A METHOD FOR TWO-DIMENSIONAL POTENTIAL PROBLEMS

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ABSTRACT
In the research of this article, we propose a new boundary-type meshless method---the average source boundary node method for the potential problem(ASBNM). This method is based on the average source technology and couples it with a completely regularized boundary integral equation. In the solution process, this method only uses boundary nodes, without any the concept of unit or integral. In addition, it has the characteristics of simple method and easy programming, so it is suitable for two-dimensional boundary value problems. Numerical examples prove the effectiveness and accuracy of this method.

Keywords: *Potential problems,boundary node,average source boundary node method, average source, technology, meshless method.*

1. INTRODUCTION
In practical engineering, stable heat conduction, stable seepage, elastic torsion, fluid flow, wave propagation, electromagnetic field, nonhomogeneous and nonlinear problems[1,2] can all be reduced to solving potential problems or are closely related to solving potential problems.Numerical methods to solve the above problems mainly include finite difference method, finite element method and boundary element method.These methods require the help of the network framework to implement ,and construct the interpolation approximation function of the basic physical quantity and perform energy integration on it.However, in the case of complex computational domain problems, the pre-processing generated by the network is quite difficult and time-consuming.
In order to simplify the calculation process and avoid the trouble caused by the grid, the meshfree method came into being.It uses the weight function of the node to represent the physical quantity at the node, thereby forming a matrix equation for solving.Now the more mature meshfree method is the element-free Galerkin method.According to the need for intra-domain collocation, the meshless method can be divided into two categories: the regional meshless method and the boundary meshless method.According to whether distribute point in the domain,the meshless method can be divided into two categories: the regional-type meshless method and the boundary-type meshless method.The boundary-type meshless method is relatively more competitive, because it inherits the advantages of boundary-type numerical methods, such as the dimensionality reduction of the boundary element method and the ease of solving external boundary value problems.At present, a variety of boundary-type meshless methods have been developed, such as funda-mental solution (MFS), the modified MFS (MMFS), the boundary collocation method (BCM),the regularized meshless method (RMM),the boundary knot method (BKM)and various derivative methods, etc.
It should be noted that for the boundary-type meshless method that relies on the boundary integral equation, regardless of its form, the key and difficult problem is how to accurately and efficiently estimate the diagonal elements of the influence matrix.But it is very difficult to solve this problem because it needs to deal with the singularities caused by the basic solution and its derivatives.
This paper proposes a new boundary-type meshless method, namely the average source boundary node method(ASBNM).The average source boundary node method couples the average source technology and the completely regularized boundary integral equation. Its advantage is that it does not need to divide the traditiona mesh, nor does it involve the concept of elements and integrals. It has the advantages of simple method and high accuracy. The method is suitable for two-dimensional boundary value problems.

2. BOUNDARY INTEGRAL EQUATION
In this paper, we always assume that Ω is a bounded domain in R^2 , Ω_c its open complement, and $\Gamma = \partial\Omega$ their common boundary.

2.1 Boundary value problem

Considering the two-dimensional potential problem in the domain $\hat{\Omega}(\hat{\Omega}=\Omega \text{ or } \Omega_c)$, the governing equation by the Laplace equation follows

$$\nabla^2 u(\boldsymbol{x})=0 \qquad (\boldsymbol{x}=(x_1,x_2)\in\hat{\Omega}) \tag{1}$$

with boundary conditions

$$u(\boldsymbol{x})=\bar{u}(\boldsymbol{x}) \qquad (\boldsymbol{x}\in\Gamma_1) \tag{2}$$

$$q(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial n(\boldsymbol{x})}=\bar{q}(\boldsymbol{x}) \qquad (\boldsymbol{x}\in\Gamma_2) \tag{3}$$

When $\hat{\Omega}=\Omega_c$, in order to ensure the uniqueness of the solution to the external problem, the following restrictive conditions must be attached to the behavior of the solution at infinity [15]:

$$|u(\boldsymbol{x})|=O(1) \qquad (\rho=\sqrt{x_1{}^2+x_2{}^2}\to\infty) \tag{4}$$

where $\Gamma=\Gamma_1\cup\Gamma_2$ is the boundary of wi $\hat{\Omega}$ th $\Gamma_1\cap\Gamma_2=\varnothing$; $\bar{u}(\boldsymbol{x})$ and $\bar{q}(\boldsymbol{x})$ are the prescribed boundary functions and $\boldsymbol{n}(\boldsymbol{x})$ is the unit outward normal vector at point $\boldsymbol{x}=(x_1,x_2)\in\Gamma$.The fundamental solution of the governing equation (1) can be expressed as

$$u^*(\boldsymbol{x},\boldsymbol{y})=-\frac{1}{2\pi}\ln|\boldsymbol{x}-\boldsymbol{y}| \tag{5}$$

and $q^*(\boldsymbol{x},\boldsymbol{y})$ is the derived fundamental solution

$$q^*(\boldsymbol{x},\boldsymbol{y})=\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}} \tag{6}$$

2.2 Regularized boundary integral equation

Considering the two-dimensional potential problem in the domain $\hat{\Omega}(\hat{\Omega}=\Omega \text{ or } \Omega_c)$, the regularized boundary integral equation can be expressed in terms of the boundary potential u and flux $q=\partial u/\partial \boldsymbol{n}$ as follows[17]

$$u(\boldsymbol{y})=\int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{x})d\Gamma-\int_\Gamma[u(\boldsymbol{x})-u(\boldsymbol{y})]q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma,\ \boldsymbol{y}\in\Gamma \tag{7}$$

For a given internal point $y\in\hat{\Omega}$,the boundary integral equation can be written as

$$u(\boldsymbol{y})=\int_\Gamma q(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma-\int_\Gamma u(\boldsymbol{x})q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma \tag{8}$$

$$\nabla u(\boldsymbol{y})=\int_\Gamma q(\boldsymbol{x})\nabla u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma-\int_\Gamma u(\boldsymbol{x})\nabla q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma \tag{9}$$

In (Eqs.(7)-(9)), $\boldsymbol{y}=(y_1,\cdots,y_{\boldsymbol{d}})$ and $\boldsymbol{x}=(x_1,\cdots,x_d)$ are the source and the field points, respectively.

In order to sidestep the direct computation of the weak singular integral in Eq. (7), based on the following integral identities

$$\int_\Gamma n_i(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma=\int_\Gamma(x_i-y_i)q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma,\ \boldsymbol{y}\in\hat{\Omega}, i=1,2 \tag{10}$$

and a limit procedure[18-20],

$$\lim_{\boldsymbol{y}\to\hat{\boldsymbol{x}}}\int_{\Gamma}\frac{\boldsymbol{x_k}-\boldsymbol{y_k}}{|\boldsymbol{x}-\boldsymbol{y}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}}=\int_{\Gamma}\frac{\boldsymbol{x_k}-\hat{\boldsymbol{x}}_{\boldsymbol{k}}}{|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}}(k=1,2) \tag{11}$$

we obtain a completely regularized boundary integral equation as follows

$$\begin{aligned}u(\boldsymbol{y})=&\int_{\Gamma}[q(\boldsymbol{x})-q(\boldsymbol{y})n(\boldsymbol{y})\cdot n(\boldsymbol{x})]u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma-\int_{\Gamma}[u(\boldsymbol{x})-u(\boldsymbol{y})]q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma\\&+q(\boldsymbol{y})\int_{\Gamma}n(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma,\boldsymbol{y}\in\Gamma\end{aligned} \tag{12}$$

3. AVERAGE SOURCE BOUNDARY NODE METHOD

Assume that $f(\boldsymbol{x},\boldsymbol{y})$ is the integrand of the above boundary integral equation. When $\boldsymbol{y}\notin\Gamma_j$, $f(\boldsymbol{x},\boldsymbol{y})$ is a smooth function of x on Γ_j .According to the literature [13], the average source boundary node form of the boundary integral can be obtained

$$\int_{\Gamma_j} f(\boldsymbol{x},\boldsymbol{y})d\Gamma_x=\int_{-1}^{1}J(\xi)f(\boldsymbol{x}(\xi),\boldsymbol{y})d\xi\approx\omega_j f(\boldsymbol{x}_j,\boldsymbol{y}) \tag{13}$$

where $\boldsymbol{x}_j$ is the mid-point of the segment Γ_j , and $\omega_j=2J(\boldsymbol{x}_j)$,where $J(\boldsymbol{x}_j)$ is the Jacobian value at the jth node $\boldsymbol{x}_j$.

Therefore, according to the average source technology [12-14], the above boundary integral equation can be transformed into the following average source boundary node method, which form can be expressed as

$$\begin{aligned}u_i&=\sum_{j=1}^{N}G_{ij}\phi_j,\\ q_i&=\sum_{j=1}^{N}H_{ij}\phi_j\end{aligned} \tag{14, 15}$$

where N is the total number of collocation points on all cells, $\phi_{\boldsymbol{j}}$ the unknown density at the jth node, and G_{ij} , H_{ij} are in-fluence coefficients corresponding to Eq. (12) , respectively, given by

$$G_{ij}=\begin{cases}-\omega_j\ln r_{ij} & (i\neq j)\\ \sum_{k=1,k\neq i}^{N}\left[\boldsymbol{n}_i\cdot\boldsymbol{n}_k\omega_k\ln r_{ik}-\boldsymbol{n}_i\cdot(\boldsymbol{x}_k-\boldsymbol{x}_i)\omega_k\frac{(\boldsymbol{r}_{ik}\cdot\boldsymbol{n}_k)}{r^2_{ik}}\right] & (i=j)\end{cases} \tag{16}$$

$$H_{ij}=\begin{cases}-\omega_j\dfrac{(\boldsymbol{r}_{ij}\cdot\boldsymbol{n}_i)}{r^2_{ij}}, & (i\neq j)\\ \hat{k}+\sum_{k=1,k\neq i}^{N}\omega_k\dfrac{\boldsymbol{r}_{ik}\cdot\boldsymbol{n}_k}{r^2_{ik}} & (i=j)\end{cases} \tag{17}$$

In Eqs.(16)-(17), $\boldsymbol{r}_{ik}$ is the vector from the ith node to the kth node, r_{ik} is the distance from the ith node to the kth node, $\boldsymbol{n}_k$ is the unit normal to the boundary Γ at the kth node, respectively, $\omega_{\boldsymbol{k}}=J(\boldsymbol{x_k})/\pi$, with $J(\boldsymbol{x}_k)$ being the value of the Jacobian at ith node $\boldsymbol{x_k}$.

For any internal point $\boldsymbol{y} \in \hat{\Omega}$ far away from the boundary Γ, (Eqs.(8)–(9)) can be reduced to the following form

$$u(y) = -\sum_{j=1}^{N} q_j \omega_j \ln r_{ij} + \sum_{j=1}^{N} u_j \omega_j \frac{(\boldsymbol{r}_{ij} \cdot \boldsymbol{n}_i)}{r^2_{ij}} \qquad (y \in \hat{\Omega}) \tag{18}$$

$$\frac{\partial u}{\partial y_k}(\boldsymbol{y}) = \sum_{j=1}^{N} q_j \omega_j \frac{r^j{}_k}{r^2{}_{ij}} - \sum_{j=1}^{N} u_j \omega_j \left[\frac{n^j{}_k}{r^2{}_{ij}} - 2\frac{(\boldsymbol{r}_{ij} \cdot \boldsymbol{n}_j)}{r^4{}_{ij}} r^j{}_k \right], k = 1, 2 \tag{19}$$

In Eqs.(18)-(19), $r_{\boldsymbol{k}}^{\boldsymbol{j}} = x_k^j - y_k^j$, where x_k^j is the kth component of the coordinates of the jth node $\boldsymbol{x_j}$, and n_k^j is the kth component of the unit normal at the jth node.

4. NUMERICAL EXAMPLES

In this section, some examples of discontinuous boundary conditions and multi-connection area problems are used to verify the effectiveness and accuracy of the method, and compared with the boundary element method, which uses precise geometric elements [16] and constant Meta interpolation calculation.To assess the accuracy of the results, the relative error is defined as

$$\mathrm{Re}\,lative \quad error = \sqrt{\sum_{k=1}^{M} (I^k{}_{num} - I^k{}_{exact})^2 \Big/ \sum_{k=1}^{M} (I^k{}_{exact})^2} \tag{20}$$

where M is the total number of calculation points, and I_{num}^k and I_{exact}^k denote the numerical solution at the kth calculation point and the exact solution at the kth calculation point, respectively.

4.1. Test problem 1: Mixed boundary conditions for irregular regions

First, we consider the boundary value problem with mixed boundary conditions on irregular regions, as shown in Figure 1, in which the parametric representation of the boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$ where

$$\Gamma_1 = \left\{ (r\cos\varphi, r\sin\varphi) : r = e^{\sin\varphi}(\sin^2(2\varphi)) + e^{\cos\varphi}(\cos^2(2\varphi)), 0 \le \varphi \le \pi \right\},$$

$$\Gamma_2 = \left\{ (r\cos\varphi, r\sin\varphi) : r = e^{\sin\varphi}\sin^2(2\varphi) + e^{\cos\varphi}\cos^2(2\varphi), \pi \le \varphi \le 2\pi \right\}.$$

The boundary conditions are: $\bar{u}$ is known on boundary Γ_1 and $\bar{q}$ is known on boundary Γ_2, where

$$\bar{u} = e^{x_1+3}\cos x_2,$$

$$\bar{q} = e^{x_1+3} n_1 \cos x_2 - e^{x_1+3} n_2 \sin x_2$$

he analytical solution of this problem is given by

$$u(x_1, x_2) = e^{x_1+3}\cos x_2$$

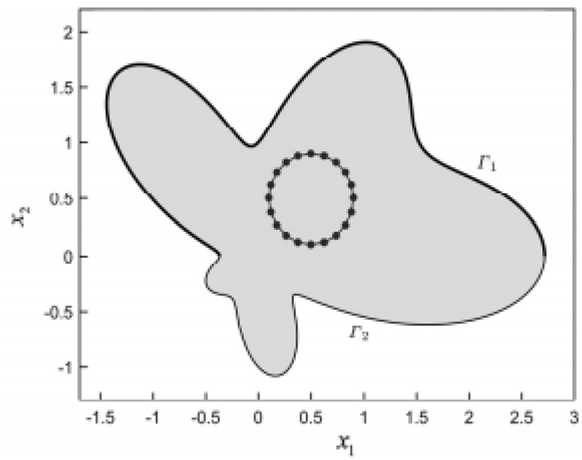


Fig.1 Problem sketch

When solving, 30 "calculation points in the domain" are evenly distributed on the circle with the center at $(0.5, 0.5)$ and the radius of 0.4. Figure 2,figure 3 and figure 4 show the convergence curves of the relative errors of the numerical solutions of the potential solution u , the flux solution $\partial u / \partial x_1$, and the boundary flux $\partial u / \partial n$ as the number of boundary nodes increases. It can be seen that whether it is an interior point or a boundary point, as the number of boundary nodes increases, the relative error decreases rapidly. It can be seen that the method in this paper has good convergence characteristics when solving this problem.

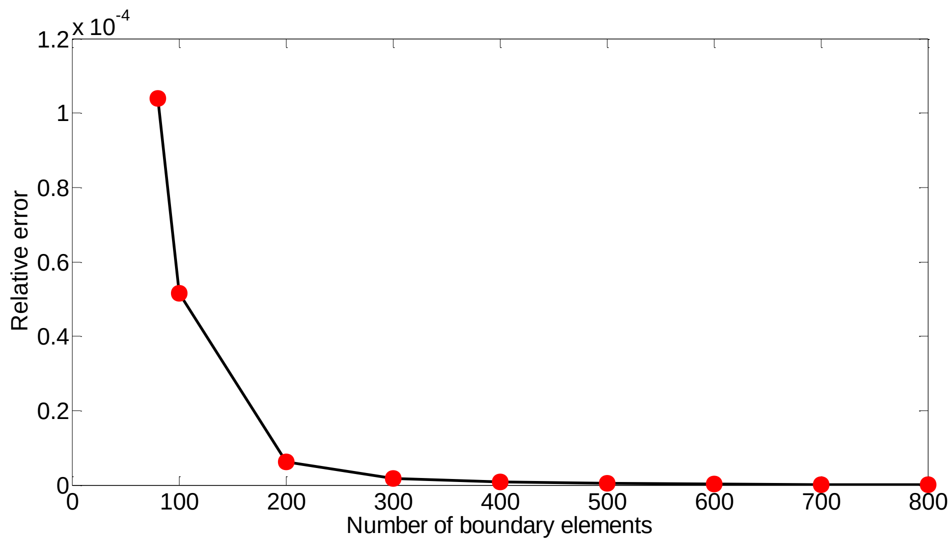


Fig.2 The relative errors for the potential solution u

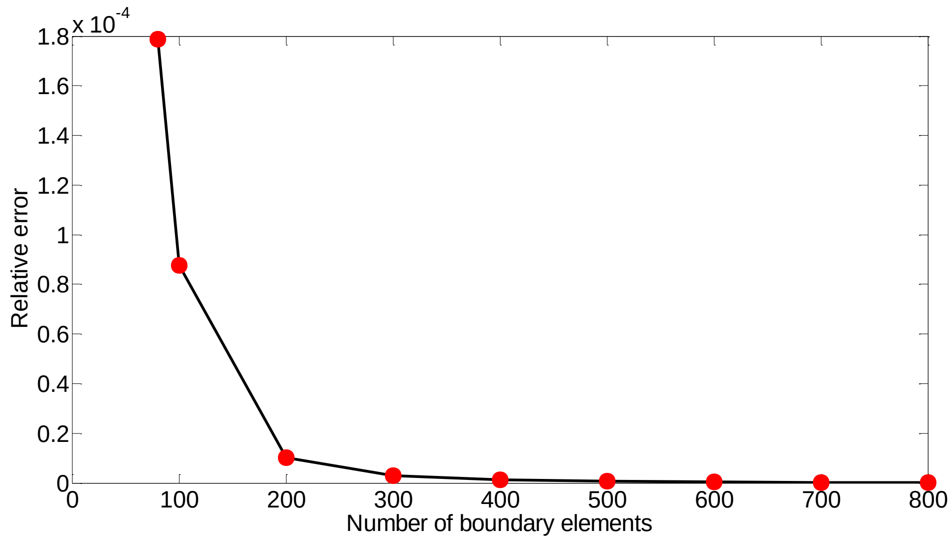


Fig.3 The relative errors for the flux solution $\partial u / \partial x_1$

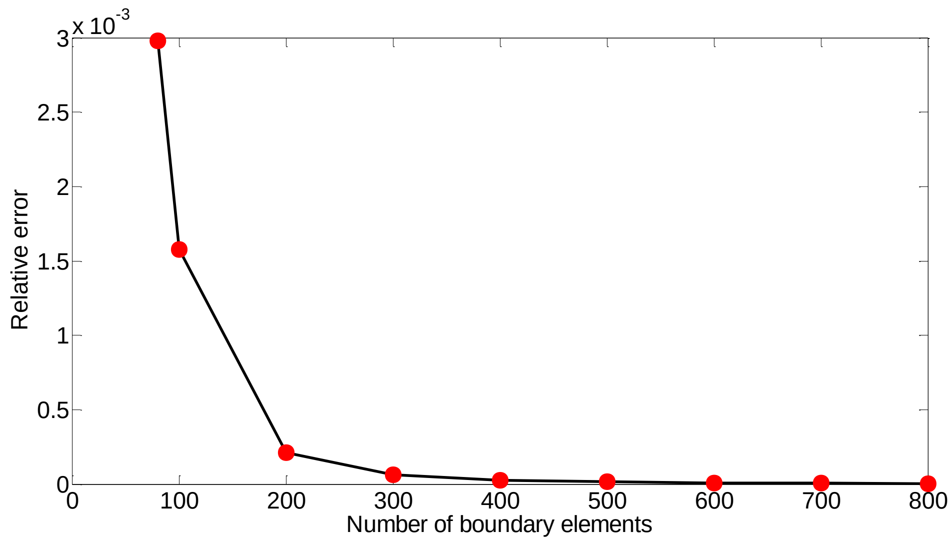


Fig.4 The relative errors for the boundary flux $\partial u / \partial n$

4.2. Test problem 2: A square domain with Dirichlet discontinuous boundary conditions

We investigate a square domain $[0,\pi]\times[0,\pi]$ subject to the Dirichlet discontinuous boundary conditions (BCs) given as follows：

$$\begin{cases} u(\pi, x_2) = 1, u(x_1, \pi) = 0 \\ q(x_1, 0) = q(0, x_2) = 0 \end{cases}$$

which is an example used in Ref. [7]. An analytical solution is available as follows:

$$u(x_1, x_2) = \sum_{n=1}^{\infty} D_n \cosh(\frac{(2n-1)x_1}{2}) \cos(\frac{(2n-1)x_2}{2})$$

where

$$D_n = 4(-1)^{n+1} / [(2n-1)\pi \cosh(\frac{(2n-1)\pi}{2})]$$

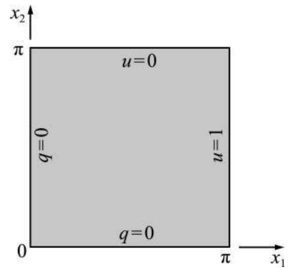
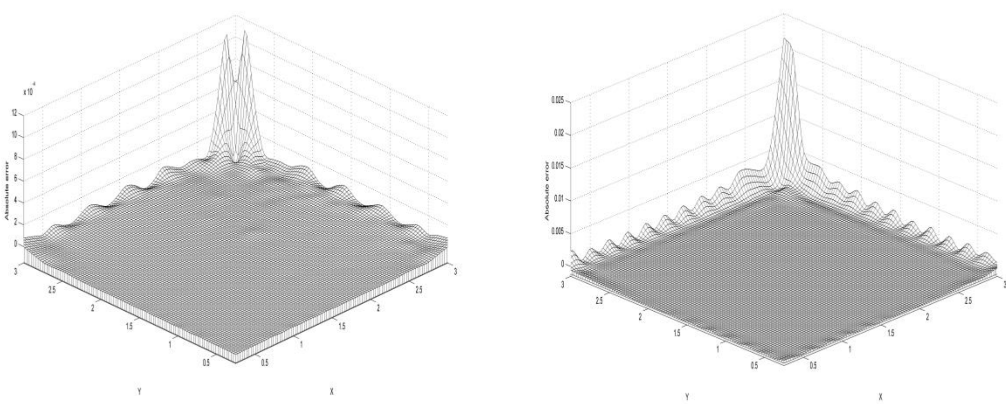


Fig.5 Problem sketch

To investigate the error behavior comprehensively,Fig.6(a)-(b) display the absolute error surfaces, respectively, for the potential solution u and the flux solution $\partial u / \partial x_1$, in which the error surfaces were produced at 25×25 interior points evenly-spaced over the interested square domain covering $[0.15, 3.0] \times [0.15, 3.0]$ with using 80 boundary nodes. Fig.7 illustrates the relative error of the potential solution u and the flux solution $\partial u / \partial x_1$ versus the number of boundary nodes.These two figures clearly show that the accuracy and convergence rate of the ASBNM are satisfactory.



(a)Potential *(b)Flux*

Fig.6 Absolute error surfaces for field potential solutionl u and the flux $\partial u / \partial x_1$ (80 nodes)

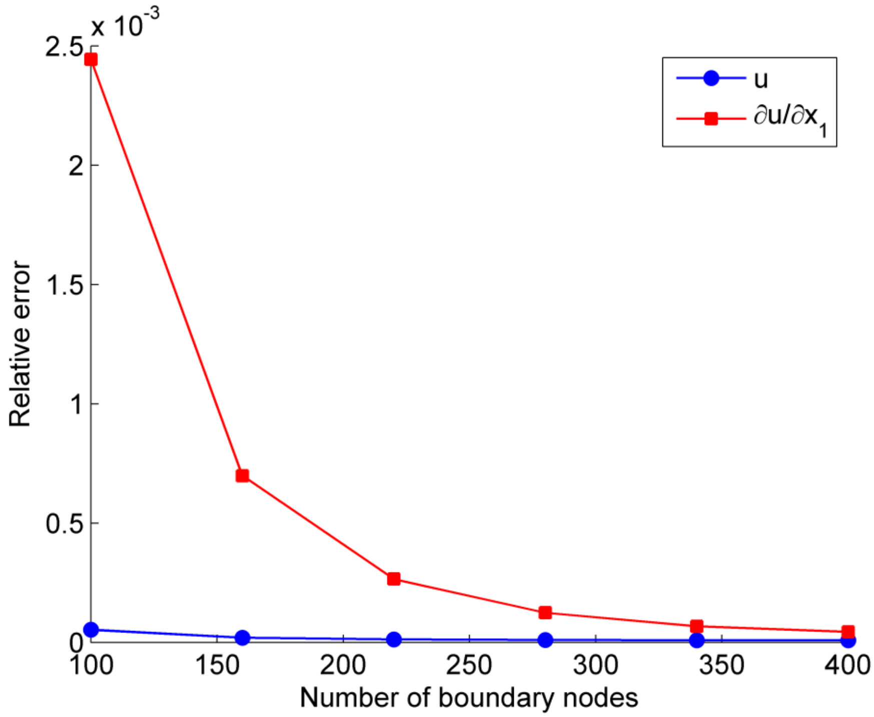


Fig.7 Relative error of the field potential u *and its derivative*

$\partial u / \partial x_1$ *vs.* the number of boundary nodes

5. CONCLUSION

In this study, a new meshfree boundary method, termed average source boundary node method (ASBNM), is proposed to solve 2D potential problems.This method is based on the average source technology and couples it with a completely regularized boundary integral equation. For the ASBNM in this study, as long as the parameter representation of the domain boundary is given, the real geometric shape of the boundary can be used without approximate calculation, as shown in the test example in this paper. Therefore, this method is not only simple and easy to implement, but also can be directly extended to other plane problems, such as elasticity problems and Stokes problems, etc.

6. REFERENCES

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