the function $f = uv$ is not convex function on the interval $[0,1]$.

Example2. For the convex functions $u(x) = x^2, v(x) = (1-x)^2$ defined on $[0,1]$ we have

$$\frac{1}{b-a}\int_a^b u(x)v(x)dx = \int_0^1 (x^2-x)^2 dx = \frac{1}{30}$$

$u(0)v(0) = 0,$

$u(1)v(1) = 0.$

Hence the Hadamard inequality

$$\frac{1}{b-a}\int_a^b u(x)v(x)dx = \int_0^1 (x^2 - x)^2 dx = \frac{1}{30} \le \frac{u(0)v(0) + u(1)v(1)}{2} = 0$$

for the non-convex function $f = uv$ does not hold.. On the other hand, since

$u^2(0) = 0,$

$u^2(1) = 1,$

$v^2(0) = 1,$

$v^2(1) = 0$

the inequality

$$\frac{1}{b-a}\int_a^b u(x)v(x)dx = \int_0^1 (x^2 - x)^2 dx = \frac{1}{30} \le \frac{\sqrt{u^2(0) + u^2(1)}\sqrt{v^2(0) + v^2(1)}}{2} = \frac{1}{2}$$

holds. It means that although the function $f = uv$ is no convex function we have that the inequality (4) is true for these functions. Our aim is to investigate the inequality (4) when $f = uv$ is non-convex function.

2. MAIN RESULTS

Our main result is the following theorem.

Theorem. Let u and v are nonnegative convex functions defined on the interval $[a,b]$. Then the inequality (4) holds.

To prove of the theorem we need the following lemma.

Lemma. Let u is a nonnegative convex function defined on the interval $[a,b]$. Then the function u^2 is also convex function on the interval $[a,b]$

Proof. For arbitrary $x, y \in [a,b]$ and $k \in [0,1]$ we have

$(u(x) - u(y))^2 \ge 0$

$u^2(x) - 2u(x)u(y) + u^2(y) \ge 0$

$$2u(x)u(y) \le u^2(x) + u^2(y) \tag{5}$$

Multiplying both sides of the inequality (5) by $k(1-k)$ we get

$2k(1-k)u(x)u(y) \le k(1-k)u^2(x) + k(1-k)u^2(y)$

Therefore

$2k(1-k)u(x)u(y) \le (k - k^2)u^2(x) + [(1-k) - (1-k)^2]u^2(y)$

So

$2k(1-k)u(x)u(y) \le ku^2(x) - k^2u^2(x) + (1-k)u^2(y) - (1-k)^2u^2(y)$

Hence

$k^2u^2(x) + 2k(1-k)u(x)u(y) + (1-k)^2u^2(y) \le ku^2(x) + (1-k)u^2(y)$

Therefore

$$[ku(x) + (1-k)u(y)]^2 \le ku^2(x) + (1-k)u^2(y) \tag{6}$$

Since $u(x)$ is a nonnegative convex function we have

$u(kx + (1-k)y) \le ku(x) + (1-k)u(y)$

$$u^2(kx + (1-k)y) \le [ku(x) + (1-k)u(y)]^2 \tag{7}$$

From (6) and (7) we get

$$u^2(kx+(1-k)y) \le ku^2(x)+(1-k)u^2(y) \tag{8}$$

The inequality (8) proves that the function u^2 is a convex function.

Proof of the theorem.

By the lemma the functions u^2 and v^2 are convex functions. By the Hadamard inequality for these functions we have

$$\frac{1}{b-a}\int_a^b u^2(x)dx \le \frac{u^2(a)+u^2(b)}{2} \tag{9}$$

$$\frac{1}{b-a}\int_a^b v^2(x)dx \le \frac{v^2(a)+v^2(b)}{2} \tag{10}$$

Multiplying the inequalities (9) and (10) we get

$$\frac{1}{(b-a)^2}\int_a^b u^2(x)dx\int_a^b v^2(x)dx \le \frac{u^2(a)+u^2(b)}{2}\,\frac{v^2(a)+v^2(b)}{2} \tag{11}$$

By Cauchy-Scwartz inequality

$$\left(\int_a^b u(x)v(x)dx\right)^2 \le \int_a^b u^2(x)dx\int_a^b v^2(x)dx \tag{12}$$

Hence by (11) and (12) we get

$$\frac{1}{(b-a)^2}\left(\int_a^b u(x)v(x)dx\right)^2 \le \frac{u^2(a)+u^2(b)}{2}\,\frac{v^2(a)+v^2(b)}{2}$$

The last inequality means that the inequality (4) holds.

Example3. Prove the inequality $\int_\pi^{2\pi}\frac{\sin x+8}{x}dx \le 2\sqrt{10}$

Solution. Since the functions $u(x)=\sin x+8$ and $v(x)=\frac{1}{x}$ are nonnegative convex functions on $[\pi,2\pi]$ we have by inequality (4)

$$\frac{1}{\pi}\int_\pi^{2\pi}\frac{\sin x+8}{x}dx \le \frac{\sqrt{(\sin\pi+8)^2+(\sin 2\pi+8)^2}\sqrt{\frac{1}{\pi^2}+\frac{1}{4\pi^2}}}{2}$$

Hence

$$\frac{1}{\pi}\int_\pi^{2\pi}\frac{\sin x+8}{x}dx \le \frac{\sqrt{8^2+8^2}\sqrt{5}}{4\pi}$$

Therefore

$$\int_\pi^{2\pi}\frac{\sin x+8}{x}dx \le 2\sqrt{10}$$

Note that $f(x)=\frac{\sin x+8}{x}$ is not a convex function on $[\pi,2\pi]$.

3. REFERENCES

[1] J.E.Pecaric, F.Proschan and Y.L.Tong, "Convex Functions, Partial Orderings and Statistical Applications",Academic Press, New York (1991)