ON THE LOCAL PROPERTY OF $\left|N, p_n, \alpha_n\right|_k$ SUMMABILITY OF A FACTORED FOURIER SERIES

U.K. Misra[1], Mahendra Misra[2], B.P. Padhy[3] & S.K. Buxi[4]

[1]Department of Mathematics,Berhampur University,Berhampur – 760007,Orissa, India,
[2]Principal,Government Science College,Mlkangiri, Orissa, India.
[3]Roland Institute of Technology,Golanthara-760008,Orissa, India.
[4]Department of Mathematics,Gunupur Science College,Gunupur,Odisha, India.
Email: [1]umakanta_misra@yahoo.com , [2]mahendramisra@2007.gmail.com , [3]iraady@gmail.com

ABSTRACT

In this paper we have established a theorem on the local property of $\left|N, p_n, \alpha_n\right|_k$ summability of factored Fourier series.

Keywords: $\left|N, p_n\right|_k$ *-summability,* $\left|N, p_n, \alpha_n\right|_k$ *-summability,* $\left|\overline{N}, p_n\right|_k$ *-summability and Fourier series.*

AMS CLASSIFICATION NO: 40D25

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$.Let $\{p_n\}$be a sequence of positive real constants such that

(1.1) $$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \ as\ n \to \infty \ (P_{-i} = p_{-i} = 0, i \geq 1)$$

The sequence –to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu$$

defines $\left|N, p_n\right|$ -means of the sequence $\{s_n\}$generated by the sequence of coefficients $\{p_n\}$.The series $\sum a_n$ is said to be summable $\left|N, p_n\right|_k$, $k \geq 1$, if

(1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty .$$

For k=1, $\left|N, p_n\right|_k$ -summability is same as $\left|N, p_n\right|$ -summability.

When $p_n = 1$ *for all n and* $k = 1$, $\left|N, p_n\right|_k$ -summability is same as $\left|C,1\right|$ -summability.

Let $\{\alpha_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $\left|N, p_n, \alpha_n\right|_k$, $k \geq 1$, if

(1.4) $$\sum_{n=1}^{\infty} \alpha_n^{k-1} \left|t_n - t_{n-1}\right|^k < \infty .$$

A sequence $\{\lambda_n\}$is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$ .Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

(1.5) $$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

2. KNOWN THEOREMS

Dealing with the $\left|\overline{N}, p_n\right|_k$ -summability of an infinite series Bor**[1]** proved the following theorem:

THEOREM:

Let $k \geq 1$ and let the sequences $\{p_n\}$ *and* $\{\lambda_n\}$ be such that

(2.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.2) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty,$$

and

(2.3) $$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta\lambda_n| < \infty,$$

where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|\overline{N}, p_n\right|_k$ of the factored Fourier series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

In this paper we have proved a theorem on the local property of $\left|N, p_n, \alpha_n\right|_k$ summability of factored Fourier series:

3. MAIN THEOREM

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that

(3.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(3.2) $$\frac{P_{n-r-1}}{P_n} = O\left(\frac{p_{n-r-1}}{P_{n-1}} \frac{P_r}{p_r}\right),$$

(3.3) $$\sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \frac{p_{n-r}}{P_n} = O\left(\frac{p_r}{P_r}\right),$$

(3.4) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k}{n} < \infty,$$

and

(3.5) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\Delta\lambda_n|^k}{n} < \infty,$$

where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|N, p_n, \alpha_n\right|_k$, $k \geq 1$ of the factored Fourier series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property, where $\{\alpha_n\}$ is a sequence of positive numbers .

In order to prove the above theorem we require the following lemma:

4. LEMMA

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that the conditions **(3.1)-(3.5)**are satisfied. If $\{s_n\}$ is bounded then the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ is $|N, p_n, \alpha_n|_k$-summable, where $\{\alpha_n\}$ is a sequence of positive numbers .

5. PROOF OF THE LEMMA

Let $\{T_n\}$denote the $|N, p_n|$ -mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ .Then by definition we have

$$
\begin{aligned}
T_n &= \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}\sum_{r=0}^{\nu} a_r \lambda_r X_r \\
&= \frac{1}{P_n}\sum_{r=0}^{n} a_r \lambda_r X_r \sum_{\nu=r}^{n} p_{n-\nu} \\
&= \frac{1}{P_n}\sum_{r=0}^{n} a_r \lambda_r X_r \sum_{\nu=r}^{n} p_{n-\nu} \\
&= \frac{1}{P_n}\sum_{r=0}^{n} a_r P_{n-r} \lambda_r X_r
\end{aligned}
$$

Hence

$$
\begin{aligned}
T_n - T_{n-1} &= \frac{1}{P_n}\sum_{r=1}^{n} P_{n-r} a_r \lambda_r X_r - \frac{1}{P_{n-1}}\sum_{r=1}^{n-1} P_{n-r-1} a_r \lambda_r X_r \\
&= \sum_{r=1}^{n}\left(\frac{P_{n-r}}{P_n} - \frac{P_{n-r-1}}{P_{n-1}}\right) a_r \lambda_r X_r \\
&= \frac{1}{P_n P_{n-1}}\sum_{r=1}^{n}\left(P_{n-r}P_{n-1} - P_{n-r-1}P_n\right) a_r \lambda_r X_r \\
&= \frac{1}{P_n P_{n-1}}\left[\sum_{r=1}^{n-1}\Delta\left\{\left(P_{n-r}P_{n-1} - P_{n-r-1}P_n\right)\lambda_r X_r\right\}\right]\sum_{\nu=1}^{r} a_\nu \\
&= \frac{1}{P_n P_{n-1}}\left[\sum_{r=1}^{n-1}\left(p_{n-r}P_{n-1} - p_{n-r-1}P_n\right)\lambda_r X_r s_r \right. \\
&\qquad + \sum_{r=1}^{n-1}\left(P_{n-r-1}P_{n-1} - P_{n-r-2}P_n\right)\Delta\lambda_r X_r s_r \\
&\qquad \left. + \sum_{r=1}^{n-1}\left(P_{n-r-1}P_{n-1} - P_{n-r-2}P_n\right)\lambda_{r+1}\Delta X_r s_r \right]
\end{aligned}
$$

(by Abel's transformation)

$$
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} + T_{n,5} + T_{n,6} \quad (say).
$$

To complete the theorem by using Minokowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty}(\alpha_n)^{k-1}\left|T_{n,i}\right|^k < \infty \quad for\ i = 1,2,3,4,5,6.
$$

Now, we have

$$
\sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|T_{n,1}\right|^k \quad = \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|\frac{1}{P_n P_{n-1}}\sum_{r=1}^{n-1} p_{n-r}P_{n-1}\lambda_r X_r s_r\right|^k
$$

$$\leq \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \frac{1}{P_n} \left(\sum_{r=1}^{n-1} p_{n-r} |\lambda_r|^k |s_r|^k X_r^k \right) \left(\frac{1}{P_n} \sum_{r=1}^{n-1} p_{n-r} \right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m} |\lambda_r|^k X_r^k \sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \left(\frac{p_{n-r}}{P_n} \right)$$

$$= O(1) \sum_{r=1}^{m} |\lambda_r|^k X_r^k \frac{p_r}{P_r} \quad \text{,by(3.3)}$$

$$= O(1) \sum_{r=1}^{m} |\lambda_r|^k X_r^{k-1} \frac{p_r}{P_r} \frac{P_r}{r p_r}, \text{ as } X_n = \frac{P_n}{n p_n}$$

$$= O(1) \sum_{r=1}^{m} X_r^{k-1} \frac{|\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty \text{ ,by (3.4).}$$

Next,

$$\sum_{n=2}^{m+1} (\alpha_n)^{k-1} |T_{n,2}|^k = \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} p_{n-r-1} P_n \lambda_r X_r s_r \right|^k$$

$$\leq \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \frac{1}{P_{n-1}} \left(\sum_{r=1}^{n-1} p_{n-r-1} |\lambda_r|^k |s_r|^k X_r^k \right) \left(\frac{1}{P_{n-1}} \sum_{r=1}^{n-1} p_{n-r-1} \right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m} |\lambda_r|^k X_r^k \sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \left(\frac{p_{n-r-1}}{P_{n-1}} \right)$$

$$= O(1) \sum_{r=1}^{m} |\lambda_r|^k X_r^k \frac{p_r}{P_r} \quad \text{,by(3.3)}$$

$$= O(1) \sum_{r=1}^{m} |\lambda_r|^k X_r^{k-1} \frac{p_r}{P_r} \frac{P_r}{r p_r}, \text{ as } X_n = \frac{P_n}{n p_n}$$

$$= O(1) \sum_{r=1}^{m} X_r^{k-1} \frac{|\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty \text{ ,by (3.4).}$$

Further,

$$\sum_{n=2}^{m+1} (\alpha_n)^{k-1} |T_{n,3}|^k = \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} P_{n-r-1} P_{n-1} \Delta\lambda_r X_r s_r \right|^k$$

$$\leq \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \frac{1}{P_n} \left(\sum_{r=1}^{n-1} P_{n-r-1} |\Delta\lambda_r|^k |s_r|^k X_r^k \right) \left(\frac{1}{P_n} \sum_{r=1}^{n-1} P_{n-r-1} |\Delta\lambda_r| \right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m} |\Delta\lambda_r|^k X_r^k \sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \left(\frac{P_{n-r-1}}{P_n} \right)$$

$$\left(Since\ \frac{1}{P_n} \sum_{r=1}^{n-1} P_{n-r-1} |\Delta\lambda_r| \leq \sum_{r=1}^{n-1} |\Delta\lambda_r| = O(1) \right)$$

$$= O(1) \sum_{r=1}^{m} |\Delta\lambda_r|^k X_r^k \frac{p_r}{P_r} \quad \text{,by(3.3)}$$

$$= O(1)\sum_{r=1}^{m} |\Delta\lambda_r|^k X_r^{k-1} \frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{as } X_n = \frac{P_n}{np_n}$$

$$= O(1)\sum_{r=1}^{m} X_r^{k-1} \frac{|\Delta\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty \text{ ,by (3.5).}$$

Now,

$$\sum_{n=2}^{m+1} (\alpha_n)^{k-1} |T_{n,4}|^k = \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} P_{n-r-2} P_n \, \Delta\lambda_r X_r s_r \right|^k$$

$$\leq \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \frac{1}{P_{n-1}} \left(\sum_{r=1}^{n-1} P_{n-r-2} |\Delta\lambda_r|^k |s_r|^k X_r^k \right) \left(\frac{1}{P_{n-1}} \sum_{r=1}^{n-1} P_{n-r-2} |\Delta\lambda_r| \right)^{k-1}$$

$$= O(1)\sum_{r=1}^{m} |\Delta\lambda_r|^k X_r^k \sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \left(\frac{P_{n-r-2}}{P_{n-1}} \right) \text{ ,(as above)}$$

$$= O(1)\sum_{r=1}^{m} |\Delta\lambda_r|^k X_r^k \frac{p_r}{P_r} \text{ , by (3.3)}$$

$$= O(1)\sum_{r=1}^{m} |\Delta\lambda_r|^k X_r^{k-1} \frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{as } X_n = \frac{P_n}{np_n}$$

$$= O(1)\sum_{r=1}^{m} X_r^{k-1} \frac{|\Delta\lambda_r|^k}{r}$$

$$= O(1) \quad as\ m \to \infty \text{ , by (3.5).}$$

Again

$$\sum_{n=2}^{m+1} (\alpha_n)^{k-1} |T_{n,5}|^k = \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} P_{n-r-1} P_{n-1} \lambda_{r+1} \Delta X_r s_r \right|^k$$

$$= \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \sum_{r=1}^{n-1} \frac{P_{n-r-1}}{P_n} \lambda_{r+1} \Delta X_r s_r \right|^k$$

$$= \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \sum_{r=1}^{n-1} \frac{p_{n-r-1}}{P_{n-1}} \frac{P_r}{p_r} \lambda_{r+1} \Delta X_r s_r \right|^k \text{ ,by(3.2)}$$

$$= \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \sum_{r=1}^{n-1} \frac{p_{n-r-1}}{P_{n-1}} \frac{P_r}{p_r} \lambda_{r+1} s_r \frac{1}{r} \right|^k \text{ , by(3.1)}$$

$$= \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left| \sum_{r=1}^{n-1} \frac{p_{n-r-1}}{P_{n-1}} \frac{P_r}{p_r} \lambda_{r+1} s_r X_r \frac{p_r}{P_r} \right|^k \text{ ,as } X_n = \frac{P_n}{np_n}$$

$$= \sum_{n=2}^{m+1} (\alpha_n)^{k-1} \left\{ \sum_{r=1}^{n-1} \frac{p_{n-r-1}}{P_{n-1}} |\lambda_{r+1}|^k |s_r|^k X_r^k \right\} \left\{ \sum_{r=1}^{n-1} \frac{p_{n-r-1}}{P_{n-1}} \right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m} |\lambda_{r+1}|^k X_r^k \sum_{n=r+1}^{m+1} (\alpha_n)^{k-1} \left(\frac{p_{n-r-1}}{P_{n-1}} \right)$$

$$= O(1)\sum_{r=1}^{m}\left|\lambda_{r+1}\right|^{k} X_r^{k-1}\frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{ as } X_n = \frac{P_n}{np_n} \text{ and by}\textbf{(3.3)}$$

$$= O(1)\sum_{r=1}^{m}\frac{\left|\lambda_{r+1}\right|^{k}}{r} X_r^{k-1},$$

$$= O(1) \quad as\ m \to \infty \text{ ,}\textbf{by (3.4).}$$

Finally,

$$\sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|T_{n,6}\right|^{k} = \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|\frac{1}{P_nP_{n-1}}\sum_{r=1}^{n-1}P_{n-r-2}P_n\lambda_{r+1}\Delta X_r s_r\right|^{k}$$

$$= \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|\sum_{r=1}^{n-1}\frac{P_{n-r-2}}{P_{n-1}}\lambda_{r+1}\Delta X_r s_r\right|^{k}$$

$$= \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\frac{P_r}{p_r}\lambda_{r+1}\Delta X_r s_r\right|^{k} \quad \text{,}\textbf{by(3.2)}$$

$$= \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\frac{P_r}{p_r}\lambda_{r+1}s_r\frac{1}{r}\right|^{k} \quad \textbf{, by(3.1)}$$

$$= \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left|\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\frac{P_r}{p_r}\lambda_{r+1}s_r X_r\frac{p_r}{P_r}\right|^{k} \quad \text{,as } X_n = \frac{P_n}{np_n}$$

$$= \sum_{n=2}^{m+1}(\alpha_n)^{k-1}\left\{\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\left|\lambda_{r+1}\right|^{k}\left|s_r\right|^{k}X_r^{k}\right\}\left\{\sum_{r=1}^{n-1}\frac{p_{n-r-2}}{P_{n-2}}\right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m}\left|\lambda_{r+1}\right|^{k}X_r^{k}\sum_{n=r+1}^{m+1}(\alpha_n)^{k-1}\left(\frac{p_{n-r-2}}{P_{n-2}}\right)$$

$$= O(1)\sum_{r=1}^{m}\left|\lambda_{r+1}\right|^{k} X_r^{k-1}\frac{p_r}{P_r}\frac{P_r}{rp_r}, \text{ as } X_n = \frac{P_n}{np_n} \text{ and by}\textbf{(3.3)}$$

$$= O(1)\sum_{r=1}^{m}\frac{\left|\lambda_{r+1}\right|^{k}}{r} X_r^{k-1},$$

$$= O(1) \quad as\ m \to \infty \text{ ,}\textbf{by (3.4).}$$

This completes the proof of the Lemma.

5. PROOF OF THE THEOREM

Since the behavior of the Fourier series, as far as convergence is concerned , for a particular value of x depends on the behavior of the function in the immediate neighborhood of this point only, hence the truth of the theorem is necessary consequence of the Lemma.

REFERENCES

[1]. **H.Bor**., 1992, On the Local Property of $\left|\overline{N}, p_n\right|_k$ Summability of factored Fourier series ,journal of mathematical analysis and applications- **163(220-226)**.