A TWO-ORDER AND TWO-SCALE COMPUTATION METHOD OF DAMAGE IN COMPOSITE MATERIALS

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ABSTRACT

This paper is aimed at developing a multi-scale analysis for obtaining numerical approximate solution to a boundary value problem describing damage phenomena in a brittle composite material. The multi-scale analysis based on mathematical homogenization is generalized to account for damage effects in heterogeneous media, and the stress update process is given. Numerical results of our model are found to be effective.

Keywords: *Damage parameter, Homogenization method, Multi-scale analysis, auxiliary functions, Finite element method.*

1. INTRODUCTION

Damage in composite materials occurs through different mechanisms that are complex and usually involve interaction between microconstituents. Various damage models for brittle composites can be classified into micromechanical and macromechanical approaches. In the macromechanical damage approach, homogenization method is introduced via obtaining an average field equation by constructing properly local smoothing operator [1, 2, 3, 4, 5]. The micromechanical damage approach, on the other hand, treats each microphase as a statistically homogeneous medium. Local damage variables are defined to represent the state of damage in each phase and phase effective material properties are defined thereafter [6, 7, 8, 9].

Multi-scale analysis for the periodic structures of composite materials has been discussed in [10, 11, 12, 13, 14, 15, 16]. Homogenized solution in a coarse mesh reflects macro-characteristic. High-order correction terms capture local micro-characteristic, and those terms only need compute in one period saving computation scale greatly. This provides a feasible framework for damage computing of heterogeneous composite materials.

The primary objective of the present manuscript is to extend the framework of the classical mathematical homogenization theory to multi-scale analysis accounting for damage effects. This is accomplished by introducing a double scale asymptotic expansion of displacement. The remainder of this paper is organized as follows: In section 2, we introduce the problem and multi-scale finite element approximate displacement solution. In Section 3, the stress update process are presented. In last section, some numerical results are reported, which support strongly that the multi-scale analysis is effective.

2. MULTI-SCALE ANALYSIS FOR DAMAGED COMPOSITES

Consider the boundary value problem describing damage phenomena in a brittle composite material [1], here $\Omega \subset R^n$ having an εY-periodic structure, that is a large number of periodically distributed inclusions with a scaling parameter ε,

$$\begin{cases} L_\varepsilon(u^\varepsilon) = \dfrac{\partial}{\partial x_j}[(1-\omega^\varepsilon(x,t))a_{ijhk}(\xi)\dfrac{1}{2}(\dfrac{\partial u_h^\varepsilon(x,t)}{\partial x_k}+\dfrac{\partial u_k^\varepsilon(x,t)}{\partial x_h})] = f_i(x,t) & in \quad \Omega \\ u^\varepsilon(x,t) = u(x,t) & on \quad \Gamma_1 \\ \sigma_i^\varepsilon(x,t) = \nu_j(1-\omega^\varepsilon(x,t))a_{ijhk}(\xi)\dfrac{1}{2}(\dfrac{\partial u_h^\varepsilon(x,t)}{\partial x_k}+\dfrac{\partial u_k^\varepsilon(x,t)}{\partial x_h}) = p_i(x,t) & on \quad \Gamma_2 \end{cases} \quad (1)$$

where $\omega^{\varepsilon}(x,t)$ is a scalar damage parameter, $u_h^{\varepsilon}(x,t)$ are components of displacement vector, $\xi = x/\varepsilon$, $a_{ijhk}(\xi)$ represents components of elastic stiffness satisfying conditions of symmetry

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$$

and positivity

$$\exists C_0 > 0 \qquad L_{ijkl}\xi_{ij}^{\varepsilon}\xi_{kl}^{\varepsilon} \geq C_0\xi_{ij}^{\varepsilon}\xi_{ij}^{\varepsilon} \qquad \forall \xi_{ij}^{\varepsilon} = \xi_{ji}^{\varepsilon}$$

for each $\varepsilon > 0$. Strain fields and Stress fields are defined as following:

$$\varepsilon_{ij}^{\varepsilon}(x,t) = (\frac{1}{2}(\frac{\partial u_h^{\varepsilon}(x,t)}{\partial x_k} + \frac{\partial u_k^{\varepsilon}(x,t)}{\partial x_h}) \tag{2}$$

$$\sigma_{ij}^{\varepsilon}(x,t) = (1-\omega^{\varepsilon}(x,t))a_{ijhk}(\xi)\frac{1}{2}(\frac{\partial u_h^{\varepsilon}(x,t)}{\partial x_k} + \frac{\partial u_k^{\varepsilon}(x,t)}{\partial x_h}) \tag{3}$$

We use two-order and two-scale asymptotic expansion to approximate the displacement solution of (1) when damage does not occur($\omega^{\varepsilon}(x,t) = 0$)

$$u_2^{\varepsilon}(x,t) \approx u^0(x,t) + \varepsilon N_{\alpha_1}(\xi)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}} + \varepsilon^2 N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2 u^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \tag{4}$$

where $u^0(x,t)$ is the homogenization solution and defined on global Ω, $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi)$ are n-order matrix valued functions with 1-periodicity, and they are defined on 1-square Q normalized basic cell

$$N_{\alpha_1}(\xi) = \begin{Bmatrix} N_{\alpha_1 11}(\xi) & \cdots & N_{\alpha_1 1n}(\xi) \\ \vdots & \cdots & \vdots \\ N_{\alpha_1 n1}(\xi) & \cdots & N_{\alpha_1 nn}(\xi) \end{Bmatrix} = \{N_{\alpha_1 1}(\xi)\cdots N_{\alpha_1 n}(\xi)\}$$

$$N_{\alpha_1\alpha_2}(\xi) = \begin{Bmatrix} N_{\alpha_1\alpha_2 11}(\xi) & \cdots & N_{\alpha_1\alpha_2 1n}(\xi) \\ \vdots & \cdots & \vdots \\ N_{\alpha_1\alpha_2 n1}(\xi) & \cdots & N_{\alpha_1\alpha_2 nn}(\xi) \end{Bmatrix} = \{N_{\alpha_1\alpha_2 1}(\xi)\cdots N_{\alpha_1\alpha_2 n}(\xi)\} \tag{5}$$

In fact, $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$ and $u^0(x,t)$ are determined as follows:

1) $N_{\alpha_1 m}(\xi)(\alpha_1, m = 1,\cdots,n)$ are the solutions of following problems

$$\begin{cases} \frac{\partial}{\partial \xi_j}[a_{ijhk}(\xi)\frac{1}{2}(\frac{\partial N_{\alpha_1 hm}(\xi)}{\partial \xi_k} + \frac{\partial N_{\alpha_1 km}(\xi)}{\partial \xi_h})] = -\frac{\partial a_{ijm\alpha_1}(\xi)}{\partial \xi_j} & in \quad \xi \in Q \\ N_{\alpha_1 m}(\xi) = 0 & on \quad \xi \in \partial Q \end{cases} \tag{6}$$

2) From $N_{\alpha_1 m}(\xi)(\alpha_1, m = 1,\cdots,n)$, the homogenization elasticity parameters $\{\hat{a}_{ijhk}\}$ are calculated as follows

$$\hat{a}_{ijhk} = \frac{1}{|Q|}\int_Q [a_{ijhk}(\xi) + a_{ijpq}(\xi)\frac{1}{2}(\frac{\partial N_{hpk}(\xi)}{\partial \xi_q} + \frac{\partial N_{hqk}(\xi)}{\partial \xi_p})]d\xi \tag{7}$$

3) $N_{\alpha_1\alpha_2 m}(\xi)(\alpha_1, \alpha_2, m = 1,\cdots,n)$ are the solutions of following problems

$$\begin{cases} \frac{\partial}{\partial \xi_j}[a_{ijhk}(\xi)\frac{1}{2}(\frac{\partial N_{\alpha_1\alpha_2 hm}(\xi)}{\partial \xi_k} + \frac{\partial N_{\alpha_1\alpha_2 km}(\xi)}{\partial \xi_h})] = \hat{a}_{i\alpha_2 m\alpha_1} & \\ -a_{i\alpha_2 m\alpha_1}(\xi) - a_{i\alpha_2 hk}(\xi)\frac{\partial N_{\alpha_1 hm}(\xi)}{\partial \xi_k} - \frac{\partial}{\partial \xi_j}(a_{ijh\alpha_2}(\xi)N_{\alpha_1 hm}(\xi)) & in \quad \xi \in Q \\ N_{\alpha_1\alpha_2 m}(\xi) = 0 & on \quad \xi \in \partial Q \end{cases} \tag{8}$$

4) $u^0(x,t) \in H^1(\Omega)$ is the solution of the homogenization problem defined on global Ω with the homogenized parameters $\{\hat{a}_{ijhk}\}$

$$\begin{cases} \frac{\partial}{\partial x_j}[\hat{a}_{ijhk}\frac{1}{2}(\frac{\partial u_h^0(x,t)}{\partial x_k}+\frac{\partial u_k^0(x,t)}{\partial x_h}] = f_i(x,t) & in \quad \Omega \\ u^0(x,t) = u(x,t) & on \quad \Gamma_1 \\ \sigma_i(x,t) = \nu_j \hat{a}_{ijhk}\frac{1}{2}(\frac{\partial u_h^0(x,t)}{\partial x_k}+\frac{\partial u_k^0(x,t)}{\partial x_h}) = p_i(x,t) & on \quad \Gamma_2 \end{cases} \tag{9}$$

Where boundary constraint is $\Gamma_1 \cap \Gamma_2 = 0,\ \Gamma_1 \cup \Gamma_2 = \partial\Omega$.

All definitions of the auxiliary functions of two-order and two-scale approximate solution are given. Now we can obtain two-order and two-scale approximate displacement solution according to (4) using multi-scale finite element method [14, 15, 16]. The strains and stresses are approximately evaluated according to formulae (2)-(3).

3. COMPUTATIONAL ALGORITHMS FOR STRESSES

In this section, we use incremental theory due to the nonlinear character of the problem (1) when damage occurs. Problem (1) is described as linear set of equations in one increment step. In one incremental step, the isotropic damage state variable $\omega^\varepsilon(x,t)$ is assumed to be a monotonically increasing function of deformation history parameter $\kappa^\varepsilon(x,t)$ [17, 18, 19], which characterizes the ultimate deformation experienced throughout the loading history. In general, the evolution of matrix damage can be expressed as

$$\omega^\varepsilon(x,t) = f(\kappa^\varepsilon(x,t)) \tag{10}$$

The deformation history parameter $\kappa^\varepsilon(x,t)$ is determined by the evolution of damage equivalent strain, denoted by $\vartheta^\varepsilon(x,t)$, as follows

$$\kappa^\varepsilon(x,t) = \max(\vartheta^\varepsilon(x,\tau)\,|\,(\tau \le t), \kappa_i) \tag{11}$$

Where the threshold value for damage initiation in the matrix, κ_i represents the extreme value of the equivant strain prior to the initiation of damage. In the present manuscript the damage equivalent strain, $\vartheta^\varepsilon(x,t)$, is defined as square root of the damage energy rate [19]

$$\vartheta^\varepsilon(x,t) = \sqrt{\frac{1}{2}\sigma_{ij}^\varepsilon \varepsilon_{ij}^\varepsilon} \tag{12}$$

Taking

$$\omega^\varepsilon(x,t) = \frac{atan(\alpha\frac{\kappa^\varepsilon}{\kappa_0} - \beta) + atan(\beta)}{\frac{\pi}{2} + atan(\beta)} \tag{13}$$

Where α , β are material parameters; and κ_0 denotes the threshold of the stain history parameter beyond which the damage will develop very quickly. For simplicity, we set $\kappa_i = 0$. The details about damage parameter can be seen in [1].

The stress update process can be stated as follows:

Given: displacement vector ${}_t u^\varepsilon$, strain vector ${}_t \varepsilon^\varepsilon$, deformation history parameter ${}_t \kappa^\varepsilon$, damage parameter ${}_t \omega^\varepsilon$ and displacement increment Δu^ε calculated from the finite element analysis of the incremental equation.

Find:

i) Calculate strain increment $\Delta\varepsilon_{ij}^\varepsilon$ according to Δu^ε in one step, and then update whole strains through $\varepsilon_{ij}^\varepsilon = {}_t\, \varepsilon_{ij}^\varepsilon + \Delta\varepsilon_{ij}^\varepsilon$;

ii) Compute the damage equivalent strain ϑ^ε defined by (12) in terms of $\varepsilon_{ij}^\varepsilon$ and σ_{ij}^ε;

iii) If $\vartheta^{\varepsilon} >_t \kappa^{\varepsilon}$, then update κ^{ε} defined by (11) in terms of ϑ^{ε} and update for ω^{ε} in (??), Otherwise, $\omega^{\varepsilon} =_t \omega^{\varepsilon}$;

vi) Update stresses $\sigma_{ij}^{\varepsilon} = (1-\omega^{\varepsilon}) a_{ijhk}^{\varepsilon} \varepsilon_{ij}^{\varepsilon}$

4. NUMERICAL RESULTS

To illustrate the effectivity of the two-order and two-scale approximate displacement for strength and damage analysis of composites, we continue to consider the problem (1). Here domain Ω is shown in Figure 1(a), the periodicity unit cell Q is shown in Figure 1(b), $\varepsilon = 1/5$. The elastic constants are seen in Table 1.

Table 1. Some material parameters

	Young's modulus	*Poisson's ratio*
Matrix	241000	0.3
Inclusion	1625	0.25
Interface	241000	0.3

Table 2. Comparison with the numbers of elements and nodes

	Original equation	*Unit cell*	*Homogenized equation*
atrix	34400	2006	6000
Inclusion	7181	461	1331

Since it is difficult to find the analytic solution of (1), we have to replace $u^{\varepsilon}(x)$ with its FE solution in a very refined mesh. Now we implement the tetrahedron partition for Ω, and adopt adaptive partition method around the interface of matrix and inclusion. The numbers of tetrahedron and nodes are shown in Table 2. We take the undersurface is the Dirichlet zero boundary, four faces are the free boundaries, and the top face is the force boundary condition, denoting p. We consider two cases:

$$Case1: \; f(x) = sin\pi x \cdot sin\pi y \cdot sin\pi z, \; p = -700;$$

$$Case2: \; f(x) = 500000, \qquad p = -35;$$

Figure 2-3 describe the main stresses occuring damage. q0 is the stress obtained by homogenized displacement, q1 is the stress obtained by one-order and two-scale approximate displacement, q2 is the stress obtained by two-order and two-scale approximate displacement and Rq is the stress obtained by the displacement in a very refined mesh. We can see that the stress q2 is in good agreement with Rq. But the stress q0 and q1 have less effect approaching stress Rq. So the multi-scale method we present improve on the traditional damage computation precision comparison to that by homogenization. From Table 2, we can see the mesh partition numbers of two-order and two-scale approximate displacement solution are much less than that of refined FE solution. It means the approximate solution we present for stress can greatly save computer memory and CPU time, it is very important in engineering computation. All information show two-order and two-scale approximate solution for stress is effective to damage problem of composite materials.

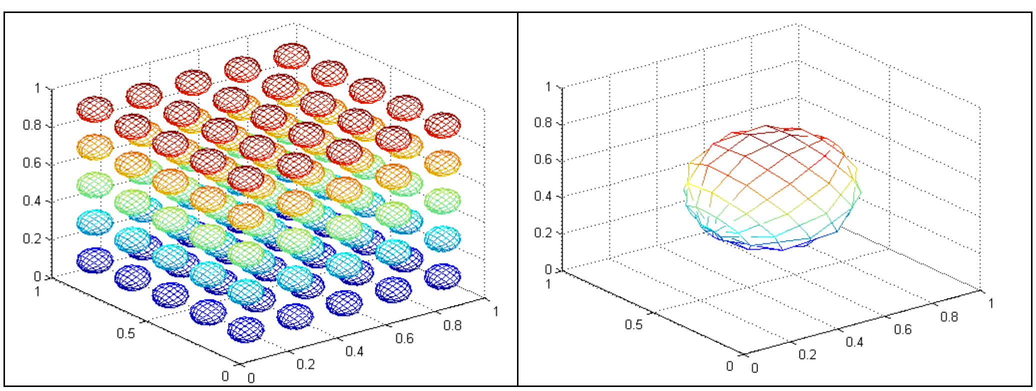

Figure 1. (a) unit cell Q=[0,1]^2 *Figure 1. (b) unit cell Q=[0,1]^2*

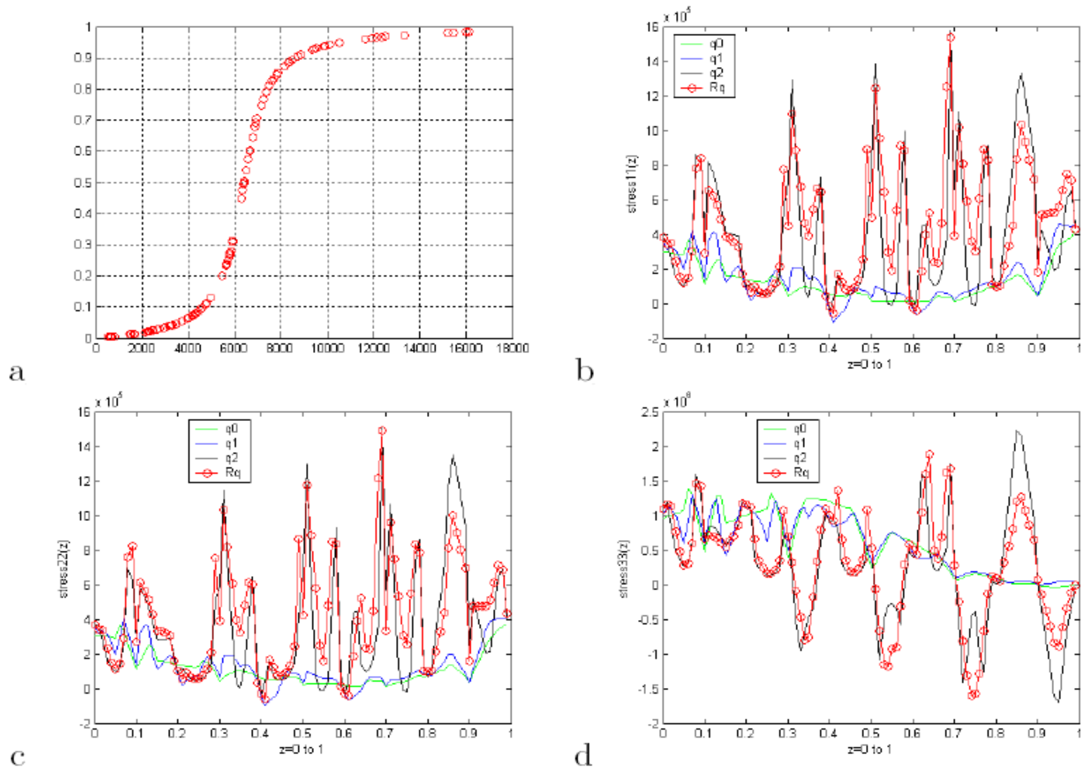

Figure 2. Case 1 (a) damage parameter ω (b) σ_{11} (c) σ_{22} (d) σ_{33}

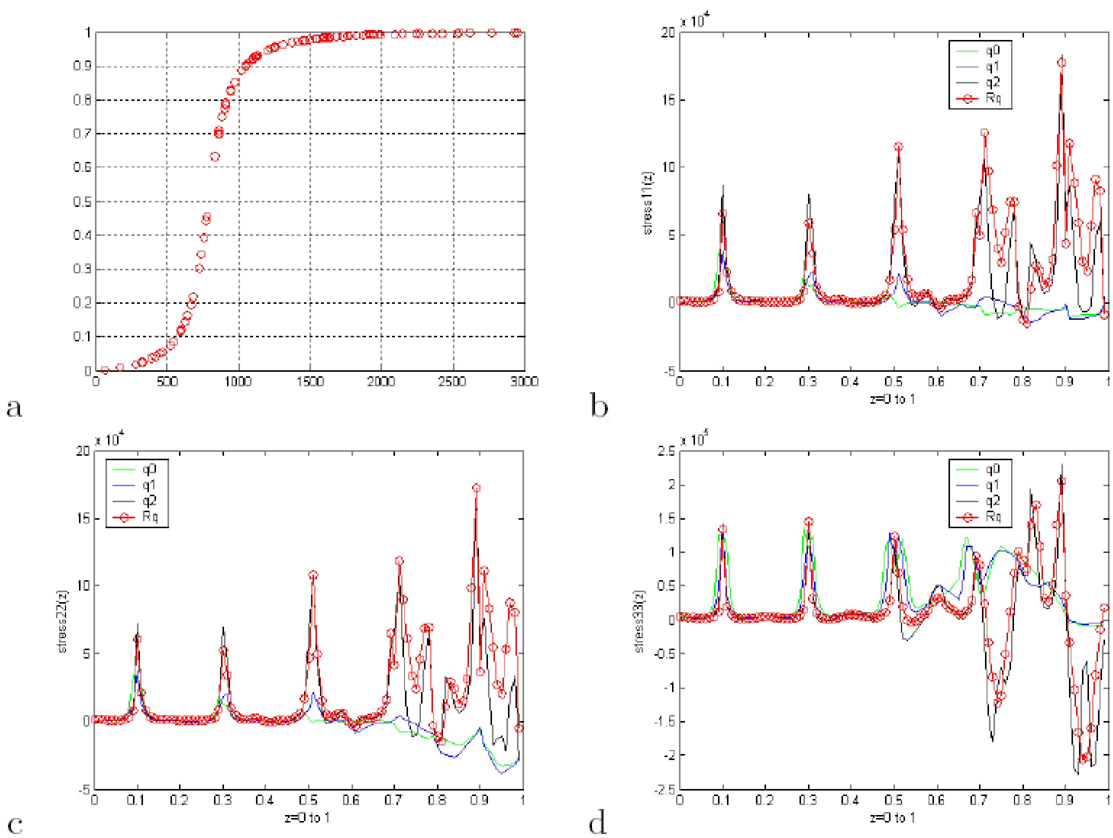

Figure3. Case 2 (a) damage parameter ω (b) σ_{11} (c) σ_{22} (d) σ_{33}

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6. REFERENCES

[1]. J. Fish, Q. Yu, and K. Shek, Computational damage mechanics for composite material based on mathematical homogenization, Int. J. Numer. Meth. Engrg., **45**, 1657-1679 (1999).

[2]. P. Ladeveze and E.L. Dantec, Damage modeling of the elementary ply for laminated composites, Composites Science and Technology, **43**, 257-267 (1992).

[3]. D.H. Allen, R.H. Jones and J.G. Boyd, Micromechanical analysis of a continuous fiber metal matrix composite including the effects of matrix viscoplasicity and evolving damage, J. Mech. Phys. Solids, **42(3)**, 505-529 (1994).

[4]. G.Z. Voyiadjis and P.I. Kattan, Micromechanical characterization of damage plasicity in metal matrix composites, Damage in Composite Materials, Elsevier Science: Amsterdam, 67-102 (1993).

[5]. G.Z. Voyiadjis and T. Park, A plasticity- damage theory for large deformation of solids- I: Theoretical Formulation. Int. J. Engrg. Sci., **30(9)**, 1089-1106 (1992).

[6]. G.Z. Voyiadjis and T. Park, Elasto-plastic stress and strain concentration tensors for damaged fibrous composites. In Damage and Interfacial Debonding in Composites, Elsevier Science: Amsterdam, 81-106 (1996).

[7]. A. Bensoussan, J. Lions and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Amsterdam, Nortth-Holland, 1978.

[8]. D. Cioranescu and P. Donato, An introduction to homogenization, Oxford University Press, Oxford, 1999.

[9]. J. Z. Cui and L. Q. Cao, Two-scale asymptotic analysis methods for a class of elliptic boundary value problems with small periodic coefficients, Math. Num. Sin., **21(1)**, 19-28 (1999).

[10]. O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, Mathematical problems in elasticity and homogenization, Amsterdam, North-Holland, 1992.

[11]. L. Q. Cao, J. Z. Cui and J. L. Luo, Multiscale asymptotic expansion and a post-processing algorithm for second-order elliptic problems with highly oscillatory coefficients over general convex domains, J. Comp. Appl. Math., **157(1)**, 1-29 (2003).

[12]. L. Q. Cao and J. Z. Cui, Finite element computation for elastic structures of composite materials formed by entirely basic configuration, J. Num. Math. Appl., **20(4)**, 25-37 (1998).

[13]. L. Q. Cao, J. Z. Cui and D. C. Zhu, Multiscale asymptotic analysis and numerical simulation for the second order Helmholtz with rapidly oscillating coefficients over general convex domains. SIAM J. Num. Anal., **40(2),** 543-577 (2003).

[14]. J. W. Ju, On energy-based coupled elastoplastic damage theories: constitutive modeling and computational aspects, International Journal of Solids and Structures, **25(7),** 803-833 (1989).

[15]. Krajcinovic, Damage Mechanics. Amsterdam, North-Holland, 1996.

[16]. Simo and J. W. Ju, Strain- and stress-based continuum damage models-I. Formulation, International Journal of Solids and Structures, **23(7)**, 821-840 (1987).