NUMERICAL STUDY OF CONVECTION – DIFFUSION PROBLEM IN TWO- DIMENSIONAL SPACE

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ABSTRACT

The convection-diffusion problem in two-dimensional space is solved on a unit square mesh with the prescribed boundary conditions by finite difference method where in central difference scheme is employed. In the process finite difference scheme of Standard five point formula was employed. . Initial approximations to temperature distribution function were given on the basis suitable to physical nature of the problem by intuition. The results thus obtained are plotted through graphs and the physical nature of the problem is discussed. It is observed that there is a boundary layer at the specific values of arguments.

Key words: *Peclet number, exponential boundary layer, singular perturbation, sub characteristics, smooth region.*

1. INTRODUCTION

Consider the elliptic operator whose second order derivatives are multiplied by a parameter ε that is close to zero. These derivatives model diffusion while first-order derivatives are associated with the convective or transport process. In classical problems ε is not close to zero. This kind of problem was studied in the paper [14] in one-dimensional space. Here the two-dimensional convection-diffusion problem is studied. Diffusion term play an important role at the boundary layer near the arguments x=1, y=1 i.e. which makes rapid changes in the solution at the boundary layer. In the two-dimensional convection- diffusion problem the differential equation got converted to difference equation. The corresponding finite difference scheme is solved by using standard five point formula with the selected initial approximations. Here we have selected the relation between mesh size (h) and the perturbation parameter (ε) such a way that the numerical solution gives a stable solution. To summarize, when a standard numerical method is applied to a convection-diffusion problem, when there is too little diffusion then the computed solution is often oscillatory, while if there is superfluous (excess) diffusion term, the computed layers are smeared.
We can see that the solution of this problem has a convective nature on most of the domain of the problem, and the diffusive part of the differential operator is influential only in the certain narrow sub-domain. In this region the gradient of the solution is large. This nature is evidenced by steep down fall of solution near the boundary.
Convection is the process in which heat moves through a gas or a liquid as the hotter part rises and the cooler, heavier part sinks, where as in the diffusion a gas or liquid diffuses or is diffused in a substance, it becomes slowly mixed with that substance. In the linear convection-diffusion problem with variable co-efficient, transport mechanism dominates where as diffusion effects are confined to a reasonably small part of the domain. The co-efficient of diffusion causes the oscillations at the boundary layer. The solution pattern shows that at the boundary layer diffusion term play significant role. For the low Peclet number we may get the stable solution.

2. MOTIVATION AND HISTORY

The numerical solution of convection-diffusion problems dates back to the 1950s Allen and southwell [1], but only in 1970s it did acquire a research momentum that continued to this day. In the literature this field is still with many openings. Perhaps the most common source of convection-diffusion problem is the Navier–Stokes equation having nonlinear terms with large Reynolds number. Morton [10] pointed out that this is by no means the only place where they arise. He listed ten examples involving convection-diffusion equations that include the drift-diffusion equations of semiconductor device modeling and the Black–Sholes equation from financial modeling. He also observed that accurate modeling of the interaction between convective and diffusive processes is the most ubiquitous and challenging task in the numerical approximation of partial differential equations.

In this paper, the diffusion coefficient ε is a small positive parameter and coefficient of convection a(x,y) is continuously differentiable function that was denoted by Holder, continuous on $\bar{\Omega}$ the closure of Ω .

3. ANALYTICAL SOLUTION

In two dimensions, the convection-diffusion equation takes the form

$$-\varepsilon\,\Delta u(x,y) + a(x,y)\nabla u\,(x,y) + b(x,y)u(x,y) = f(x,y)$$

on $\Omega \subset R^2$ with u(x,y) = g(x,y) on $\partial\Omega$ (1)

Where 0 < ε <<1, and the functions a, b and f which are assumed to be Holder continuous on $\vec{\Omega}$, the closure of Ω. Here we also assume that $b \geq 0$ on $\vec{\Omega}$. Here Ω is any bounded domain in R^2 with a piecewise Lipchitz–continuous boundary $\partial\Omega$. Let us suppose that g is continuous except perhaps for a jump discontinuity at a single point.

The differential operator L is elliptic so (1) possess a solution in the region defined. Here L also satisfies the Maximum principle which is discussed in [9]. Assume that the absolute value of 'a' is close to 1 so that convection dominates diffusion. In the problem that we consider, the solution u(x,y) of (1) has an asymptotic structure similar to that of one-dimensional problem[14]. We can write u as the sum of the solutions to a first-order partial differential equation, at layer(s) and order O(ε) term.

To make this more precise, we divide the boundary $\partial\Omega$ into 3 parts

Inflow boundary $\partial^-\Omega = \{ x \in \partial\Omega : a.n < 0\}$,

Outflow boundary $\partial^+\Omega = \{ x \in \partial\Omega : a.n > 0\}$

Tangential flow boundary $\partial^0\Omega = \{x \in \partial\Omega : a.n = 0\}$, ……. (2)

Where **n** is the outward-pointing unit normal to $\partial\Omega$.

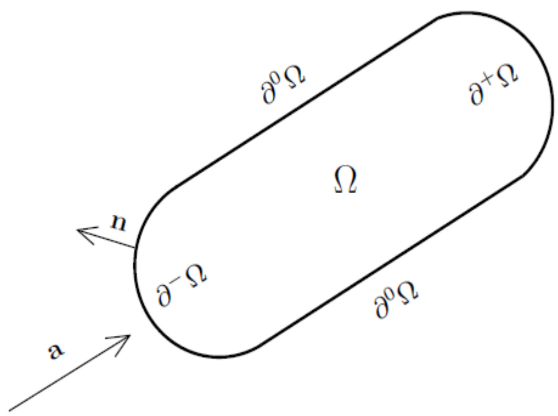

Figure 1 Partition of $\partial\Omega$.

A typical solution u will have boundary layers–narrow regions close to $\partial\,\Omega$ where $|\nabla u|$ is large along $\partial^+\Omega$ and $\partial^0\Omega$. As in one-dimensional problems exceptional Dirichlet boundary conditions on g can eliminate these layers. On most of Ω , u is approximately equal to $u_0(x,y)$. Then the solution of the reduced problem

$a(x,y)\nabla u_0\,(x,y) + b(x,y)u(x,y) = f(x,y)$ on Ω, u_0=g on $\partial^-\Omega$ (3)

This first-order partial differential equation (3) characteristic curves are the parameterized curves (x (t), y(t)) in Ω defined by

$$x^1(t) = a_1(x, y), y^1(t) = a_2(x, y) \qquad (4)$$

With initial data (x (0), y (0)) = (x′ ,y′) where (x′ ,y′) is any point in $\partial^-\Omega$. Thus one such curve emanates into Ω from each point in $\partial^-\Omega$.

4. Exponential Boundary Layers :

Consider the boundary value Problem (1)

$-\varepsilon \Delta u + b(x, y) \nabla u + c(x, y) u = f(x, y)$ in $\Omega = (0, 1) \times (0, 1)$,

$u = 0$ on the boundary Γ

Assume that the data are smooth and that $c \geq 0$ with

$b = (b_1, b_2)$ where $b_1 > 0$ and $b_2 > 0$.

Then the sub characteristics behave as in Figure. 2 and the reduced problem is defined by

$$b \cdot \nabla u_0 + cu_0 = f, \; u_0|_{x=0} = u_0|_{y=0} = 0. \qquad (5)$$

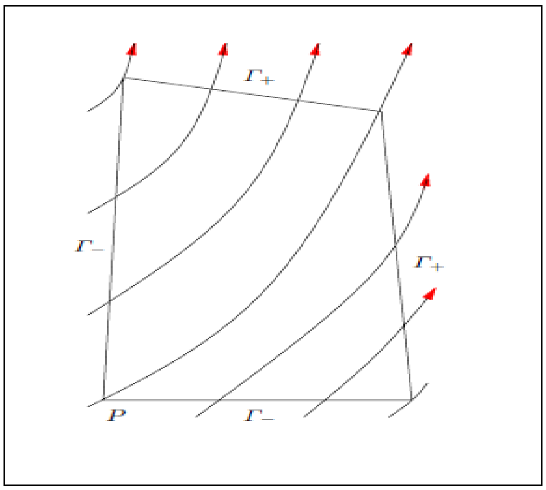

Figure..**2 Sub Characteristics through a Corner**

We expect exponential boundary layers at x = 1 and at y = 1. The asymptotic approximation with transformation ξ = (1-x) / ε and η = (1-y)/ε takes the form:

$$u_{Asy}^*(x, y) = u_0(x ,y) - u_0(1, y) \exp\left[-b_1(1, y)\frac{1-x}{\varepsilon}\right] - u_0(x,1)\exp\left[-b_2(x,1)\frac{1-y}{\varepsilon}\right] \qquad (6)$$

equation (6) is inaccurate near the corner point (1,1) because the boundary layer terms overlap there . Consequently we add a corner layer correction which is the solution of

$$-\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right) - b_1(1,1)\frac{\partial \omega}{\partial \xi} - b_2(1,1)\frac{\partial \omega}{\partial \eta} = 0 \quad \text{on } (0,\infty) \text{ X } (o, \infty)$$

With the use of the transformations ξ = (1-x) / ε and η = (1-y)/ε

We obtain

$$u_{Asy}(x,y) = u_{Asy}^*(x,y) + u_0(1,1) \exp\left[-b_1(1,1)\frac{(1-x)}{\varepsilon}\right]\exp\left[-b_2(1,1)\frac{1-y}{\varepsilon}\right]$$

if $u_0 \in c^2(\Omega) \cap C(\vec{\Omega})$, the classical comparison principle gives

$$\|u - u_{asy}\| \leq C\varepsilon \qquad (7)$$

Here C is generic constant which is independent on ε .Layers along $\partial^+\Omega$ are called regular or exponential boundary layers. Writing $\vec{n}$ =(n_1 , n_2) for the unit outward-pointing normal to the $\partial\,\Omega$, then near $\partial^+\Omega$, exponential layers are essentially multiples of the function Exp[-(**a.n**) d(x,y), $\partial^+\Omega$ / ε] Where d((x,y), denote the distance from the point (x,y) to the out-flow boundary $\partial^+\Omega$. Thus in cross-section perpendicular to $\partial^+\Omega$ these layers are very similar to the boundary layers that in one -dimension. Their first order derivatives in the direction perpendicular to the boundary have magnitude $O(\frac{1}{\varepsilon})$, and the width of the layer is $O(\varepsilon \ln(1/\varepsilon))$.

5. FINITE DIFFERENCE METHOD:

Consider the Two-dimensional convection-diffusion problem

$$-\varepsilon\,\Delta u(x,y)+\frac{\partial u}{\partial x}=1 \quad \text{Equivalently}$$

$$-\varepsilon[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}]+\frac{\partial u}{\partial x}=1 \text{ defined in the region } \Omega = (0,1)\text{ X }(0,1)$$

u(x,y) =0 on the boundary $\partial\Omega$ (8)

i.e., u(0,0) = 0 , u(i,0) = 0 , u(0,j) = 0 , u(1,1) = 0 i=1,2,3……….n

As a closed form solution, in general, is not possible so we solve the problem by using Discretization method. Discretize the above differential equation by using central difference approximations

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{h^2}[u_{i+1,j}-2u_{i,j}+u_{i-1,j}],\ \frac{\partial^2 u}{\partial y^2}=\frac{1}{k^2}[u_{i,j+1}-2u_{i,j}+u_{i,j-1}] \quad (9)$$

and

$$\frac{\partial u}{\partial x}=\frac{u_{i+1,j}-u_{i-1,j}}{2h} \quad (10)$$

Apply equation (9), (10) in (8) to get a difference equation is of the form with h = k on the square Region .

$$\frac{-\varepsilon}{h^2}[u_{i+1,j}-2u_{i,j}+u_{i-1,j}+u_{i,j+1}-2u_{i,j}+u_{i,j-1}]+\frac{u_{i+1,j}-u_{i-1,j}}{2h}=1$$

The final transformed difference scheme is

$$u_{i,j}=\frac{1}{8\varepsilon}[2h^2-(-2\varepsilon+h)\,u_{i+1,j}+(2\varepsilon+h)\,u_{i-1,j}+2\varepsilon u_{i,j+1}+2\varepsilon\,u_{i,j-1}] \quad (11)$$

Select $\varepsilon = 0.05$, h = 0.01 so that we can expect a stable solution. Apply the standard five point formula on (11) by selecting the initial approximations we can get values of u at each nodal point. The associated graph as plotted below for $\varepsilon = 0.05$

The values of u have been computed for the ranges of x =0 to x =1 and y=0 to y=1with spacing h=k=0.01. There are as many as 99x99 entries in the tabulated output. Here we are furnishing only selected values of u corresponding to : x=0 to x=0.1 , x=0.2,0.3, 0.4, 0.5, 0.6 ,0.7 ,0.8 ,0.9. Next values of u from x=0.9 x=1 are presented.

U-values

x-values ↓	y-values → 0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0	0	0	0	0	0	0	0	0	0	0	0
0.01	0	0.08516	0.15355	0.20461	0.24031	0.26388	0.2612	0.2617	0.2644	0.2766	0.281
0.02	0	0.15252	0.27489	0.29882	0.4282	0.469045	0.27869	0.28759	0.29274	0.29561	0.29717
0.03	0	0.19734	0.35617	0.29882	0.55394	0.60592	0.49433	0.50929	0.51781	0.5225	0.525
0.04	0	0.22099	0.39993	0.29882	0.62246	0.68051	0.63776	0.6564	0.6669	0.67259	0.6756
0.05	0	0.23024	0.41771	0.55635	0.65112	0.71183	0.71583	0.73637	0.74784	0.75404	0.75431
0.06	0	0.23269	0.42273	0.56482	0.65963	0.72121	0.74865	0.76998	0.78186	0.78825	0.80403
0.07	0	0.23306	0.42358	0.56482	0.66121	0.72297	0.75851	0.78009	0.79209	0.79854	0.80403
0.08	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76038	0.78201	0.79403	0.80049	0.80403
0.09	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.1	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.2	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.3	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.4	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.5	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.6	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.7	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.8	0	0.23309	0.42365	0.56482	0.66134	0.72312	0.76053	0.78215	0.7942	0.80066	0.80061
0.9	0	0.23293	0.42386	0.56211	0.65395	0.71856	0.74116	0.66777	0.79405	0.80056	0.7012
0.91	0	0.23274	0.42196	0.5593	0.64698	0.71414	0.72493	0.59541	0.79402	0.79984	0.701
0.92	0	0.23234	0.42014	0.5539	0.63427	0.70592	0.69724	0.55345	0.794	0.79899	0.7
0.93	0	0.2315	0.41656	0.54389	0.61204	0.69115	0.6521	0.50432	0.79399	0.79734	0.6989
0.94	0	0.22981	0.40978	0.52605	0.57492	0.66568	0.58233	0.5	0.78802	0.77796	0.68989
0.95	0	0.22652	0.39743	0.49574	0.51627	0.62374	0.48118	0.48997	0.78238	0.75976	0.6885
0.96	0	0.23038	0.376	0.49574	0.42941	0.5583	0.44878	0.48112	0.67	0.72924	0.68232
0.97	0	0.20944	0.34084	0.43377	0.31047	0.46257	0.3052	0.46128	0.6022	0.68025	0.60723
0.98	0	0.19102	0.28684	0.27189	0.16263	0.33297	0.3	0.42134	0.49614	0.49614	0.49978
0.99	0	0.11989	0.11186	0.14338	0.16163	0.17356	0.1794	0.18236	0.1838	0.18446	0.18477
1	0	0	0	0	0	0	0	0	0	0	0

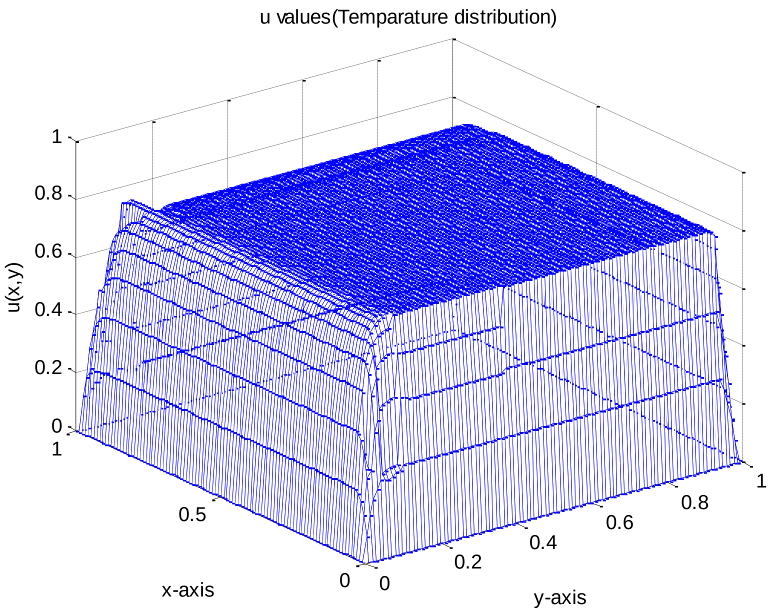

Figure.3

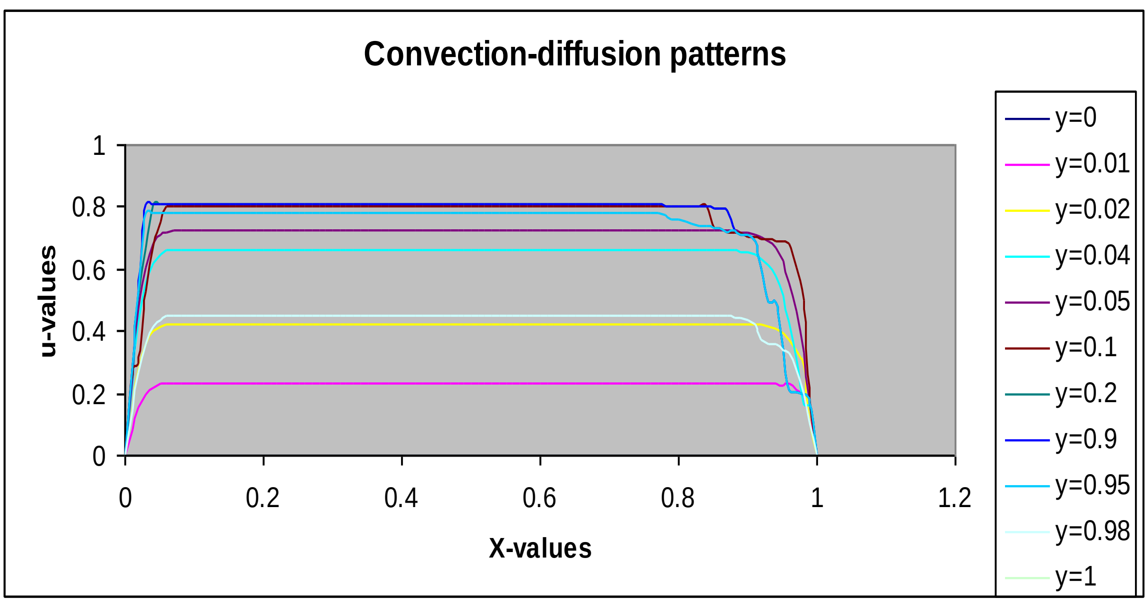

Figure. 4

6. RESULT ANALYSIS

The inflow boundary $\partial^- \Omega$ is the side $x = 0$ of $\vec{\Omega}$; the tangential flow boundary comprises of the sides $y = 0$ and $y = 1$; the outflow boundary is the remaining side

$x = 1$.From (4) each sub characteristic is parameterized by $x'(t) = 1, y'(t) = 0$

so that we can get $x = t$ as admissible solution and the sub characteristics are the lines $y = k$ (arbitrary).

On most of Ω From Figure.3, Figure. 4 it is evidenced that $u(x, y) \approx x$ in the region. The side $x = 1$ of $\vec{\Omega}$ is the outflow boundary $\partial^+\Omega$ and an exponential layer appears there. The tangential flow boundaries $y = 0$ and $y = 1$ have characteristic boundary layers that grow in strength as *x* moves from 0 to 1 because of the increasing discrepancy between u_0 and the boundary conditions. On most of the region Convection process dominates where as Diffusion process is visible only at the neighborhood of the corner point (1,1). For low Peclet number Convection process dominates in the region identified. When values of x are in the range 0.08-0.81 values of u are found to be constant for any choice of values of y means there is no effect of diffusion. Infact naturally u lies in the smooth region , as mentioned above , prior to the boundary layer region. For high Peclet number solutions are essentially of pure convection flows. The solution possesses an interior layer starting at (0, 0.8). On the boundary x=1 and on the right part of the boundary y=0 exponential layers are developed.

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