MOTION AROUND OBLATE VARIABLE MASS BODY

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ABSTRACT

In inter planetary missions, the motion around the sun or variable mas body is possible. In this work we will study the motion around the central oblate body of varying mass in the Hamiltonian mechanics framework. The asphericity of the central body is taken into account tell J_2. The Hamiltonian of the problem will be constructed in terms of Delaunay variable as canonical variables. According to the appearing of time explicitly, transformation to the augmented phase space must be dune. The Hamiltonian of the problem, in the augmented phase space, will be doubly averaged to eliminate, in successive, the short and long periodic variables. The elements of transformation and the inverse transformation will be outlined.The variation in the orbital elements will be studied analytically and graphically.

Keywords: *Varying mass, Canonical Transformation, Doubly averaged Hamiltonian.*

1. INTRODUCTION

In the case of varying mass problem, an ample bibliography is found in the published works of Polyakova, [1] and Prieto, [2]. The specific case which results in a slow isotropic mass loss has also been the focus of exhaustive studies carried out by researchers like, for instance, Hadjidemetriou, [3, 4], to name but a few. The vast majority of these, in search of the stellar application, have taken the so-called Eddington-Jeans law, Jeans, [5, 6], as a law of the variation of mass

$$\dot{m} = -\alpha\, m^n \,, \tag{1}$$

where α $(\alpha << 1) \in \mathbb{R}^+$ is very small real positive number, and $n \in [1.4, 4.4]$.

In the framework of celestial mechanics the problem of varying mass, has been exhaustively addressed by, Docobo et al, [7], Andrade and Docobo[8], Andrade [9], Andrade and Docobo [10], Rahoma et al. and [11], Rahoma et al., [12].

El-Saftawy and Algethami [13] published a paper treating the problem of varying mass in the canonical framework taking into consideration the periastron effect. In that work, the authors introduced the varying mass as unspecified new canonical variable with a new unspecified conjugate momentum in the extended phase space.

The Hamiltonian represents the two-body problem with varying mass, expressed in terms of Delaunay variable, which was derived firstly by Deprit, 1983, [14] is:

$$\mathcal{H}_1\left(l_1, L_1, L_2; t\right) = -\frac{1}{2}\frac{\mu^2}{L_1^2} + \frac{\dot{\mu}}{\mu} L_1\, e\, Sin\, E\,, \tag{2}$$

where E is the eccentric anomaly and the usual Delaunay variables related to the classical orbital elements through the relations:

$$l_1 = Mean\ anomaly,\quad l_2 = \omega,\quad l_3 = \Omega,\quad L_1 = \sqrt{\mu a},\quad L_2 = L_1\sqrt{1-e^2}\,,\quad L_3 = L_2\, Cos\, I\,,$$

with a, e, I, ω and Ω are, in respective, the semi-major axis, eccentricity, inclination, argument of pericenter and longitude of ascending node. $l_i's$ are considered as the coordinates while $L_i's$ are their corresponding conjugate momenta.
In this work, the variation of the gravitational parameter, μ is assumed due to the variation in the mass of the central body.

2. DEVELOPMENT OF THE HAMILTONIAN.

In this section, the Hamiltonian of our problem, will be developed to describe the total energy of the problem under consideration. The contribution for all parts of the perturbing potential will be derived.

2.1. Contribution of Varying Mass.

The Hamiltonian $\mathcal{H}_1$ represented by eqn. (2) depends implicitly on time through the variable mass in μ and its time derivative $\dot{\mu}$.
Since the variable mass can be expressed as a Taylor series expansion as:

$$\mu = \mu_0 + \frac{d\mu}{dt}\Big|_{t=t_0}\frac{(t - t_0)}{1!} + \frac{d^2\mu}{dt^2}\Big|_{t=t_0}\frac{(t - t_0)^2}{2!} + \ldots\ldots\ldots, \tag{3}$$

From Jeans law of varying mass, Eqn. (1) yields:

$$\frac{d\mu}{dt}\Big|_{t=t_0} = \dot{\mu}_0 = -\alpha\,\mu^n\Big|_{t=t_0} = -\alpha\,\mu_0^n\ ,$$

where μ_0 is the gravitational parameter of the system at time t_0 .

$$\frac{d^2\mu}{dt^2}\Big|_{t=t_0} = \ddot{\mu}_0 = -\alpha\, n\,\mu^{n-1}\,\dot{\mu}\Big|_{t=t_0} = \alpha^2 n\,\mu_0^{2n-1}.$$

Substituting the last two equations into Eqn. (3), retaining orders up to $O(\alpha)$, yields:

$$\mu = \mu_0 - \alpha\,\mu_0^n\,(t - t_0). \tag{4}$$

With the help of Eqn. (4), we can develop the Hamiltonian (2), up to $O(\alpha)$, to be:

$$\mathcal{H}_{VM} = -\frac{\mu_0^2}{2L_1^2} + \alpha\left[\frac{1}{L_1^2}\mu_0^{n+1}(t - t_0) - L_1\, e\,\mu_0^{n-1} Sin\, E\right] \tag{5}$$

2.2 Contribution of the Nonsphericity Variable Mass.

The Hamiltonian due to the nonsphericity of the central body was derived and published in many publications such as El-Saftawy, 1998 [15]. The contribution of the nonsphericity of the central body in the Hamiltonian function, up to J_2 in the zonal harmonics, will be:

$$\mathcal{H}_{Ob} = -\frac{\mu^4 r_e^2 J_2}{4L_1^6}\left(\frac{a}{r}\right)^3 \left[3S^2 \operatorname{Cos} 2(f + l_2) - (3S^2 - 2)\right], \tag{6}$$

where r_e is the equatorial redius of the centeral body, r is the distance of the orbiter from the center of the centeral body, f is the true anomaly and $S = \operatorname{Sin} I$.

The Hamiltonian $\mathcal{H}_{Ob}$, is implicitly dependent on time throught the true anomaly and the gravitational prameter, μ. Substituting from Eqn. (4) into Eqn. (6), rataining terms of order α, we get:

$$\mathcal{H}_{Ob} = -\frac{r_e^2 J_2}{4L_1^6}\left[\mu_0^4 - 4\alpha\, \mu_0^{3+n}(t - t_0)\right]\left(\frac{a}{r}\right)^3 \left[3S^2 \operatorname{Cos} 2(f + l_2) - (3S^2 - 2)\right]$$

The Hamiltonian of the problem is a combination between the Hamiltonians, due to the non-sphericity of the central body, $\mathcal{H}_{Ob}$, in addition to the Hamiltonian resulting from the variation of its mass, $\mathcal{H}_{VM}$. The final Hamiltonian of the problem, in terms of Delaunay variables, is:

$$\begin{aligned}\mathcal{H} = &-\frac{\mu_0^2}{2L_1^2} - \frac{r_e^2 J_2}{4L_1^6}\left[\mu_0^4 - 4\alpha\, \mu_0^{3+n}(t - t_0)\right]\left(\frac{a}{r}\right)^3 \left[3S^2 \operatorname{Cos} 2(f + l_2) - (3S^2 - 2)\right] + \\ &+\alpha\left[\frac{1}{L_1^2}\mu_0^{n+1}(t - t_0) - L_1\, e\, \mu_0^{n-1} Sin\, E\right]\end{aligned} \tag{7}$$

The equation of motion for the system will be:

$$\dot{l}_i = \frac{\partial \mathcal{H}}{\partial L_i} \quad \text{and} \quad \dot{L}_i = -\frac{\partial \mathcal{H}}{\partial l_i} \qquad i = 1,2,3 \tag{8}$$

2.3 The Hamiltonian in the Extended Phase Space.

Since the Hamiltonian represented by Eqn. (7) is explicitly time dependent, so we have to extend the phase space by introducing a new pair of variables (l_4, L_4). The first is l_4, [$l_4 = \mu_0^n (t - t_0)$], assigned as the variable mass and the second is its conjugate momentum, L_4, which describe the momentum rising due to the variation of the mass.

The new systems of canonical equations of motion will be:

$$\dot{l}_i = \frac{\partial \mathcal{K}}{\partial L_i} \quad \text{and} \quad \dot{L}_i = -\frac{\partial \mathcal{K}}{\partial l_i}, \qquad i = 1,2,3,4 \tag{9}$$

with $\mathcal{K}$ is the new Hamiltonian in the extended phase space that expressed as:

$$\begin{aligned}\mathcal{K} &= \mu_o^n L_4 + \mathcal{H} \\ &= -\frac{\mu_0^2}{2L_1^2} + \mu_o^n L_4 - \frac{r_e^2 J_2}{4L_1^6}\left[\mu_0^4 - 4\alpha\, \mu_0^3\, l_4\right]\left(\frac{a}{r}\right)^3 \left[3S^2 Cos\, 2(f + l_2) - (3S^2 - 2)\right] + \\ &+\alpha\left[\frac{\mu_0}{L_1^2} l_4 - L_1\, e\, \mu_0^{n-1} Sin\, E\right]\end{aligned} \tag{10}$$

To suit the transformation method, introduced by Lie-Deprit-Kamel ([16] and [17]), the Hamiltonian (10) must be developed in the expandable form as:

$$\mathcal{K} = \sum_{j \geq 0} \frac{\varepsilon^j}{j!} \mathcal{K}_j . \tag{11}$$

Now, let $\varepsilon = J_2$ be considered as the small parameter of the problem, then J_2 is assumed 1st order quantity. According to the value of the small parameter α, as published by Andrade and Docobo, 2003 [18], $\alpha = 10^{-6}$. Therefore the small parameter α, can assumed as second order of J_2 and the Hamiltonian (10) can be written in the form (11), with:

$$\mathcal{K}_o = -\frac{\mu_0^2}{2 L_1^2} + \mu_o^n L_4 . \tag{12.1}$$

$$\mathcal{K}_1 = -\frac{A_1}{L_1^6} \left(\frac{a}{r} \right)^3 \left[3 S^2 Cos\, 2 \left(f + l_2 \right) - \left(3 S^2 - 2 \right) \right]. \tag{12.2}$$

$$\mathcal{K}_2 = A_2 \left[\frac{1}{L_1^2} l_4 - L_1\, e\, \mu_0^{n-2} Sin\, E \right]. \tag{12.3}$$

$$\mathcal{K}_3 = \frac{A_3\, l_4}{L_1^6} \left(\frac{a}{r} \right)^3 \left[3 S^2 Cos\, 2 \left(f + l_2 \right) - \left(3 S^2 - 2 \right) \right], \tag{12.4}$$

where A's are zero order, quantities defined as:

$$A_1 = \frac{r_e^2\, \mu_0^4}{4} . \qquad \& \qquad A_2 = \frac{2 \alpha\, \mu_0}{J_2^2} . \qquad \& \qquad A_3 = \frac{6\, r_e^2\, \mu_0^3\, \alpha}{J_2^2} .$$

The second term in Eq. (12.1) is rising from varying mass on the Keplerian case. The part of Hamiltonian, (12.2), is the same as the effect of the zonal harmonics, J_2, in constant mass problem. Eq. (12.3) is exactly the contribution of pure varying mass problem, while Eq. (12.4) represents the coupling effect between the oblateness of the central body and its varying mass.

3. THE SOLUTION OF THE AVERAGED PROBLEM.

If the Hamiltonian function $\mathcal{K} \equiv \mathcal{K}(u_i, U_i)$, with the integrable part of the Hamiltonian, $\mathcal{K}_0$, is function of $U_i, (i = 1,4)$. So the variables $u_i\ (i = 1,4)$ can be considered as the fast variables.

Hori's method, [19], developed by Kamel, [20], is used to eliminate the short period termes from the Hamiltonian. If we denote the transformed Hamiltonian by, $\mathcal{K}^*$, which can be written up to k^{th} orders as follows:

$$\mathcal{K}_0^* = \mathcal{K}_0 \left(U'_1, U'_4 \right) \tag{13.1}$$

$$\mathcal{K}_k^* = \tilde{\mathcal{K}}_k + \mathcal{L}_k \mathcal{K}_0^* . \tag{13.2}$$

$$\tilde{\mathcal{K}}_k = \mathcal{K}_k + \sum_{j=1}^{k-1}\left[\binom{k-1}{j} G_j \mathcal{K}^*_{k-j} + \binom{k-1}{j-1} \mathcal{L}_j \mathcal{K}_{k-j}\right] \tag{13.3}$$

where $\mathcal{L}_j$ is the Lie derivatives generated by the *j* component of the generating function $\mathcal{S}$, and G_j is function of $\mathcal{L}_j$ and can be calculated using:

$$G_j = \mathcal{L}_j - \sum_{m=0}^{j-2}\binom{j-1}{m} \mathcal{L}_{m+1}\, G_{j-m-1}\,, \qquad 1 \le j \le k\,.$$

Since $u_i\,(i=1,4)$ is considired as the fast variables in $\mathcal{K}$, hence we choose $\mathcal{K}^*_k$ to be the secular part of $\tilde{\mathcal{K}}_k$. Then by averaging over $u_i\,(i=1,4)$ we get:

$$\mathcal{K}^*_k = \langle \tilde{\mathcal{K}}_k \rangle_{u_i'}\,. \quad (i=1,4) \tag{13.4}$$

While the periodic part of $\tilde{\mathcal{K}}_k$ can be calculated from:

$$\mathcal{P}_k = \tilde{\mathcal{K}}_k - \mathcal{K}^*_k = \left(\mathcal{K}_0\,;\mathcal{S}_k\right), \tag{13.5}$$

and then the generating function, for different order, can be calculated using:

$$\mathcal{S}_k = \left(\frac{\partial \mathcal{K}_0}{\partial U'_1}\right)^{-1} \int \mathcal{P}_k\, du'_1\,. \tag{13.6}$$

Noting that the prime over the variable to indicate the new variable in the transformed phase space.

3.1 Short Period Transformation.

The Hamiltonian (12) is including l_1, l_2 and l_4 as variable and theire conjugate momentum L_1, L_2 and L_4. The zero order Hamiltonian, $\mathcal{K}_0$, is function of L_1 and L_4. Therefore we can consider l_1 and l_4 as fast variables.

3.11Zero Order.

From Eqn.(13.1) directly we have:

$$\mathcal{K}^*_0 = -\frac{\mu_0^2}{2L'^2_1} + \mu_o^n\, L'_4\,. \tag{14}$$

3.12 First Order.

The first order transformed Hamiltonian, can be calculated, using equations (13.2) – (13.4), with *k* = 1, and by averaging the Hameltonian, $\mathcal{K}_1$, over the mean anomaly l_1. Therefore, the first order transformed Hamiltonian, $\mathcal{K}^*_1$ is

$$\mathcal{K}^*_1 = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{K}_1\, d\,l_1\,. \tag{15.1}$$

After calculating the required averages needed in the last equation we get:

$$\mathcal{K}_1^* = A_1 \eta'^{-3,-3}\left(3S'^2 - 2\right). \tag{15.2}$$

Where $\eta^{i,j}$ is defined as $L_1^i L_2^j$. The primes over the variable indicats for the new phase space variables, but for the sake of writing simplicity, we well drop it in the nixt subsections.

The periodic part of the first order can calculated, using Eqn. (13.5), as:

$$\mathcal{P}_1 = \mathcal{K}_1 - \mathcal{K}_1^* = -\left(\mathcal{K}_0 ; \mathcal{S}_1\right)$$

$$= -\frac{A_1}{L_1^6}\left(\frac{a}{r}\right)^3\left[3S^2 \operatorname{Cos} 2\left(f + l_2\right) - \left(3S^2 - 2\right)\right] - A_1 \eta^{-3,-3}\left(3S^2 - 2\right). \tag{15.3}$$

The generating function at this order can be derived, using Eqn. (13.6), to be in the form:

$$\mathcal{S}_1 = \left(\frac{\partial \mathcal{K}_0}{\partial L'_1}\right)^{-1} \int \mathcal{P}_1 dl'_1 .$$
$$= \frac{A_1}{\mu_0^2}\left\{\sum_{i=1}^{3}\left[B_{1,i}^C \, Cos\, i\, f + B_{1,i}^S \, Sin\, i\, f\right] + B_{1,0}\left(f - l_1\right)\right\}, \tag{15.4}$$

where the coeffecients $B_{1,i}^C$, $B_{1,i}^S$ and $B_{1,0}$ are function of $L_i\,'s$ and l_2 and given by:

$$B_{1,1}^C = -\frac{3eS^2}{2L_2^3} Sin\, 2l_2 \quad \& \quad B_{1,2}^C = -\frac{3S^2}{2L_2^3} Sin\, 2l_2 \quad \& \quad B_{1,3}^C = -\frac{eS^2}{2L_2^3} Sin\, 2l_2$$

$$B_{1,1}^S = -\frac{3eS^2}{2L_2^3} Cos\, 2l_2 + e\left(3S^2 - 2\right) \quad \& \quad B_{1,2}^S = -\frac{3S^2}{2L_2^3} Cos\, 2l_2 \quad \& \quad B_{1,3}^S = -\frac{eS^2}{2L_2^3} Cos\, 2l_2$$

$$B_{1,0} = \frac{1}{L_2^3}\left(3S^2 - 2\right).$$

3.13 Second Order.

The procedure to calculate the second order transformed Hamiltonian can be outlined, using equations (13-2) – (13-4), as follows:

$$\mathcal{K}_2^* = \tilde{\mathcal{K}}_2 + \left(\mathcal{K}_0^*, \mathcal{S}_2\right), \tag{16.1}$$

with,

$$\tilde{\mathcal{K}}_2 = \mathcal{K}_2 + \left(\mathcal{K}_1 + \mathcal{K}_1^* ; \mathcal{S}_1\right) \tag{16.2}$$

Because $\mathcal{K}_2^*$ is arbitrary, then we can shoose it to be the avarages of $\tilde{\mathcal{K}}_2$, so:

$$\mathcal{K}_2^* = \left\langle \mathcal{K}_2 + \left(\mathcal{K}_1 + \mathcal{K}_1^* ; \mathcal{S}_1\right)\right\rangle_{l_1} = \frac{1}{2\pi}\int_0^{2\pi}\left[\mathcal{K}_2 + \left(\mathcal{K}_1 + \mathcal{K}_1^* ; \mathcal{S}_1\right)\right] d\,l_1 \,, \tag{16.3}$$

where the average of the Poisson Bracket $\left(\mathcal{K}_1 + \mathcal{K}_1^* ; \mathcal{S}_1\right)$ can be easily calculated using Eqns. (12.2), (15.2) and (15.4). After performing the required mathematical manipulations, we get:

$$\left\langle\left(\mathcal{K}_1 + \mathcal{K}_1^* ; \mathcal{S}_1\right)\right\rangle_{l_1} = \frac{A_1^2}{\mu_0^2}\sum_{i=0}^{2}\gamma_{2i}^C \, Cos\, 2i\, l_2 \,, \tag{16.4}$$

with,

$$\gamma_0^C = \frac{3}{128}\Big[S^4\left(-1839\,\eta^{-1,-9} + 82\,\eta^{-3,-7} - 1152\,\eta^{-4,-6} + 45\,\eta^{-5,-5}\right) + 32S^2\left(147\,\eta^{-1,-9} - \right.$$
$$\left. -58\,\eta^{-3,-7} + 48\,\eta^{-4,-6} + 7\,\eta^{-5,-5}\right) + 32\left(-49\eta^{-1,-9} + 6\,\eta^{-3,-7} - 16\,\eta^{-4,-6} + 3\,\eta^{-5,-5}\right)\Big].$$

$$\gamma_2^C = \frac{1}{32}\Big[S^4\left(-972\,\eta^{-1,-9} + 4571\,\eta^{-3,-7} - 1971\,\eta^{-5,-5}\right) + S^2\left(648\,\eta^{-1,-9} - 2850\,\eta^{-3,-7} + \right.$$
$$\left. +1122\,\eta^{-5,-5}\right)\Big].$$

$$\gamma_4^C = \frac{1}{128} S^4\left(87\,\eta^{-1,-9} + 586\,\eta^{-3,-7} - 673\,\eta^{-5,-5}\right).$$

The average of $\mathcal{K}_2$, $\left\langle \mathcal{K}_2 \right\rangle_{l_1}$, can be calculated using Eqn. (12.3) and the results is:

$$\left\langle \mathcal{K}_2 \right\rangle_{l_1} = \frac{A_2}{L_1^2}\, l_4 \tag{16.5}$$

From Eqns. (16.4) and (16.5), the second order Hamiltonian will be constructed in the form:

$$\mathcal{K}_2^* = \frac{A_1^2}{\mu_0^2}\sum_{i=0}^{2}\gamma_{2i}^C \, Cos\, 2i\, l_2 + \frac{A_2}{L_1^2}\, l_4 = \mathcal{K}_{2Ob}^* + \mathcal{K}_{2VM}^* \tag{16.6}$$

The second order transformed Hamiltonian is composed of two parts. The first part is originated from pure oblateness, $\mathcal{K}_{2Ob}^*$, perturbations and the second part is originated from the perturbations due to variation of the mass, $\mathcal{K}_{2VM}^*$.

The next step is to construct the generating function for the second order. With the help of Eqn. (13.5) we can separate the periodic part to be:

$$\begin{aligned}\mathcal{P}_2 &= \tilde{\mathcal{K}}_2 - \mathcal{K}_2^* = \left(\mathcal{K}_0\,;\,\mathcal{S}_2\right)\\ &= -A_2\,L_1\,e\,\mu_0^{n-2} Sin\,E + \frac{A_1^2}{\mu_0^2}\left\{\sum_{k=-4}^{8}\left[\Gamma_k^C\,Cos\,k\,f + \Gamma_k^S\,Sin\,k\,f\right] + \left(f - l_2\right)\left[\Gamma_{2,0}^C Cos\,2\,f + \right.\right.\\ &\left.\left. + \Gamma_{2,0}^S\,Sin\,2\,f\right] - \sum_{i=0}^{2}\gamma_{2i}^C\,Cos\,2i\,l_2\right\}\\ &= \mathcal{K}_{2\mathcal{P}Ob} + \mathcal{K}_{2\mathcal{P}VM}\end{aligned}$$

The periodic part of the Hamiltonian, at this order, is composed also from two parts. The first part is rising from the oblatness of the body, $\mathcal{K}_{2\mathcal{P}Ob}$, while the second part is rising from the variation of the mass, $\mathcal{K}_{2\mathcal{P}VM}$.

The second order generating function will also be composed of two parts. The part rising from oblateness is called $\mathcal{S}_{2Ob}$, while the part rising from variation of the mass is called $\mathcal{S}_{2VM}$.

The first part of the generating function, $\mathcal{S}_{2Ob}$, can be calculated from:

$$\mathcal{K}_{2\mathcal{P}Ob} = -\left(\mathcal{K}_0\,;\,\mathcal{S}_{2Ob}\right) \quad \Rightarrow$$

$$\frac{\partial\mathcal{K}_0}{\partial L_1}\frac{\partial\mathcal{S}_{2Ob}}{\partial l_1} + \frac{\partial\mathcal{K}_0}{\partial L_4}\frac{\partial\mathcal{S}_{2Ob}}{\partial l_4} = \frac{A_1^2}{\mu_0^2}\left\{\sum_{k=-4}^{8}\left[\Gamma_k^C\,Cos\,k\,f + \Gamma_k^S\,Sin\,k\,f\right] + \left(f - l_2\right)\left[\Gamma_{2,0}^C.\right.\right.$$
$$\left.\left. .Cos\,2\,f + \Gamma_{2,0}^S\,Sin\,2\,f\right] - \sum_{i=0}^{2}\gamma_{2i}^C\,Cos\,2i\,l_2\right\}.$$

The coeffecients Γ_k^C, Γ_k^S , $\Gamma_{2,0}^C$ and $\Gamma_{2,0}^S$ are functions of the action variables only. After performing the required integrals and some mathematical manipulations, the required generating function, $\mathcal{S}_{2Ob}$, can be written as:

$$\mathcal{S}_{2Ob} = \frac{A_1^2}{\mu_0^4}\left\{\sum_{i=1}^{9}\left[D_{2,i}^C Cos\,i\,f + D_{2,i}^S Sin\,i\,f\right] + \sum_{i=0}^{3}\left(f - l_1\right)\left[D'^C_{2,i}\,Cos\,i\,f + D'^S_{2,i}\,Sin\,i\,f\right]\right\}, \tag{16.7}$$

where the coeffecients $D_{2,i}^C$, $D_{2,i}^S$, $D'^C_{2,i}$ and $D'^S_{2,i}$ are given by:

$$D_{2,i}^C = \sum_{j=-2}^{2}\chi_{2\,j,i} Sin\,2\,j\,l_2 \qquad \& \qquad D_{2,i}^S = \sum_{j=-2}^{2}\chi_{2\,j,i} Cos\,2\,j\,l_2$$

$$D'^C_{2,i} = \sum_{j=0}^{1}\chi'^C_{2\,j,i}\,Cos\,2\,j\,l_2 \qquad \& \qquad D'^S_{2,i} = \chi'^S_{2i} Sin\,2l_2,$$

the coeffecients $\chi_{2\,j,i}$, $\chi'^C_{2\,j}$ and χ'^S_{2i} are functions in the action variables only and are inserted in Appindix.

The second part of the generating function, $\mathcal{S}_{2VM}$, can be calculated from:

$$\mathcal{K}_{2\mathcal{P}VM} = -\left(\mathcal{K}_0 \,;\, \mathcal{S}_{2VM}\right) \;\Rightarrow\; \frac{\partial \mathcal{K}_0}{\partial L_1}\frac{\partial \mathcal{S}_{2VM}}{\partial l_1} + \frac{\partial \mathcal{K}_0}{\partial L_4}\frac{\partial \mathcal{S}_{2VM}}{\partial l_4} = -\, A_2\, L_1\, e\, \mu_0^{n-2}\, Sin\, E$$

Then the generating function rising from vriation of the mass, $\mathcal{S}_{2VM}$, for this order, is:

$$\mathcal{S}_{2VM} = A_2\, L_1^4\, e\, \mu_0^{n-4}\left(\frac{a}{r}\right) Cos\, E \tag{16.8}$$

From Eqns. (16.7) and (16.8) we get:

$$\mathcal{S}_2 = \frac{A_1^2}{\mu_0^4}\left\{\sum_{i=1}^{9}\left[D_{2,i}^{C}\, Cos\, i\, f + D_{2,i}^{S}\, Sin\, i\, f\right] + \sum_{i=0}^{3}\left(f - l_1\right)\left[D'^{\,C}_{2,i}\, Cos\, i\, f + D'^{\,S}_{2,i}\, Sin\, i\, f\right]\right\} + \\ + A_2\, L_1^4\, e\, \mu_0^{n-4}\left(\frac{a}{r}\right) Sin\, E \tag{16.9}$$

3.2 Elments of the Short Period Transformation and it's Inverse

The elements of the short period transformation, and their inverses, are to be constructed wing the relations.

$$l_i = l'_i + J_2\frac{\partial \mathcal{S}_1}{\partial L'_i} + \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2}{\partial L'_i} + \left(\frac{\partial \mathcal{S}_1}{\partial L'_i};\mathcal{S}_1\right)\right] \tag{17.1}$$

$$L_i = L'_i - J_2\frac{\partial \mathcal{S}_1}{\partial l'_i} - \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2}{\partial l'_i} + \left(\frac{\partial \mathcal{S}_1}{\partial l'_i};\mathcal{S}_1\right)\right] \tag{17.2}$$

And it's inverse are:

$$l'_i = l_i - J_2\frac{\partial \mathcal{S}_1}{\partial L'_i} - \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2}{\partial L'_i} - \left(\frac{\partial \mathcal{S}_1}{\partial L'_i};\mathcal{S}_1\right)\right] \tag{17.3}$$

$$L'_i = L_i + J_2\frac{\partial \mathcal{S}_1}{\partial l'_i} + \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2}{\partial l'_i} - \left(\frac{\partial \mathcal{S}_1}{\partial l'_i};\mathcal{S}_1\right)\right] \tag{17.4}$$

The required derivatives can be easily obtained.

3.3 Long Period Transformation.

In this section a canonical transformation is performed to eliminate the long - period terms, (i.e. those periodic in l'_2) from the Hamiltonian. Then the elements of the transformation and its inverse will obtained. The procedure is essentially similar to that of the short- period transformation with the changes:

$$\left(l'_i, L'_i\right) \;\to\; \left(l''_i, L''_i\right) \quad , \quad \mathcal{S}_i \;\to\; \mathcal{S}^*_i \quad and \quad \mathcal{K}^* \to \mathcal{K}^{**}$$

In the following, all the variables are understood to be double primed, but the primes will dropped for the sake of simplicity of writing.

Proceeding as in the case of the short period transformation with the averages and integrations now performed over l'_2.

3.31 Zero Order.

The zero order is unchanged the primes are skipped for the sake of writing simplicity.

$$\mathcal{K}_0^{**} = -\frac{\mu_0^2}{2L_1^2} + \mu_o^n L_4 . \tag{18.1}$$

3.3.2 First Order.

Because the first order Hamiltonian, $\mathcal{K}_1^*$, dos not depend on the angle variables, so it is also unchanged. Then $\mathcal{K}_1^{**} = \mathcal{K}_1^*$,

$$\mathcal{K}_1^{**} = A_1 \eta^{-3,-3} \left(3S^2 - 2\right). \tag{18.2}$$

The first order generating function, $\mathcal{S}_1^*$,will be calculated from the next order.

3.3.3 Second Order.

The procedure to calculate the second order transformed Hamiltonian, $\mathcal{K}_2^{**}$, can be outlined, using equations (13-2) – (13-4), as follows:

$$\mathcal{K}_2^{**} = \tilde{\mathcal{K}}_2^* + \left(\mathcal{K}_0^{**}, \mathcal{S}_2^*\right) , \tag{19.1}$$

with,

$$\tilde{\mathcal{K}}_2^{**} = \mathcal{K}_2^* + \left(\mathcal{K}_1^* + \mathcal{K}_1^{**}; \mathcal{S}_1^{**}\right) , \tag{19.2}$$

choosing $\mathcal{K}_2^{**}$ to be the avarages of $\tilde{\mathcal{K}}_2^*$, so:

$$\mathcal{K}_2^{**} = \left\langle \mathcal{K}_2^* + \left(\mathcal{K}_1^* + \mathcal{K}_1^{**}; \mathcal{S}_1^{**}\right)\right\rangle_{l_2} = \frac{1}{2\pi}\int_0^{2\pi} \left[\mathcal{K}_2^* + \left(\mathcal{K}_1^* + \mathcal{K}_1^{**}; \mathcal{S}_1^{**}\right)\right] d\, l_2 \tag{19.3}$$

After performing the required mathematical manipulations, we get:

$$\begin{aligned}\mathcal{K}_2^{**} = {} & \frac{A_1^2}{\mu_0^2}\gamma_0^C + \frac{A_2}{L_1^2} l_4 = \frac{3A_1^2}{128\mu_0^2}\Big[S^4\left(-1839\eta^{-1,-9} + 82\,\eta^{-3,-7} - 1152\,\eta^{-4,-6} + \right.\\ & \left. 45\,\eta^{-5,-5}\right) + 32S^2\left(147\,\eta^{-1,-9} - 58\,\eta^{-3,-7} + 48\,\eta^{-4,-6} + 7\eta^{-5,-5}\right) + 32\big(-49.\\ & .\eta^{-1,-9} + 6\eta^{-3,-7} - 16\,\eta^{-4,-6} + 3\eta^{-5,-5}\big)\Big] + \frac{A_2}{L_1^2} l_4 .\end{aligned} \tag{19.4}$$

3.4 Elements of the Long Period Transformation and its Inverse

The elements of the short period transformation, and their inverses, are constructed using the relations.

$$l'_i = l''_i + J_2 \frac{\partial \mathcal{S}_1'}{\partial L''_i} + \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2'}{\partial L''_i} + \left(\frac{\partial \mathcal{S}_1'}{\partial L''_i}; \mathcal{S}_1'\right)\right] \tag{20.1}$$

$$L'_i = L''_i - J_2 \frac{\partial \mathcal{S}_1'}{\partial l''_i} - \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2'}{\partial l''_i} + \left(\frac{\partial \mathcal{S}_1'}{\partial l''_i}; \mathcal{S}_1'\right)\right] \tag{20.2}$$

And its inverse are:

$$l''_i = l'_i - J_2 \frac{\partial \mathcal{S}_1'}{\partial L''_i} - \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2'}{\partial L''_i} - \left(\frac{\partial \mathcal{S}_1'}{\partial L''_i}; \mathcal{S}_1'\right)\right] \tag{20.3}$$

$$L''_i = L'_i + J_2 \frac{\partial \mathcal{S}_1'}{\partial l''_i} + \frac{J_2^2}{2!}\left[\frac{\partial \mathcal{S}_2'}{\partial l''_i} - \left(\frac{\partial \mathcal{S}_1'}{\partial l''_i}; \mathcal{S}_1'\right)\right] \tag{20.4}$$

The required derivatives can be easily obtained.

4. THE VARIATION IN THE ELEMENTS.

The equation of motion, in terms of the Delaunay variable, can be constructed using Hamilton's equations of motion (9). But in terms of the orbital elements are:

$$\dot{l}_i = \frac{\partial \mathcal{K}^{**}}{\partial L_i} \text{ and } \dot{L}_i = -\frac{\partial \mathcal{K}^{**}}{\partial l_i}, \quad i = 1,2,3,4 \tag{21}$$

Where, $\mathcal{K}^{**}$ can be constructed using Eqns. (18.1), (18.2) and (19.4) and written as:

$$\begin{aligned}\mathcal{K}^{**} &= \sum_{i=0}^{2} \frac{J_2^i}{i!}\mathcal{K}_i^{**}.\\ &= -\frac{\mu_0^2}{2L_1^2} + \mu_o^n L_4 + \frac{r_e^2\,\mu_0^4\,J_2}{4}\eta^{-3,-3}\left(3S^2-2\right) + \frac{3r_e^4\mu_0^6 J_2^2}{4096}\Big[S^4\left(-1839\eta^{-1,-9} + 82\eta^{-3,-7} -\right.\\ &\left.-1152\eta^{-4,-6} + 45\eta^{-5,-5}\right) + 32S^2\left(147\eta^{-1,-9} - 58\eta^{-3,-7} + 48\eta^{-4,-6} + 7\eta^{-5,-5}\right) + 32\left(-49\eta^{-1,-9} +\right.\\ &\left.+6\eta^{-3,-7} - 16\,\eta^{-4,-6} + 3\eta^{-5,-5}\right)\Big] + \frac{\alpha\,\mu_0}{L_1^2} l_4.\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{e} &= \frac{1}{eL_1}\left[\frac{L_2}{L_1}\frac{\partial \mathcal{K}^*}{\partial l_2} - \left(\frac{L_2}{L_1}\right)^2\frac{\partial \mathcal{K}^*}{\partial l_1}\right] = \frac{1}{e}\frac{L_2}{L_1^2}\frac{\partial \mathcal{K}^*}{\partial l_2}\\ &= \frac{J_2^2 r_0^4\,S^2\sqrt{a\mu_0}\ \text{Sin}\,2\omega}{512a^6 e\left(1-e^2\right)^4}\Big[3e^4\left(657S^2-374\right) + e^2\left(629S^2-606\right) - 4\left(407S^2-270\right) +\\ &\qquad + \left(-760+673e^2\right)S^2\,e^2\,\text{Cos}\,2\,\omega\Big]\end{aligned} \tag{23.1}$$

$$\dot{I} = \frac{1}{L_2 S}\left[\frac{\partial \mathcal{K}^*}{\partial l_3} - C\frac{\partial \mathcal{K}^*}{\partial l_2}\right] = -\frac{C}{L_2 S}\frac{\partial \mathcal{K}^*}{\partial l_2} = -\frac{C\,e}{S\left(1-e^2\right)}\dot{e}. \tag{23.2}$$

$$\dot{\omega}=\frac{\partial\mathcal{K}^{*}}{\partial L_2}$$

$$=\frac{3a^2J_2r_0^2\sqrt{a\mu_0}}{4a^6\left(1-e^2\right)^2}\left(2-3S^2\right)+\frac{J_2^2r_0^4\sqrt{a\mu_0}}{4096a^6\eta^{11}}\Big\{4S^2\eta\Big[8508-13394S^2+15e^4\left(657S^2-374\right)+$$

$$+e^2\left(12287S^2-8730\right)\Big]\mathrm{Cos}\,2\omega+\left(3365e^4-2628e^2-1520\right)S^4\eta\,\mathrm{Cos}\,4\omega-768e^2. \qquad (23.3)$$

$$.\Big[24-9\eta-6S^2\left(12-7\eta\right)+2S^4\left(27-2\eta\right)\Big]+3e^4\Big[9S^4\left(768-25\eta\right)+96\left(32-5\eta\right)-$$

$$-32S^2\left(288+35\eta\right)\Big]+24.\Big[384\left(1+4\eta\right)-32S^2\left(119\eta+36\right)+S^4\left(864+1969\eta\right)\Big\}.$$

where, $\eta=\sqrt{1-e^2}$.

$$\dot{\Omega}=\frac{\partial\mathcal{K}^{*}}{\partial L_3}=0\,. \qquad (23.4)$$

$$\dot{l}_4=\frac{\partial\mathcal{K}^{*}}{\partial L_4}=\mu_0{}^{n} \qquad (23.5)$$

$$\dot{L}_4=\frac{\partial\mathcal{K}^{*}}{\partial l_4}=\frac{\alpha\mu_0}{a} \qquad (23.6)$$

5. GRAPHICAL ILLUSTRATION FOR THE SOLUTIONS.

In what follows, graphical presentations are given for the orbital variations with different, bounded, orbital values to illustrate the kinds and orders of the variations.

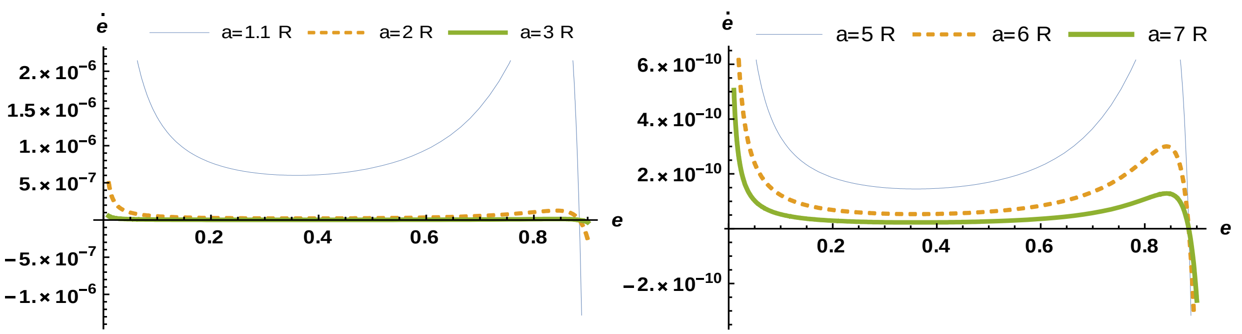

Fig.1 shows the variation of e against e for different orbital size with ω = Pi/3 and i =Pi/4

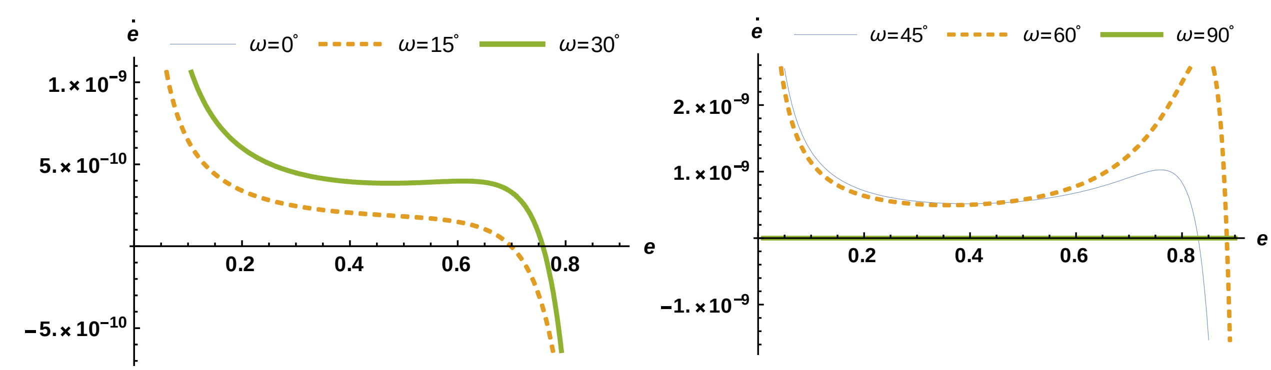

Fig.2 shows the variation of e against e for different ω with a = 4R and i =Pi/4.

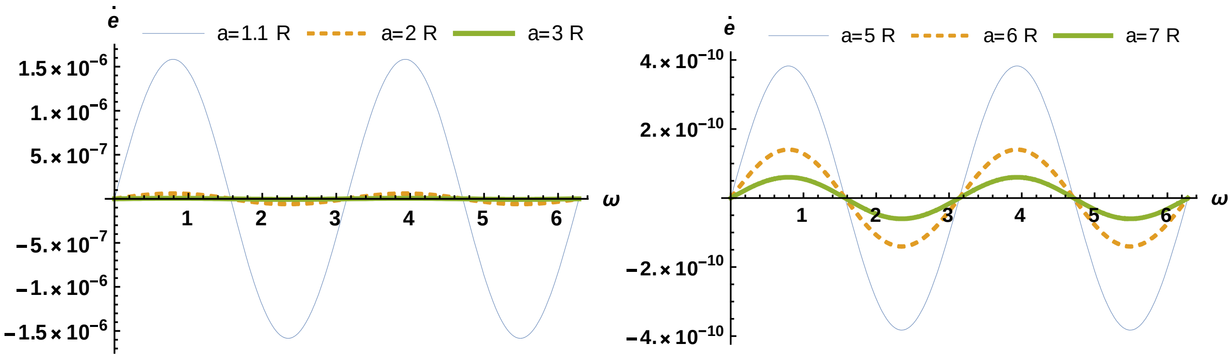

Fig.3 shows the variation of e against ω for different orbital size with e = 0.1 and i =Pi/4.

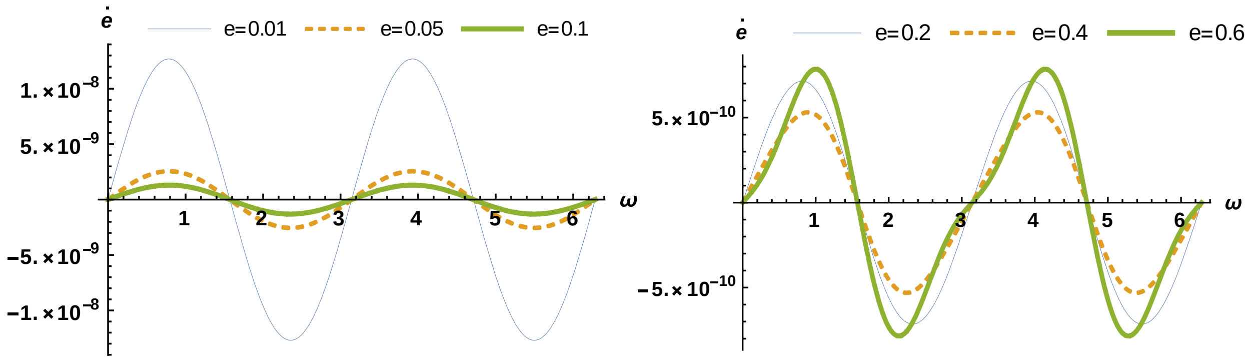

Fig.4 shows the variation of e against ω for different orbital shape with a = 4R and i =Pi/4.

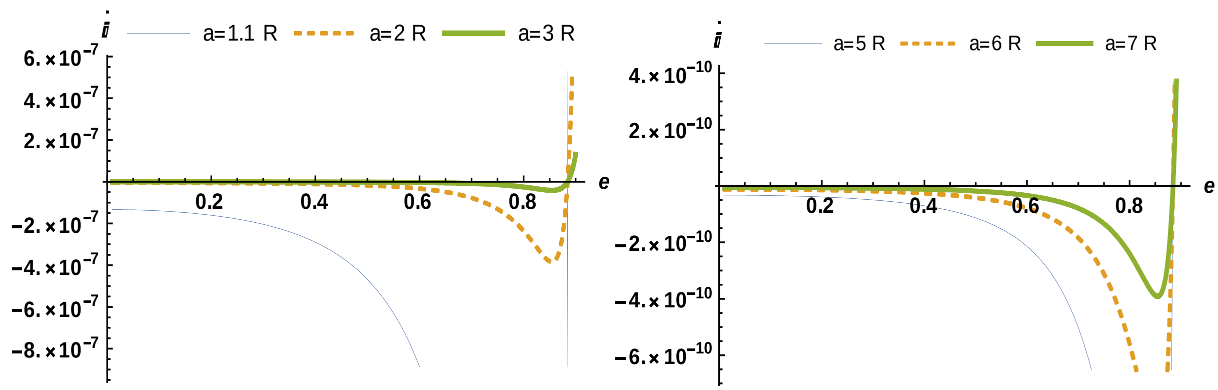

Fig.5 shows the variation of i against e for different orbital size with ω = Pi/3 and i =Pi/4.

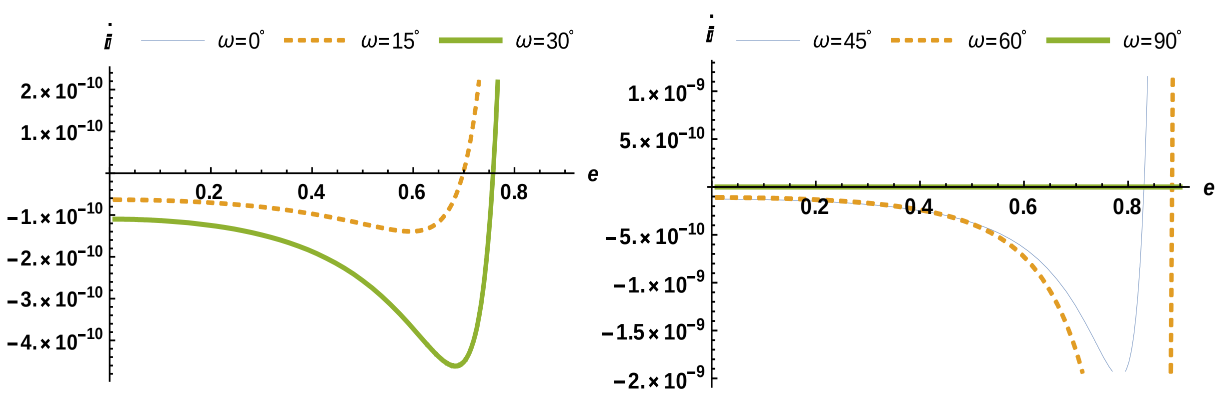

Fig.6 shows the variation of i against e for different ω with a = 4R and i =Pi/4.

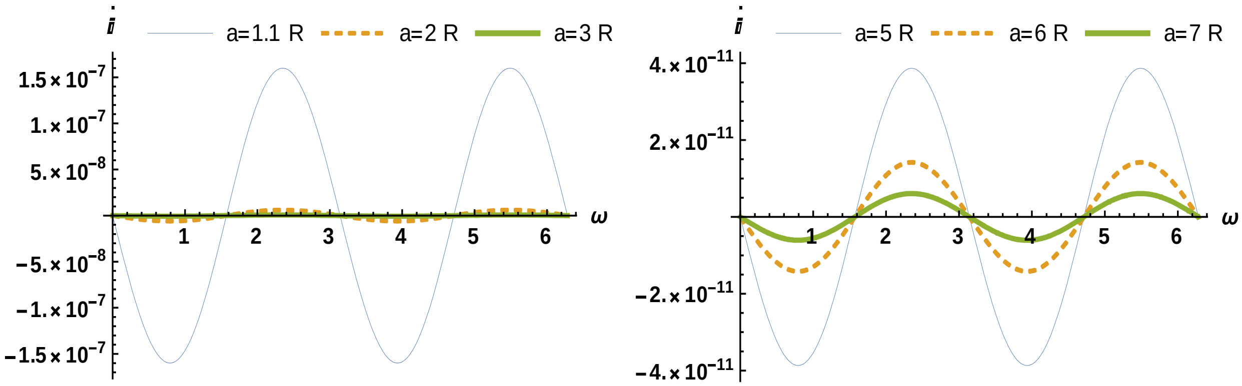

Fig.7 shows the variation of i against ω for different orbital size with e = 0.1 and i =Pi/4.

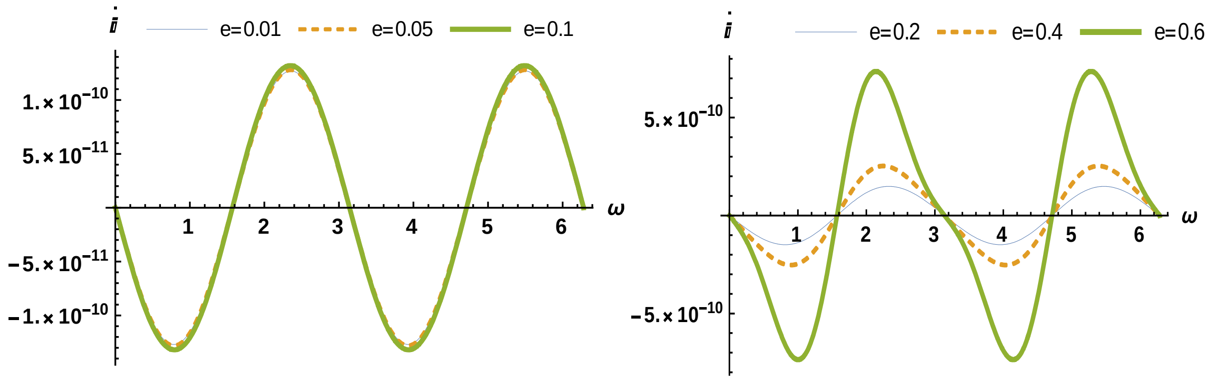

Fig.8 shows the variation of i against ω for different orbital shape with a = 4R and i =Pi/4.

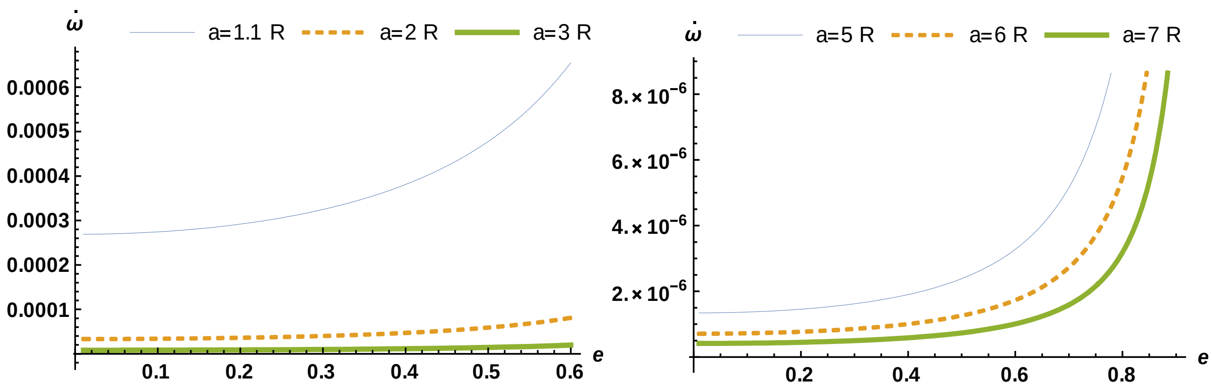

Fig.9 shows the variation of ω against e for different orbital size with ω = Pi/3 and i =Pi/4.

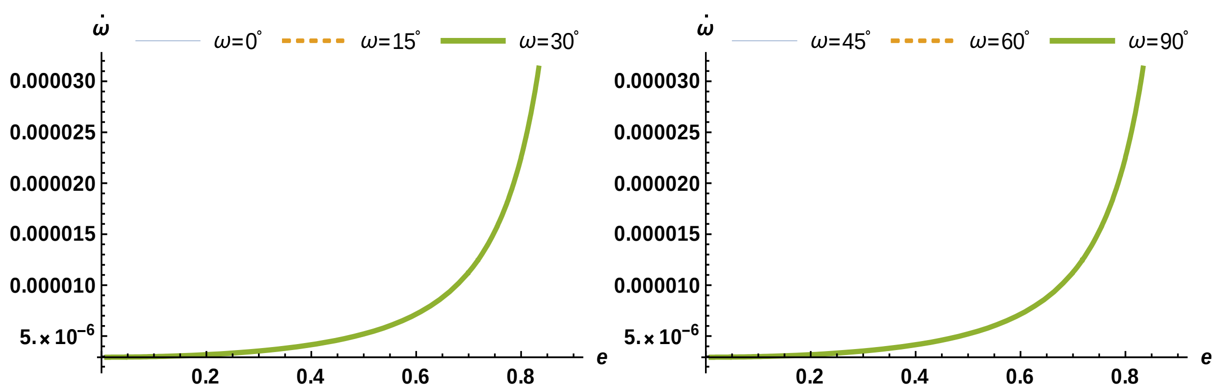

Fig.10 shows the variation of ω against e for different ω with a = 4R and i =Pi/4.

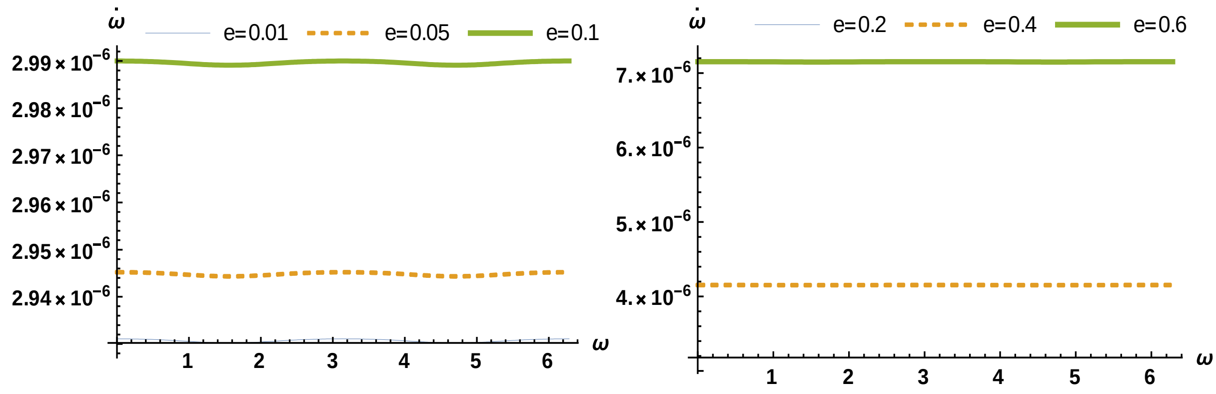

Fig.11 shows the variation of ω against ω for different orbital shape with a = 4R and i =Pi/4.

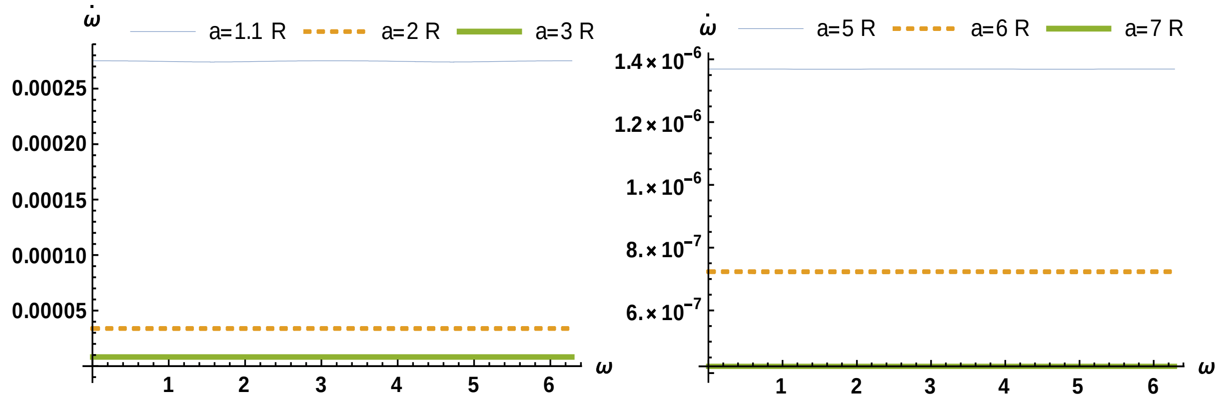

Fig.12 shows the variation of ω against ω for different orbital size with a = 4R and i =Pi/4.

6. CONCLUSION AND DISCUSIONS.

In this work, we studeed the motion around oblate varying mass body up to order J_2^2 assuming the second zonal harmonic, J_2 , of order 10^{-3}. The total energy of the system represented by the Hamiltonian was derived up to O(3) as in Eqn. (12). Two canonical transformations was constructed to eleminate in sucssive the short and long period terms. The trasformed Hamiltonian, K^{**} as in eqn. (22), was derived with the ellements of transformations and its inverse transformation. As it cleare from Eqn. (23.6), there is a momentun is rising due to the variation of the mass which is depend, directly, on the initial mass of the varying mass and also depend, inversly, on the simi-major axis of the orbiter.

From Figures 1 – 4, we represent the variation of eccentricity and we note the following:

i) The order of variations varies from 10^{-6} to 10^{-10}.
ii) It is inverse proportional with *e*, except near the parabolic orbit.
iii) It is periodic with ω with no considerable change of the order.

iv) The major changes of the order occur with the semi-major axis *a*.
v) The variation of *e* is secular with e while it is periodic with ω.

From Figures 5 – 8, we represent the variation of inclination and we note the following:

i) The order of variations varies from 10^{-11} to 10^{-7}
ii) It is always negative and decreasing with *e* and there are considerable change of order for different orbital size.
iii) It is periodic with ω with no considerable change of the order.

From Figures 9 – 12, we represent the variation of ω and we note the following:
i) The variation is secular and the order of variations varies from 10^{-4} to 10^{-8}.
ii) It is decreasing with *e* with a considerable change of order for increasing the orbital size.
iii) It is not affected with ω.
iv) Again, and as expected, the major changes of the order occur with the semi-major axis *a*.

Finally, the variation of the mass is not affect the variation in the oblateness at this order and we expects that change to be in higher orders.

7. REFERENCES

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Appindix

$\chi_{-4,i} = 0.$ [i = 1 : 9]. & $\chi_{-2,1} = \frac{3e}{8L_2^7}\left(1-4C^2+3C^4\right)\left(1-\eta^{-1,1}\right)$

$\chi_{-2,i} = 0.$ [i = 2 : 9].

$$\chi_{0,1} = \frac{3}{4eL_2^7}\Big[\left(7-10C^2-57C^4\right)+\left(3-18C^2+27C^4\right)\eta^{-1,1}-\left(11-34C^2-21C^4\right)\eta^{-2,2}+ +\left(1-6C^2+9C^4\right)\eta^{-3,3}\Big]$$

$$\chi_{0,2} = \frac{3}{8L_2^7}\Big[\left(1+6C^2-31C^4\right)+4\left(1-6C^2+9C^4\right)\eta^{-1,1}-\left(5-18C^2+5C^4\right)\eta^{-2,2}\Big]$$

$\chi_{0,3} = -\frac{e}{4L_2^7}\left(1-6C^2+9C^4\right)\left(1-\eta^{-1,1}\right)$ & $\chi_{0,i} = 0.$ [i = 4 : 9].

$$\chi_{2,1} = \frac{3}{8eL_2^7}\Big[-21\left(1-12C^2+11C^4\right)+9\left(1-4C^2+3C^4\right)\eta^{-1,1}+\left(17-236C^2+219C^4\right). .\eta^{-2,2}-5\left(1-2C^2+3C^4\right)\eta^{-3,3}\Big]$$

$$\chi_{2,2} = \frac{3}{2L_2^7}\Big[\left(3+4C^2-7C^4\right)-4\left(1-C^4\right)\eta^{-2,2}\Big]$$

$$\chi_{2,3} = \frac{1}{8eL_2^7}\Big[\left(63-244C^2+181C^4\right)-9\left(1-4C^2+3C^4\right)\eta^{-1,1}-\left(35-132C^2+97C^4\right). .\eta^{-2,2}-\left(1-4C^2+3C^4\right)\eta^{-3,3}\Big]$$

$$\chi_{2,4} = \frac{3}{8L_2^7}\Big[\left(9-40C^2+31C^4\right)-3\left(1-4C^2+3C^4\right)\eta^{-1,1}-\left(3-16C^2+13C^4\right)\eta^{-2,2}\Big]$$

$\chi_{2,i} = 0.$ [i = 5 : 9]. & $\chi_{4,1} = 0.$

$\chi_{4,2} = -\frac{15}{16L_2^7}\left(1-2C^2+C^4\right)\left(1-\eta^{-2,2}\right)$ & $\chi_{4,2} = -\frac{15}{16L_2^7}\left(1-2C^2+C^4\right)\left(1-\eta^{-2,2}\right)$

$\chi_{4,3} = -\frac{3e}{4L_2^7}\left(1-2C^2+C^4\right)$ & $\chi_{4,4} = \frac{3}{16L_2^7}\left(1-2C^2+C^4\right)\left(3-\eta^{-2,2}\right)$

$\chi_{4,5} = \frac{3e}{4L_2^7}\left(1-2C^2+C^4\right)$ & $\chi_{4,6} = \frac{3}{16L_2^7}\left(1-2C^2+C^4\right)\left(1-\eta^{-2,2}\right)$

$\chi_{4,\mathrm{i}} = 0.$ [i = 7 : 9],

$$\chi'^{C}_{0,0} = \frac{3}{4L_2^7}\Big[5\left(1-2C^2-7C^4\right)-\left(5-18C^2+5C^4\right)\eta^{-2,2}\Big]$$

$\chi'^{C}_{2,0} = -\frac{3}{2L_2^7}\left(1-16C^2+15C^4\right)\eta^{-2,2}$ & $\chi'^{C}_{2,1} = \frac{9e}{L_2^7}\left(1-6C^2+5C^4\right)$ & $\chi'^{C}_{2,2} = \frac{1}{e}\chi'^{C}_{2,1}$

$\chi'^{C}_{2,3} = \frac{1}{3}\chi'^{C}_{2,1}$ & $\chi'^{S}_{2,1} = -\chi'^{C}_{2,1}$ & $\chi'^{S}_{2,2} = \frac{1}{e}\chi'^{S}_{2,1}$ & $\chi'^{S}_{2,3} = \frac{1}{3}\chi'^{S}_{2,1}$