A METHOD FOR THE GRADIENT BOUNDARY INTEGRAL EQUATION OF TWO-DIMENSIONAL POTENTIAL PROBLEMS

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ABSTRACT
In the boundary element analysis of the two-dimensional potential problem, the calculation of the boundary potential gradient of the coordinate variable is difficult. Several techniques have been proposed to address this problem so far. However, they require complex theoretical deduction and a large number of numerical manipulation. This paper proposes a new method named auxiliary boundary value problem method (ABVPM) to solve the the gradient boundary integral equation (GBIE) for two-dimensional potential problems. An ABVPM with the same solution domain as the original boundary value problem is constructed, which is an over-determined boundary value problem with known solution. Consequently, the system matrix of the GBIE, which is the most important problem for boundary analysis, will be obtained by solving the ABVPM. It can be used to solve original boundary value problem. The solution procedure is simple, because only a linear system need to be solved to obtain the solution of the original boundary value problem. It is worth noting that it is not necessary to recalculate the system matrix when solving the original boundary value problem, so the efficiency of the auxiliary boundary value method is not very poor. The proposed ABVPM circumvents the troublesome issue of computing the strongly singular integrals, with some advantages, such as simple mathematical deduction, easy programming and high accuracy. More importantly, the ABVPM provides a new way for solving the GBIE. Three benchmark examples are tested to verify the effectiveness of the proposed technique.

Keywords: *potential problem,gradient boundary integral equation,strongly singular integrals,auxiliary boundary value problem.*

1. INTRODUCTION
Numerical analysis of many problems in science and engineering, such as stable heat conduction, torsion of elastic members, stable seepage, hydrodynamic pressure, membrane balance, Helmhotz equation, electromagnetic field, heterogeneous materials and nonlinear problems [1-3], can be directly or indirectly attributed to the solution of boundary value problems of Laplace or Poisson equations. Boundary element method is a powerful numerical tool for solving such problems.
The physical quantity gradient boundary integral equation is obtained by taking the derivative of the physical quantity boundary integral equation with respect to space variables [4]. It is suitable for certain problems, such as crack problems, wave scattering problems, sheet bending problems, and narrow and thin domain problems with degenerate boundaries [5-7]. The boundary integral equation of basic physical quantities alone cannot accurately represent the solution of the original boundary value problem, it is not equivalent to the original boundary value problem. In order to avoid this situation, the gradient boundary integral equation of the basic physical quantity is usually combined with the boundary integral equation of the physical quantity, that is, the dual boundary integral equation, to express the solution of the original boundary value problem. There have been many studies on the potential gradient boundary integral equation of the potential problem [8-12]. Literature [8] established a direct variable regularized gradient boundary integral equation for two-dimensional potential problems. Literature [10] studies the potential gradient boundary integral equation of the two-dimensional potential problem. It introduces a series of transformations to transform the gradient boundary integral equation into a natural boundary integral equation with strong singular integrals, and then uses addition and subtraction techniques to regularize the strong singularity integral. It eliminates the singularity effectively and obtains good numerical results. Different from the regularized boundary integral equation method mentioned above, literature [13-15] and literature [16-19] respectively proposed a local regularization method for directly calculating the singular integral in the gradient equation. Their algorithms are general and they can calculate singular integrals of any order. The disadvantage is that the theoretical derivation of two algorithms is complicated and the amount of calculation is quite large.
This paper proposes a new algorithm for solving the potential gradient boundary integral equation of the two-dimensional potential problem. This method calculates the system matrix of the gradient boundary integral equation by constructing an auxiliary boundary value problem. There is no need to establish a regularized boundary integral equation or calculate strong singular integrals directly. Therefore, the method has the advantages of simple mathematical theory, high calculation efficiency, and high accuracy of results. It should be emphasized that the

auxiliary boundary value problem method in this paper provides a new idea and approach for calculating strong singular integral in gradient boundary integral equations. In addition, the method in this paper can be extended to other problems, such as elasticity problems, Stokes and Helmholtz problems, etc.

2. BOUNDARY VALUE PROBLEM

In this paper, we always assume that Ω is a bounded domain in R^2, Ω_c its open complement, and Γ their common boundary. $\boldsymbol{n}(\boldsymbol{x})$ is the unit outward normal vector at point $\boldsymbol{x}=(x_1,x_2)\in\Gamma$.The fundamental solution of the two-dimensional potential problem is

$$u^*(x,y)=\frac{\mathbf{1}}{\mathbf{2}\pi}\ln\frac{\mathbf{1}}{r}$$

Consider a two-dimensional potential problem in the domain $\hat{\Omega}\left(\hat{\Omega}=\Omega\ or\ \Omega_c\right)$ governed by the Laplace equation

$$\nabla^2 u(\boldsymbol{x})=0,\quad \boldsymbol{x}=(x_1,x_2)\in\hat{\Omega} \tag{1}$$

with boundary conditions

$$u(\boldsymbol{x})=\bar{u}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_1 \tag{2}$$

$$q(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}(\boldsymbol{x})}=\bar{q}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_2 \tag{3}$$

where $\Gamma=\Gamma_1\cup\Gamma_2$ is the boundary of $\hat{\Omega}$ with $\Gamma_1\cap\Gamma_2=\varnothing$; $\bar{u}(\boldsymbol{x})$ and $\bar{q}(\boldsymbol{x})$ are the prescribed boundary functions.

Lemma [11]. Let Γ be a piecewise smooth curve (open or closed), and $\hat{x}$ a point on Γ (perhaps a corner). Suppose $h=|\boldsymbol{y}-\hat{\boldsymbol{x}}|$ and $d=\inf_{\boldsymbol{x}\in\Gamma}|\boldsymbol{y}-\boldsymbol{x}|$. If $\psi(\boldsymbol{x})\in C^{0,\alpha}(\Gamma)$ and $\frac{h}{d}\le K_1$ (with constant K_1), then there holds

$$\lim_{\boldsymbol{y}\to\hat{\boldsymbol{x}}}\int_\Gamma\frac{x_k-y_k}{|\boldsymbol{x}-\boldsymbol{y}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}}=\int_\Gamma\frac{x_k-\hat{x}_k}{|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}}\ \ (k=1,2) \tag{4}$$

3. ABVPM FOR POTENTIAL GRADIENT BOUNDARY INTEGRAL EQUATION

Consider three numerical examples to verify the effectiveness of this method. The focus of the numerical experiment is to investigate the ability and accuracy of the auxiliary boundary value problem method to calculate the boundary flux $\partial u/\partial x_{\mathbf{1}}$. In order to estimate the numerical error, the following $L_{\mathbf{2}}$ norm is used

$$RelativeError=\sqrt{\sum_{k=\mathbf{1}}^{M}(u_{num}^k-u_{exact}^k)^{\mathbf{2}}}\Big/\sqrt{\sum_{k=\mathbf{1}}^{M}(u_{exact}^k)^{\mathbf{2}}} \tag{15}$$

where M is the total number of calculation points, u_{num}^k and u_{exact}^k denote the numerical and exact solution at the kth calculation point, respectively.

Test problem 1: A square domain with Dirichlet discontinuous boundary conditions

First, as shown in fig 1, we investigate a square domain $[0,1]\times[0,1]$ subject to the Dirichlet discontinuous boundary conditions given as follows:

$$u(x_1,0)=x_1, u(x_1,1)=u(0,x_2)=u(1,x_2)=0$$

An analytical solution is available as follows:

$$u(x_1,x_2)=\sum_{n=1}^{\infty}D_n\sinh\left(n\pi(1-x_2)\right)\sin\left(n\pi x_1\right)$$

where

$$D_n=\frac{2(-1)^{n+1}}{(n\pi)\sinh(n\pi)}$$

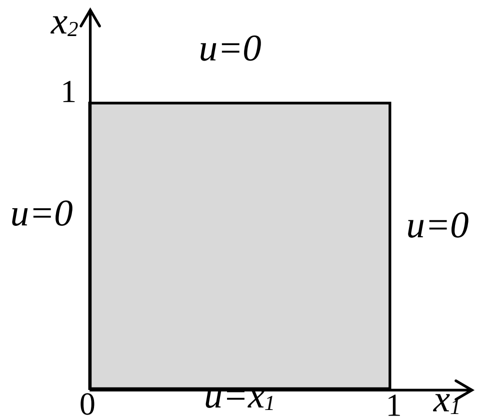


Figure 1.the square domain

There are 80 boundary elements to calculate the example,and Table 1 shows the comparison between the analytical and numerical solutions of $\partial u/\partial x_1$ on the boundary $x_1 = 1.0$ .It can be seen that results obtained by ABVPM match the exact solutions very well.As shown in fig 2, there are 1600 points in $0.15 \le x_1, x_2 \le 0.85$, and given the absolute error surface of the numerical solutions at these points.It can be seen from figure 3 that although there are very few boundary elements, the numerical results obtained by ABVPM are still quite accurate.

Table 1. boundary flux $\partial u/\partial x_1$ *on* $x_1 = 1.0$

Boundary Points	Exact solutions	Numerical results	Relative error
(1.0, 0.125)	-0.4155213E+01	-0.4122077E+01	7.974714E-03
(1.0, 0.175)	-0.2724655E+01	-0.2703559E+01	7.742648E-03
(1.0, 0.225)	-0.1940493E+01	-0.1931832E+01	4.463168E-03
(1.0, 0.275)	-0.1449872E+01	-0.1445299E+01	3.153867E-03
(1.0, 0.325)	-0.1117022E+01	-0.1114381E+01	2.363975E-03
(1.0, 0.375)	-0.8785297E+00	-0.8768937E+00	1.862193E-03
(1.0, 0.425)	-0.7007772E+00	-0.6997158E+00	1.514603E-03
(1.0, 0.475)	-0.5642567E+00	-0.5635486E+00	1.255076E-03
(1.0, 0.525)	-0.4568490E+00	-0.4563718E+00	1.044440E-03
(1.0, 0.575)	-0.3706071E+00	-0.3702902E+00	8.550815E-04
(1.0, 0.625)	-0.3000828E+00	-0.2998843E+00	6.615155E-04
(1.0, 0.675)	-0.2413954E+00	-0.2412914E+00	4.310303E-04
(1.0, 0.725)	-0.1916832E+00	-0.1916628E+00	1.064481E-04
(1.0, 0.775)	-0.1487660E+00	-0.1488300E+00	4.302583E-04
(1.0, 0.825)	-0.1109270E+00	-0.1111063E+00	1.616166E-03
(1.0, 0.875)	-0.7676643E-01	-0.7700261E-01	3.076569E-03

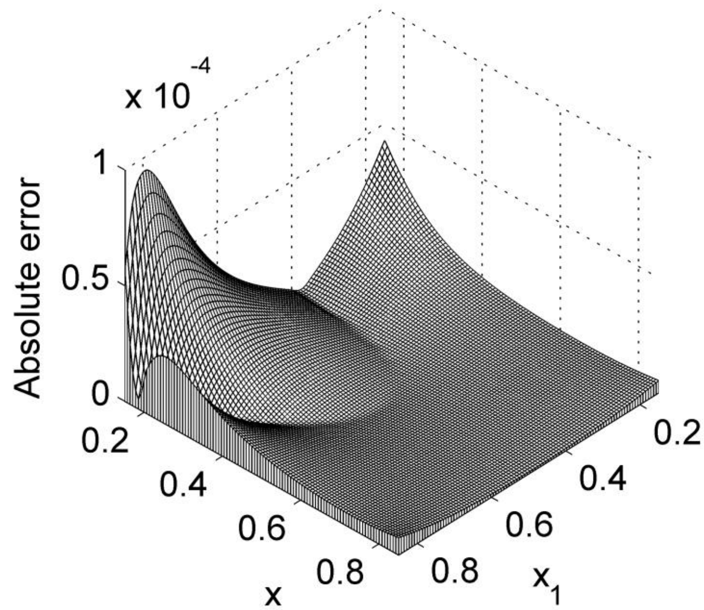


Figure 2. Absolute error surfaces for field potential solutions

Test problem 2: Mixed boundary conditions for irregular regions
In this case, an irregular domain subject to the mixed type boundary conditions is considered as fig 3, and the boundary conditions are complex and oscillatory. Problem sketch are shown in Fig.3, respectively, in which the parametric representation of the boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$ where

$$\Gamma_1 = \left\{(R\cos\varphi, R\sin\varphi) : R = e^{\sin\varphi}(\sin^2(2\varphi)) + e^{\cos\varphi}(\cos^2(2\varphi)), 0 \le \varphi \le \pi\right\}$$

$$\Gamma_2 = \left\{(R\cos\varphi, R\sin\varphi) : R = e^{\sin\varphi}(\sin^2(2\varphi)) + e^{\cos\varphi}(\cos^2(2\varphi)), \pi \le \varphi \le 2\pi\right\}$$

The boundary conditions are

$$\bar{u} = e^{x_1}\cos x_2 \quad on \ \Gamma_1$$

$$\bar{q} = e^{x_1} n_1 \cos x_2 - e^{x_1} n_2 \sin x_2 \quad on \ \Gamma_2$$

The analytical solution of this problem is given by

$$u(x_1, x_2) = e^{x_1}\cos x_2$$

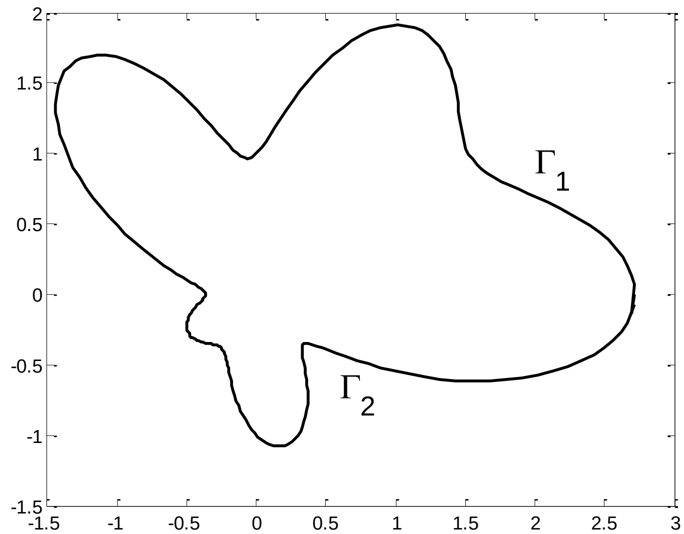


Figure 3 . The irregular regions

In the numerical calculation, the boundary is divided into 250 units. Figure 4 shows the absolute error surface of the numerical solutions of boundary flux .Figure 5 is the potential diagram of the numerical solution and the exact solution of the boundary point temperature. It can be seen that the numerical results are very consistent with the exact solution.

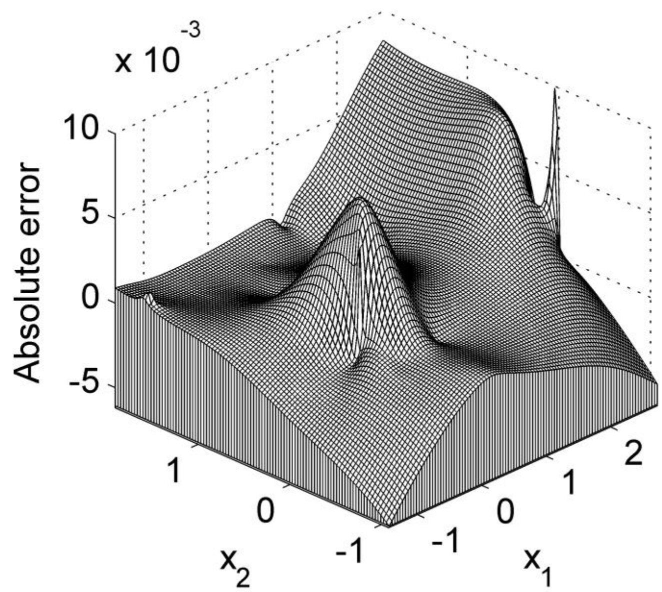
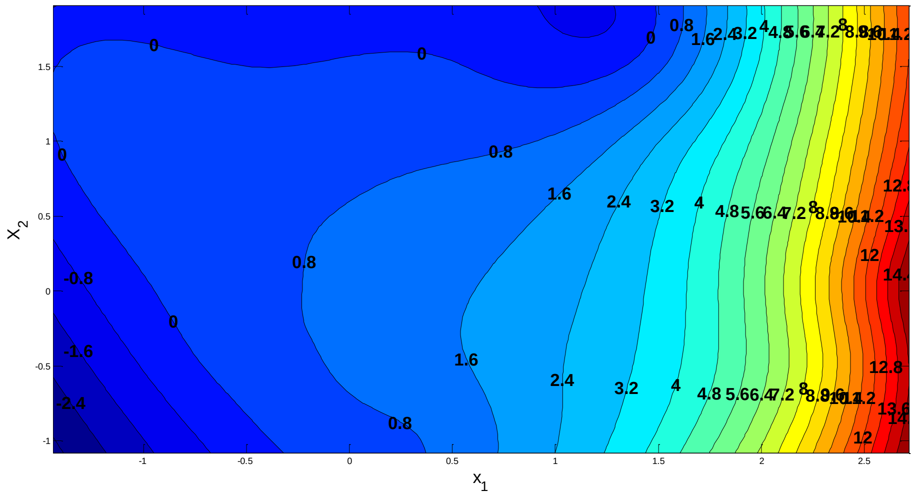
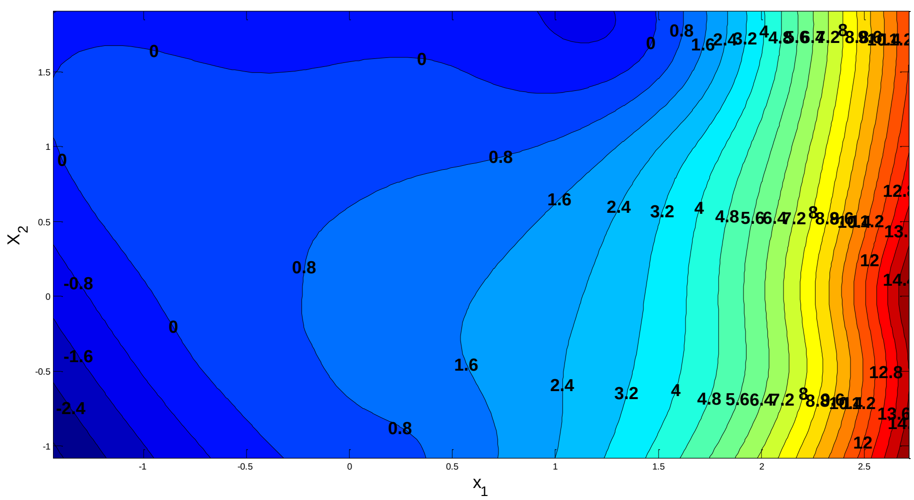


Figure 4. Absolute error surfaces for boundary flux $\partial u/\partial x_1$ *solutions*



(a)



(b)

Figure 5. The field potential solutions:(a)numerical solution and (b) exact solution

Test problem 3: 2D multiply-connected domain with mixed boundary conditions
In this case, a multiply-connected domain subject to the mixed type boundary conditions is considered. The outer boundary Γ_1 is a circle with radius $r_1 = 10$, and the inner boundary Γ_2 is a circle with radius $r_2 = 5$.The problem sketch and the boundary conditions are shown in Fig. 6.

$$\bar{u}(x_1, x_2) = x_1^2 - x_2^2 + x_2, \quad \bar{q}(x_1, x_2) = 2x_1 n_1 + (-2x_2 + 1) n_2$$

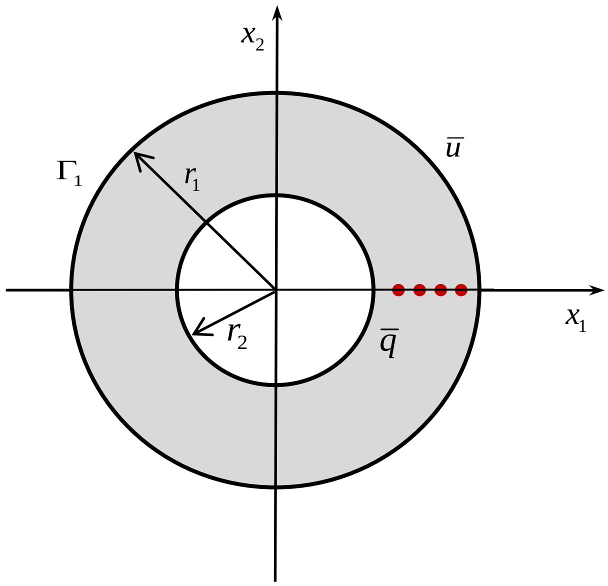


Figure 6. Problem sketch

The Γ_1 is divided into 48 elements and Γ_2 is divided into 32 elements. Table 2 gives the internal points numerical solutions and the analytical solutions, and fig 7 and fig 8 gives the relative errors of boundary flux $\partial u/\partial x_1$ on the boundary. Fig 9 gives the relative errors of boundary flux $\partial u/\partial n$ on the boundary Γ_1. As shown that the method in this paper has good stability for solving the oscillation problem in complex regions.And fig 10 gives the convergence curves of boundary flux $\partial u/\partial x_1$ and $\partial u/\partial n$.It can be seen from the fig 10 that as the number of boundary elements increases, the average relative error decreases rapidly, so the method in this paper has good convergence for solving such problems.

Table 2. Temperature solutions on internal points

Internal Points	Exact solutions	Numerical results	Relative error
(6.0, 0.0)	0.3600000E+02	0.3599477E+02	1.451833E-04
(7.0, 0.0)	0.4900000E+02	0.4898987E+02	2.068274E-04
(8.0, 0.0)	0.6400000E+02	0.6398484E+02	2.368142E-04
(9.0, 0.0)	0.8100000E+02	0.8097887E+02	2.608308E-04

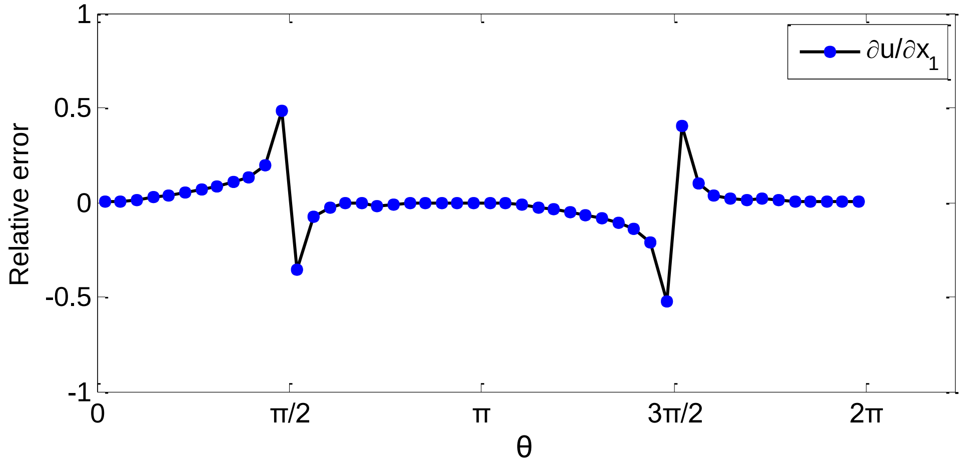


Figure 7. The relative errors for the boundary flux $\partial u/\partial x_1$ *on the boundary* Γ_1

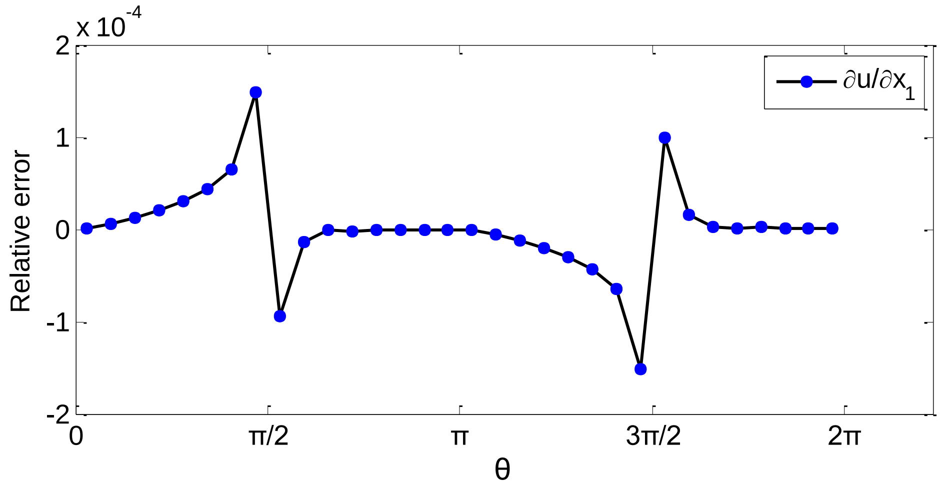


Figure 8. The relative errors for the boundary flux $\partial u/\partial x_1$ *on the boundary* Γ_2

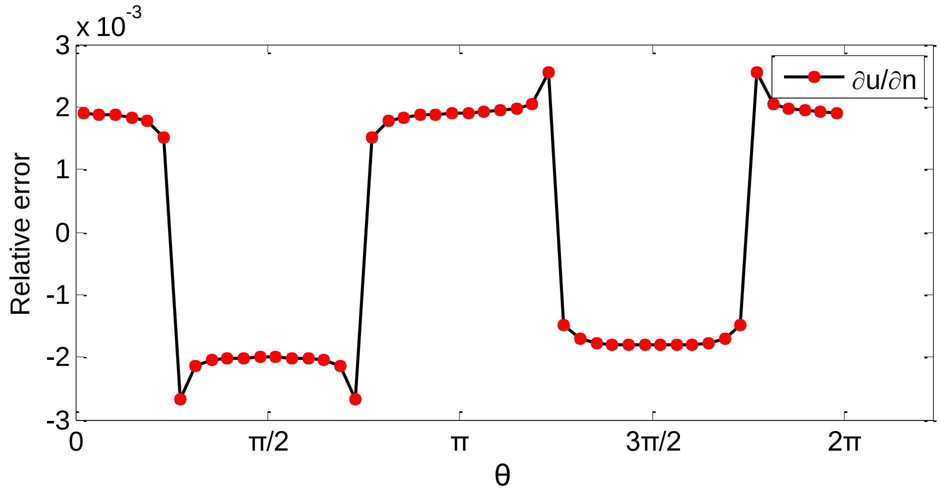


Figure 9. The relative errors for the boundary flux $\partial u/\partial n$ *on the boundary* Γ_1

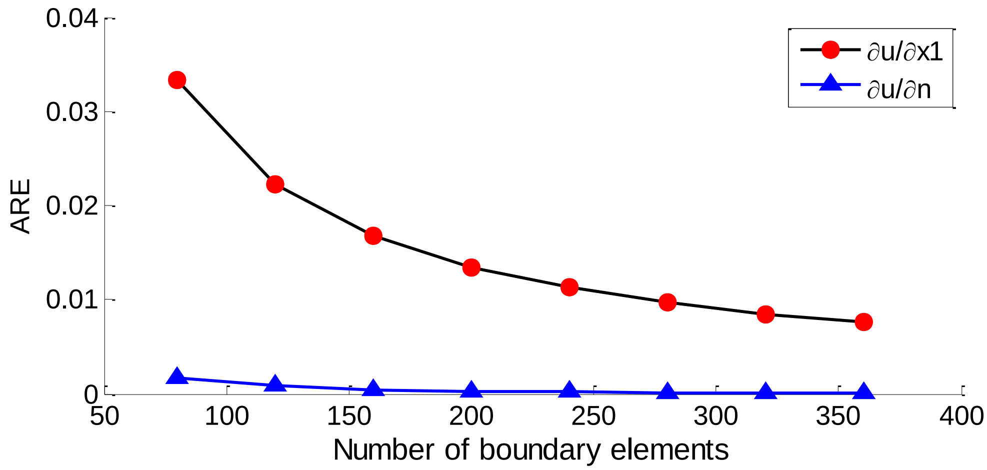


Figure 10. The convergence curves of boundary flux $\partial u/\partial x_1$ *and* $\partial u/\partial n$ *on the boundary*

4. CONCLUSION

In this paper, auxiliary boundary value problem method (ABVPM) is proposed to solve the gradient boundary integral equation of 2-D potential problems. This method is based on the regularized indirect boundary integral equations. However, the new method by solving the equations to avoid calculating strong singular integral, and it has a simple mathematical theory, easy programming, high accuracy, etc. Three numerical examples verify the effectiveness of the method. Moreover, it is observed that for the boundary value problem with discontinue boundary conditions, the ABVPM is better than traditional BEM in the theory and program.

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