PROBABILISTIC LINEAR PROBLEMS WITH BIVARIATE EXPONENTIAL DISTRIBUTED RANDOM PARAMETERS

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ABSTRACT

In this paper, chance constrained programming (CCP) problems with some dependent exponential distributed random parameters are considered. Firstly, a suggested bivariate exponential distribution model is presented. This model is an important for financial, insurance, economical problems, etc.
Secondly, a proposed method to convert (CCP) problems to the equivalent deterministic programming problems in two cases: (i) for individual constraints and some L.H.S. parameters $\tilde{a}_{ij}$ follow a suggested model, (ii) for the joint (dividual) constraints and some R.H.S. random parameters $\tilde{b}_i$ follow a suggested model also.

Keywords: *Bessel function, Box's theorem, Biswal and other method, CCP, Downton bivariate exponential distribution, El-Dash approach, Freund's bivariate exponential distribution, Joint constraints, Probabilistic programming, Stochastic programming.*

1. Introduction

A CCP approach is considered one of the probabilistic or stochastic programming [9,14,16,18,33]. It was first introduced by Charnes and Cooper (1958). They developed different kinds of decisions rules and optimizing objectives and solved some important uncertainty economical problems [23,24,28,35].
Miller and Wagner (1965) were the first to introduce the joint (dividual constraints) CCP problems when the R.H.S. parameters are independent parameters [18,20,27].
In the literature of CCP various models and approaches have been suggested by several researchers [14,15,16,17,18,19,25,28,32,33].
Nearly all researchers deal with independent random parameters $\tilde{a}_{ij}'s$, $\tilde{b}_i's$ in individual constraints:

$$P_r\left(\sum_{j=1}^{n^\backslash} \tilde{a}_{ij}\, x_j \le b_i\right) \ge \gamma_i \;, \quad i = 1, \dots, m^\backslash \tag{1.1}$$

or joint (dividual) constraints

$$P_r\left(\sum_{j=1}^{n} a_{ij}\, x_j \le \tilde{b}_i \quad , i = m^\backslash + 1, m^\backslash + 2, \dots, m\right) \ge \gamma \tag{1.2}$$

Where $0 < \gamma \;,\; \gamma_i < 1$ are tolerance measures and $x_j \ge 0$ are the decisions variables.

And most of above constraints have been deal with random exponential distributed parameters, Sengupta (1972) proved that the random variable $\left(\sum \tilde{a}_{ij}\, x_j\right)$ follow approximate noncentral Chi-square distribution with degree of freedom as function of decision's variables x_j [29,32,33], in turn, it is impossible using non-center Chi-square table [14,15] to obtain the F^{-1} (inverse function of the cumulative function of $\left(\sum_{j=1}^{n} \tilde{a}_{ij}\, x_j\right)$)of the variable $\left(\sum \tilde{a}_{ij}\, x_j\right)$.
El-Dash (1984) introduced the exact distribution of $\left(\sum \tilde{a}_{ij}\, x_j\right)$ by using Box's theorem [6]. Then she converted chance constraints to the equivalent deterministic constraints [14], when $\tilde{a}_{ij}$ follow exponential with two parameters.
Biswal and other (1998) obtained the El-Dash results by using other different techniques [5].
But the most of real life problems have some dependent exponential distributed parameters as: economical, financial, insurance problems,…..etc [8,10,23,24,28].
Recently, El-Dash and other (2018) deal with dependent exponential distributed parameters.
They presented an approximate Downton bivariate distribution [13], then they used the approximated distribution to convert the chance constraints (dividual constraints [20] and individual constraints [19] to deterministic constraints).
In this paper, the following CCP linear problem is investigated

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j \tag{1.3}$$

$$\text{S.T.} \quad P_r\left\{\sum_{j=1}^{n} a_{ij} \le \tilde{b}_i \quad , \quad i = 1,2\right\} \ge \gamma \tag{1.4}$$

$$P_r\left\{\sum_{j=1}^{2} \tilde{a}_{ij}\, x_j + \sum_{j=3}^{n} a_{ij}\, x_j \le b_i\right\} \ge \gamma_i \quad , \quad i = 3,4, \dots, m^{\backslash} \tag{1.5}$$

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i \quad , \quad i = m^{\backslash} + 1, m^{\backslash} + 2, \dots, m \tag{1.6}$$

$$x_j \ge 0 \quad , \quad j = 1,2,3, \dots, n \tag{1.7}$$

Where $\tilde{a}_{ij}$, $\tilde{b}_i$ are random parameters and c_j, a_{ij}, b_i are constants and $0 < \gamma$, $\gamma_i < 1$, $x_j \ge 0$ denote to tolerance measures and decisions variables respectively.

2. A suggested Bivariate Exponential Distribution

Let $\tilde{a}_{i1}, \tilde{a}_{i2}$ follow exponential distributions with parameter λ_j , $j = 1,2$ and density function $f(\tilde{a}_{ij})$,

$$f(\tilde{a}_{ij}) = \lambda_{ij} \text{Exp}(-\lambda_{ij}\tilde{a}_{ij}) \quad , \quad i = 1,2, \dots m \quad , \quad j = 1,2, \dots, n \tag{2.1}$$

and $\tilde{a}_{i1}, \tilde{a}_{i2}$ are dependent. We assume $f(\tilde{a}_{i1}, \tilde{a}_{i2})$, $F(a_{i1}, a_{i2})$ denote to joint density function and joint cumulative distributed function respectively, and can be stated as:

$$f(\tilde{a}_{i1}, \tilde{a}_{i2}) = \begin{cases} c\,\frac{\lambda_{i1}}{(\lambda_{i1}+\theta)} \text{Exp}[-\lambda_{i1}\tilde{a}_{i1} - (\lambda_{i2}+\theta)\tilde{a}_{i2}] \;,\; \tilde{a}_{i2} > \tilde{a}_{i1} \ge 0 & (2.2) \\ c\,\frac{\lambda_{i2}}{(\lambda_{i2}+\theta)} \text{Exp}[-(\lambda_{i1}+\theta)\tilde{a}_{i1} - \lambda_{i2}\tilde{a}_{i2}] \;,\; \tilde{a}_{i1} > \tilde{a}_{i2} \ge 0 & (2.3) \end{cases}$$

$$c = \frac{(\lambda_{i1}+\theta)(\lambda_{i2}+\theta)(\lambda_{i1}+\lambda_{i2}+\theta)}{(\lambda_{i1}+\lambda_{i2})} \quad , \quad 0 \le \theta \le 1 \tag{2.4}$$

$$F(a_{i1}, a_{i2}) = P_r(\tilde{a}_{i1} \le a_{i1} \;, \tilde{a}_{i2} \le a_{i2}) \quad , \quad a_{i1}, a_{i2} > 0$$

$$= \begin{cases} \frac{(\lambda_{i1}+\lambda_{i2}+\theta)}{(\lambda_{i1}+\lambda_{i2})}\{1 - \text{Exp}[-\lambda_{i1}a_{i1}]\}\{1 - \text{Exp}[-(\lambda_{i2}+\theta)a_{i2}]\} & (2.5) \\ \frac{(\lambda_{i1}+\lambda_{i2}+\theta)}{(\lambda_{i1}+\lambda_{i2})}\{1 - \text{Exp}[-(\lambda_{i1}+\theta)a_{i1}]\}\{1 - \text{Exp}[-\lambda_{i2}a_{i2}]\} & (2.6) \end{cases}$$

The above model is considered an extension of Freund's bivariate exponential distribution [3]. It notes that (2.2)-(2.6) when $\tilde{a}_1, \tilde{a}_2$ are independent, in turn $\theta = 0$ then:

$$f(\tilde{a}_{i1}, \tilde{a}_{i2}) = \lambda_{i1}\lambda_{i2}\, \text{Exp}(-\lambda_{i1}\tilde{a}_{i1} - \lambda_{i2}\tilde{a}_{i2}) \tag{2.7}$$

$$F(a_{i1}, a_{i2}) = \{1 - \text{Exp}[-\lambda_{i1}a_{i1}]\}\{1 - \text{Exp}[-\lambda_{i2}a_{i2}]\} \tag{2.8}$$

3. Probability Distribution of $\sum_{j=1}^{2} \tilde{a}_{ij}\, x_j$, $x_j > 0$, $j = 1, 2$

Let $(\tilde{a}_{i1}, \tilde{a}_{i2})$ follow the suggested distribution in (2.2) – (2.6). The following theorem gives the probability distribution of $\tilde{y}_i = \sum_{j=1}^{2} \tilde{a}_{ij}\, x_j$**:**

<u>Theorem (3.1):</u> Let $f(\tilde{y}_i)$, $F(y_i)$ donate to a density function and cumulative distribution function of $\tilde{y}_i$ respectively, then

(1) $$f(\tilde{y}_i) = \begin{cases} g_1(x_1, x_2)\left\{\text{Exp}\left[\frac{-\lambda_1}{x_1}\right]\tilde{y}_i - \text{Exp}\left[\frac{-(\lambda_2+\theta)}{x_2}\right]\tilde{y}_i\right\} \;,\; \tilde{a}_2 > \tilde{a}_1 & (3.1) \\ g_2(x_1, x_2)\left\{\text{Exp}\left[\frac{-(\lambda_1+\theta)}{x_1}\right]\tilde{y}_i - \text{Exp}\left[\frac{-\lambda_2}{x_2}\right]\tilde{y}_i\right\} \;,\; \tilde{a}_1 > \tilde{a}_2 & (3.2) \end{cases}$$

$$g_1(x_1, x_2) = \frac{c\lambda_1}{(\lambda_1+\theta)[(\lambda_2+\theta)x_1 - \lambda_1 x_2]} \tag{3.3}$$

$$g_2(x_1,x_2) = \frac{c\lambda_2}{(\lambda_2+\theta)[\lambda_2 x_1-(\lambda_1+\theta)x_2]} \tag{3.4}$$

$$(2) \qquad F(y_i) = \begin{cases} g_1(x_1,x_2)\left\{\frac{x_1}{\lambda_1}\left[1-\text{Exp}\left(\frac{-\lambda_1}{x_1}\right)y_i\right]-\frac{x_2}{(\lambda_2+\theta)}\left[1-\text{Exp}\left[\frac{-(\lambda_2+\theta)}{x_2}\right]y_i\right]\right\} & , \quad a_2>a_1 \\ g_2(x_1,x_2)\left\{\frac{x_1}{(\lambda_1+\theta)}\left[1-\text{Exp}\left[\frac{-(\lambda_1+\theta)}{x_1}\right]y_i\right]-\frac{x_2}{\lambda_2}\left[1-\text{Exp}\left(\frac{-\lambda_2}{x_2}\right)y_i\right]\right\} & , \quad a_1>a_2 \end{cases}$$

Proof: let $\tilde{a}_2 > \tilde{a}_1$, by using transformation technique [15] let $\tilde{k} = \tilde{a}_1 x_1$, $\tilde{y}_i = \tilde{a}_1 x_1 + \tilde{a}_2 x_2 \longrightarrow$

$$|J| = \begin{vmatrix} \frac{\partial \tilde{a}_1}{\partial \tilde{y}_i} & \frac{\partial \tilde{a}_1}{\partial \tilde{k}} \\ \frac{\partial \tilde{a}_2}{\partial \tilde{y}_i} & \frac{\partial \tilde{a}_2}{\partial \tilde{k}} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{x_1} \\ \frac{1}{x_2} & \frac{-1}{x_2} \end{vmatrix} = \left|\frac{1}{x_1 x_2}\right| = \frac{1}{x_1 x_2} \tag{3.5}$$

$$f(\tilde{y}_i,\tilde{k}) = \frac{1}{x_1x_2}\ c\ \left(\frac{\lambda_1}{\lambda_1+\theta}\right)\ \text{Exp}\left\{\frac{-\lambda_1}{x_1}\ \tilde{k} - (\lambda_2+\theta)\left(\frac{\tilde{y}_i-\tilde{k}}{x_2}\right)\right\} \longrightarrow$$

$$f(\tilde{y}_i) = \int_0^{\tilde{y}_i} f(\tilde{y}_i,\tilde{k})\, d\tilde{k}$$

$$= \frac{c\lambda_1}{(\lambda_1+\theta)[(\lambda_2+\theta)x_1-\lambda_1 x_2]}\ \left\{\text{Exp}\left[\frac{-\lambda_1}{x_1}\right]\tilde{y}_i - \text{Exp}\left[\frac{-(\lambda_2+\theta)}{x_2}\right]\tilde{y}_i\right\}$$

By same way, $f(\tilde{y}_i)$ is obtained when $\tilde{a}_1 > \tilde{a}_2$.

$$F(y_i) = P_r(\tilde{y}_i \le y_i) = \int_0^{y_i} f(\tilde{y}_i)\, d\tilde{y}_i$$

$$= \begin{cases} g_1(x_1,x_2)\left\{\frac{x_1}{\lambda_1}\left[1-\text{Exp}\left(\frac{-\lambda_1}{x_1}\right)y_i\right]-\frac{x_2}{(\lambda_2+\theta)}\left[1-\text{Exp}\left[\frac{-(\lambda_2+\theta)}{x_2}\right]y_i\right]\right\} & , \quad a_2>a_1 \qquad (3.6) \\ g_2(x_1,x_2)\left\{\frac{x_1}{(\lambda_1+\theta)}\left[1-\text{Exp}\left[\frac{-(\lambda_1+\theta)}{x_1}\right]y_i\right]-\frac{x_2}{\lambda_2}\left[1-\text{Exp}\left(\frac{-\lambda_2}{x_2}\right)y_i\right]\right\} & , \quad a_1>a_2 \qquad (3.7) \end{cases}$$

4. The equivalent deterministic constraints

Let individual constraints in (1.5)

$$P_r\left\{\sum_{j=1}^{2} \tilde{a}_{ij}\, x_j + \sum_{j=3}^{n} a_{ij}\, x_j \le b_i\right\} \ge \gamma_i \ \longrightarrow$$

$$P_r\left\{\tilde{y}_i \le b_i - \sum_{j=3}^{n} a_{ij}\, x_j\right\} \ge \gamma_i \longrightarrow F\left(b_i - \sum_{j=3}^{n} a_{ij}\, x_j\right) \ge \gamma_i \qquad ,i = 3,4,\dots,m \tag{4.1}$$

From (3.6), (3.7) substitute in L.H.S. of constraints (4.1), the following equivalent deterministic constraint are:

$$g_1(x_1,x_2)\left\{\frac{x_1}{\lambda_{i1}}\left[1-\text{Exp}\left(\frac{-\lambda i_1}{x_1}\right)\left(b_i-\sum_{j=3}^{n} a_{ij}\, x_j\right)\right]-\frac{x_2}{(\lambda_{i2}+\theta)}\left[1-\text{Exp}\left[\frac{-(\lambda_{i2}+\theta)}{x_2}\right]\left(b_i-\sum_{j=3}^{n} a_{ij}\, x_j\right)\right]\right\} \ge \gamma_i \quad , \quad \tilde{a}_2 > \tilde{a}_1 \tag{4.2}$$

or

$$g_2(x_1,x_2)\left\{\frac{x_1}{(\lambda_{i1}+\theta)}\left[1-\text{Exp}\left[\frac{-(\lambda_{i1}+\theta)}{x_1}\right]\left(b_i-\sum_{j=3}^{n} a_{ij}\, x_j\right)\right]-\frac{x_2}{\lambda_2}\left[1-\text{Exp}\left(\frac{-\lambda i_2}{x_2}\right)\left(b_i-\sum_{j=3}^{n} a_{ij}\, x_j\right)\right]\right\} \ge \gamma_i \quad , \quad \tilde{a}_1 > \tilde{a}_2 \tag{4.3}$$

In section (5), numerical example illustrates the transformation [14,15].

5. Joint constraints

In this section, joint chance constraints in (1.4) are considered where $(\tilde{b}_i, \tilde{b}_{i+1})$,follow bivariate the suggested distribution, and the density and cumulative functions $f(\tilde{b}_i, \tilde{b}_{i+1})$, $F(b_i, b_{i+1})$ respectively are:

$$f(\tilde{b}_i, \tilde{b}_{i+1}) = \begin{cases} c\frac{\lambda_i}{(\lambda_i+\theta)} \text{Exp}\left[-\lambda_i \tilde{b}_i - (\lambda_{i+1}+\theta)\tilde{b}_{i+1}\right] \;,\; \tilde{b}_{i+1} > \tilde{b}_i & (5.1) \\ c\frac{\lambda_{i+1}}{(\lambda_{i+1}+\theta)} \text{Exp}\left[-(\lambda_i+\theta)\tilde{b}_i - \lambda_{i+1}\tilde{b}_{i+1}\right] \;,\; \tilde{b}_i > \tilde{b}_{i+1} & (5.2) \end{cases}$$

$$F(b_i, b_{i+1}) = \begin{cases} \frac{(\lambda_i+\lambda_{i+1}+\theta)}{(\lambda_i+\lambda_{i+1})}\{1-\text{Exp}[-\lambda_i b_i]\}\{1-\text{Exp}[-(\lambda_{i+1}+\theta)b_{i+1}]\} \;,\; b_{i+1} > b_i & (5.3) \\ \frac{(\lambda_i+\lambda_{i+1}+\theta)}{(\lambda_i+\lambda_{i+1})}\{1-\text{Exp}[-\lambda_{i+1} b_{i+1}]\}\{1-\text{Exp}[-(\lambda_i+\theta)]\} \;,\; b_i > b_{i+1} & (5.4) \end{cases}$$

Three lemma are presented to transform the joint constraints in different types to equivalent deterministic constraints.

Lemma (5.1): Let the joint constraints:

$$P_r\left(\sum_{j=1}^n a_{1j}x_j \le \tilde{b}_i \;,\; \sum_{j=1}^n a_{2j} \le \tilde{b}_i\right) \ge \gamma_1^\backslash \tag{5.5}$$

Then the equivalent deterministic constraints are:

$$\frac{(\lambda_1+\lambda_2+\theta)}{(\lambda_1+\lambda_2)} \text{Exp}\left\{-\left[\lambda_1 \textstyle\sum_j a_{1j}x_j + (\lambda_2+\theta)\sum_j a_{2j}x_2\right]\right\} - \left(\frac{\lambda_2+\theta}{\lambda_1+\lambda_2}\right) \text{Exp}\left\{-(\lambda_1+\lambda_2+\theta)\textstyle\sum_j a_{2j}x_j\right\} \ge \gamma_1^\backslash \;,\; \tilde{b}_2 > \tilde{b}_1 \tag{5.6}$$

or

$$\frac{(\lambda_1+\lambda_2+\theta)}{(\lambda_1+\lambda_2)} \text{Exp}\left\{-\left[(\lambda_1+\theta)\textstyle\sum a_{1j}x_j + \lambda_2\sum_j a_{2j}x_j\right]\right\} - \left(\frac{\lambda_1+\theta}{\lambda_1+\lambda_2}\right) \text{Exp}\left\{-(\lambda_1+\lambda_2+\theta)\textstyle\sum a_{1j}x_j\right\} \ge \gamma_1^\backslash \;,\; \tilde{b}_1 > \tilde{b}_2 \tag{5.7}$$

and $$\sum a_{1j}x_j \ge 0 \;,\quad \sum a_{2j}x_j \ge 0 \tag{5.8}$$

Proof: When $\tilde{b}_2 > \tilde{b}_1$, then the L.H.S. of constraints (5.5)

$$P_r\left(\textstyle\sum_j a_{1j}x_j \le \tilde{b}_1 \;,\; \sum_j a_{2j}x_j \le \tilde{b}_2\right) = \int_{\sum_j a_{2j}x_j}^{\infty}\int_{\sum_j a_{1j}x_j}^{\tilde{b}_2} c\frac{\lambda_1}{(\lambda_1+\theta)} \text{Exp}\left[-\lambda_1\tilde{b}_1 - (\lambda_2+\theta)\tilde{b}_2\right] d\tilde{b}_1\, d\tilde{b}_2$$

$$= \frac{(\lambda_1+\lambda_2+\theta)}{(\lambda_1+\lambda_2)} \text{Exp}\left\{-\left[\left(\lambda_1 \textstyle\sum_j a_{1j}x_j\right) + (\lambda_2+\theta)\sum_j a_{2j}x_j\right]\right\} - \left(\frac{\lambda_2+\theta}{\lambda_1+\theta}\right) \text{Exp}\left\{-(\lambda_1+\lambda_2+\theta)\textstyle\sum_j a_{2j}x_j\right\}$$

When $\tilde{b}_1 > \tilde{b}_2$, then the L.H.S. of (5.5)

$$P_r\left(\textstyle\sum_j a_{1j}x_j \le \tilde{b}_1 \;,\; \sum_j a_{2j}x_j \le \tilde{b}_2\right) = \int_{\sum_j a_{1j}x_j}^{\infty}\int_{\sum_j a_{2j}x_j}^{\tilde{b}_1} c\frac{\lambda_2}{(\lambda_2+\theta)} \text{Exp}\left[-(\lambda_1+\theta)\tilde{b}_1 - \widetilde{\lambda_2 b_2}\right] d\tilde{b}_1\, d\tilde{b}_2$$

$$= \frac{(\lambda_1+\lambda_2+\theta)}{(\lambda_1+\lambda_2)} \text{Exp}\left\{-\left[(\lambda_1+\theta)\textstyle\sum_j a_{1j}x_j + \lambda_2\sum_j a_{2j}x_j\right]\right\} - \left(\frac{\lambda_1+\theta}{\lambda_1+\lambda_2}\right) \text{Exp}\left\{-(\lambda_1+\lambda_2+\theta)\textstyle\sum_j a_{1j}x_j\right\}$$

Lemma (5.2): Let the joint constraints:

$$P_r\left(\textstyle\sum_j a_{3j}x_j \le \tilde{b}_3 \;,\; \sum_j a_{4j}x_j \ge \tilde{b}_i\right) \ge \gamma_2^\backslash \tag{5.9}$$

Then the equivalent deterministic constraints are:

$$\frac{(\lambda_3+\lambda_4+\theta)}{(\lambda_3+\lambda_4)} \text{Exp}\left\{-\lambda_3\textstyle\sum_j a_{3j}x_j\right\}\left\{1-\text{Exp}\left[-(\lambda_4+\theta)\sum_j a_{4j}x_j\right]\right\} - \left(\frac{\lambda_3+\theta}{\lambda_3+\lambda_4}\right)\left\{1-\text{Exp}\left[-(\lambda_3+\lambda_4+\theta)\textstyle\sum_j a_{4j}x_j\right]\right\} \ge \gamma_2^\backslash \;,\; \tilde{b}_4 > \tilde{b}_3 \tag{5.10}$$

or

$$\frac{(\lambda_3+\lambda_4+\theta)}{(\lambda_3+\lambda_4)} \text{Exp}\left\{-\lambda_3\textstyle\sum_j a_{3j}x_j\right\}\left\{1-\text{Exp}\left[-(\lambda_4+\theta)\sum_j a_{4j}x_j\right]\right\} - \left(\frac{\lambda_4+\theta}{\lambda_3+\lambda_4}\right)\left\{1-\text{Exp}\left[-(\lambda_3+\lambda_4+\theta)\textstyle\sum_j a_{4j}x_j\right]\right\} \ge \gamma_2^\backslash$$

$$, \quad \tilde{b}_3 > \tilde{b}_4 \qquad (5.11)$$

and $\sum a_{3j}x_j \geq 0 \ , \quad \sum a_{4j}x_j \geq 0$

Proof: By the same way in Lemma (5.1).

Lemma (5.3): Let the joint constraints:

$$P_r\left(\sum_j a_{5j}x_j \leq \tilde{b}_5 \quad , \quad \sum_j a_{6j}x_j \geq \tilde{b}_6\right) \geq \gamma_3^{\backslash} \qquad (5.12)$$

Then, the equivalent deterministic constraints are:

$$\frac{(\lambda_5+\lambda_6+\theta)}{(\lambda_5+\lambda_6)}\left\{1 - \mathrm{Exp}\left[-\lambda_5 \sum_j a_{5j}x_j\right]\right\}\left\{1 - \mathrm{Exp}\left[-(\lambda_6+\theta)\sum_j a_{6j}x_j\right]\right\} \geq \gamma_3^{\backslash} \ , \qquad \tilde{b}_6 > \tilde{b}_5 \qquad (5.13)$$

or

$$\frac{(\lambda_5+\lambda_6+\theta)}{(\lambda_5+\lambda_6)}\left\{1 - \mathrm{Exp}\left[-(\lambda_5+\theta)\sum_j a_{5j}x_j\right]\right\}\left\{1 - \mathrm{Exp}\left[-\lambda_6 \sum_j a_{6j}x_j\right]\right\} \geq \gamma_3^{\backslash} \ , \qquad \tilde{b}_5 > \tilde{b}_6 \qquad (5.14)$$

and $\sum a_{5j}x_j \geq 0 \ , \quad \sum a_{6j}x_j \geq 0$

Proof: By the same way in Lemma (5.1).

6. Numerical example

Consider the following (CCP) model and $(\tilde{a}_1, \tilde{a}_2)$ follow the bivariate suggested distribution with parameters $\lambda_1 = 2 \ , \ \lambda_2 = 1 \ , \ \theta = 0.7 \ , \ \tilde{a}_2 > \tilde{a}_1$

$$\begin{array}{lll} & \text{Max. } H = 4x_1 + 3x_2 + x_3 & (1) \\ \text{S.T.} & P_r(\tilde{a}_1x_1 + \tilde{a}_2x_2 + 2x_3 \leq 15) \geq 0.8 & (2) \\ & 5x_1 + 7x_2 + x_3 \leq 35 & (3) \\ & 10x_1 + 3x_2 + 2x_3 \leq 30 & (4) \\ & x_1, x_2, x_3 \geq 0 & (5) \end{array}$$

Solution: the constraint (2) can be rewritten as:

$$F(\tilde{y} \leq 15 - 2x_3) \geq 0.8 \qquad (6)$$

From (2.4)-(3.4),c, $g_2(x_1, x_2)$ are computed as following:

$$c = 5.661 \ , \quad g_2(x_1, x_2) = \frac{5.661}{(1.7x_1 - 4.59x_2)}$$

From (3.7) substituting in L.H.S. of (6), the equivalent constraints of (2) is:

$$\frac{5.661}{(1.7x_1-4.59x_2)}\left\{\frac{x_1}{1.7}\left[1 - \mathrm{Exp} - \left(\frac{40.5-5.4x_3}{x_1}\right)\right] - \left[1 - \mathrm{Exp} - \left(\frac{15-2x_3}{x_2}\right)\right]\right\} \geq 0.8 \qquad (7)$$

The above constraint can be approximated to linear by using Taylor theorem (initial point: $x_1^0 = 1, x_2^0 = 0.5, x_3^0 = 1$) as following:

$$6.6x_1 + 181.81x_2 + 0.058x_3 \leq 97.57 \qquad (8)$$

By Simplex, the LP model (1,3,4,5,8) is solved and its optimal solution:

$$H^* = 15.79 \ , \ x_1^* = 0 \ , \ x_2^* = 0.53 \ , \ x_3^* = 14.2$$

7. Conclusions

In this paper, firstly suggested bivariate exponential distribution model is presented, it is considered an extension of Freund's model. This distribution is more applicable in CCP problems when some parameters are dependent random variables.

Secondly, a suggested approach to transform individual and joint (dividual) constraints to equivalent deterministic constraints is presented also. Finally, numerical example is introduced to illustrate the suggested distribution and approach.

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