# SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF INTEREST: EVOLUTION EQUATIONS AND THEIR PHYSICAL MEANING

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**ABSTRACT**

In this paper we perform an overview of some nonlinear partial differential equations (nPDEs) specifically evolution equations (EVs). With the help of nPDEs several processes in nature as well as economic, social, human medicine and technical sciences can be studied.

In opposite to linear equations the nonlinear version can not be classified exactly. Thus several special techniques are necessary to study initial values problems (IVP) and/or boundary condition (BVP) problems and eventually eigenvalue problems (EVP).

Also the qualitative behavior of solutions (more precisely classes of solutions) requires profound considerations.

The structure of the paper is as follow: First we present some interesting nPDEs importantly in natural sciences. Then we develop some tools of functional analysis especially for existence and uniqueness. We further introduce terms like weak and/or strong convergence as well as weak and/or classical solutions. A further point of interest is the positivity and the long-time behavior of solutions.

A critical part will be the proofs; we try to perform them as clear as possible to ensure the physical and mathematical correctness. We find it useful to cite some original works of historical interest.

**Keywords:** *Nonlinear partial differential equations (nPDEs), evolution equations, functional analysis, weak solutions, weak convergence, compactness, stationarity.*

**PACS Codes:** 02.30Jr, 02.20Qs, 02.30Hq, 03.40Kf.
**AMS-Code:** 35L05, 35Q53, 14H05.

## 1. Some equations of physical/chemical/technical interest

### 1.1. A liquid or a gas in a porous medium

We like to derive an equation which describes the evolution of a liquid through a porous medium or the flow of a gas through sandstone [1]. Otherwise the equation may describe the motion of ground water in rock layers or groundwater and surface water [2]. We are looking for an equation which describes the density of the liquid $n(x,t)$. Assuming conservation of mass the function $n(x,t)$ satisfies

$$n_t = \frac{\partial n}{\partial t} = div(n.v) = 0\,, \tag{1}$$

and $v$ is the mean velocity of the liquid. Further we assume that the velocity is proportional to the gradient of the pressure. It's called the law of Darcy or steady-state groundwater flow equation [3]

$$v = -k\nabla p\,,\ \ k > 0\,, \tag{2}$$

$k$ depends upon the viscosity of the liquid. As a third assumption the adiabatic state equation holds

$$p = n^{\gamma}\,,\ \gamma > 0\,. \tag{3}$$

Putting together (1), (2) and (3), one derives at the porous medium equation (PME)

$$n_t - div\left(\frac{k.\gamma}{\gamma+1}\nabla n^{\gamma+1}\right). \tag{4}$$

The PME is a quasilinear parabolic equation and will be solved in a domain $\Omega \subset R^n$ considering initial and/or boundary conditions $n = 0$ upon $\partial\Omega$, $t > 0$, $n(.,0)$ in $\Omega$. On can write the PME as

$$n_t - div\left(D(n)\nabla n\right) = 0 \text{ with } D(n) = k\gamma n^{\gamma}, \tag{4a}$$

where $D(n)$ is the diffusion term (density dependent). If the diffusion is constant, $D(n) = D_0$ one derives at the heat equation in the form $u_t - D_0 \Delta n = 0$ by using the operator $L(u) = -div\left(D(n)\nabla n\right)$.

**1.2. Real liquids and gases**

Imagine a liquid or a gas (water or air) with particle density $n(x,t)$ and the particle's velocity $v(x,t)$. The conservation law for the particle density is equal to eq.(1) The particle current will be described by a current density as $(n.v)$ which satisfies

$$(n.v)_t + div(n.v \otimes v) + grad(p) = F(v). \tag{5}$$

$(v \otimes v)$ is a matrix with components $v_i v_j$ (tensor product) and $p$ is the pressure. If $F(v) = 0$ holds one has the Euler equations, e.g. [4], [5]. It may be interesting to study the original work appeared in the year 1757 [6]. For some historical works relating hydrodynamic see [7], [8].

Connecting (1) and (5) one derives at a set of equations, called the Navier-Stokes Equations (NSE)

Some exact solutions of the NSE exist. Examples of degenerate cases with the nonlinear terms in the NSE are equal to zero, they are called Poiseuille flow, Couette flow and the oscillatory Stokes boundary layer. Navier (1822) and Stokes (1845) can be seen as the basic description of all known liquid properties in mechanical sense (it is assumed that the fluid being studied is a continuum). Note that the existence of exact solutions does not imply that they are stable: Turbulence may develop at higher Reynolds numbers [9]. The NSE is also of great interest in a purely mathematical sense. Somewhat surprisingly, given their wide range of practical uses, it has not yet been proven that in three dimensions solutions always exist (existence), or that if they do exist, then they do not contain any singularities (they are smooth).

These are called the Navier Stokes existence and smoothness problems. The relation with the Maxwell Equations lead to magnetohydrodynamic, MHD, [10], also astrophysical applications are well discussed nowadays e.g. [11], [12].

**1.3. A chemical example I – reaction of chemicals**

The concentration of a chemical in a domain $\Omega$ assuming constant diffusion is given by

$$\mathrm{C} u_t - \Delta u = f_+ + f_-, \tag{6}$$

with a constant C. The functions $f_\pm$ act as sources and sinks in the considered domain $\Omega$. Assuming that the chemical will be created by a constant rate $f_+ = f_0 > 0$ and assuming further that the sinks depend quadratically upon the density $f_- = -u^2 \leq 0$. If the chemical can not pass the domain one assumes Neumann boundary conditions

$$\frac{\partial u}{\partial v} = 0 \text{ upon } \partial\Omega,\ t > 0,\ u(.,0) = u_0 \text{ in } \Omega, \tag{7}$$

so that the chemical has the density $u_0$ at $t = 0$. Such initial value problems lead to semi-linear parabolic equations, therefore, types of

$$u_t = \frac{\partial u}{\partial t} - \Delta u = f(x,u) \tag{8}$$

are called reaction-diffusion equations, e.g. originally called the Fisher Equation [13]. It is pertinent to mention works on gene culture [14], propagation of chemical waves [15] or neutron population in a nuclear reactor [16] and [17]. It is assumed that the concentration does not change at $t \to \infty$ and $u_t = 0$ so that one has to solve the problem in a special domain and the normal derivative upon the bound of the domain:

$$\Delta u = u^2 - f_0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial v} = 0 \text{ on } \partial\Omega\ . \tag{9}$$

A solution is $u_\infty = \sqrt{f_0}$ and it is of interest to know how fast $u(.,0)$ converges into the stationary solution.

### 1.4. A chemical example II – time-dependent electrochemistry

An advance in understanding of the distribution of charges around an ion in aqueous solution was achieved by Debye and Hückel [18], [19]. The idea lays in the formulation of a model for the time-average distribution of ions in very dilute solutions of electrolytes. From this distribution they were able to calculate the electrostatic potential contributed by the surrounding ions to the total electrostatic potential at the reference ion and hence, the chemical potential change arising from the ion-ion interaction. In extensive studies the author developed a model equation derived from the nonlinear Poisson-Boltzmann Equation (precisely the electroquasistatic approximation, EQS is assumed). Then one derives a nPDE of the third order for the unknown potential function [20]:

$$\frac{\partial^2 u}{\partial x^2}+\tau\frac{\partial^3 u}{\partial x^2\partial t}-\mu^2\frac{\partial u}{\partial t}e^{-\eta u}=0,\quad u=u(x,t),\ -\infty<x<\infty. \tag{10}$$

At this stage let us impose boundary conditions so that $\lim\limits_{x\to\infty} u_0=u_L$ and $\lim\limits_{x\to\infty}\frac{du}{dx}=0$ holds; they are necessary conditions later for the function $u=u(x,t)$. Here $\tau$, $\mu$ and $\eta$ are some electrochemical parameters. We seek for solutions for which $u=F(x,t)$, $F\in C^3(D)$, $D\subset R^2$ is an open set and further we exclude $D:=\left\{(u(x,t))\in\tilde{D}: u=0,\ u_x=0, u_t=0,.....\right\}$ and a positive time $t>0$. Suitable classes of solutions are $u\in I$, $I$ an interval so that $I\subseteq D$ and $u: I\to R^2$. Note that the nPDE above was studied by application of algebraic methods, e.g. [21], [22]. An extended version of the nPDE (10) is just in study and will appear soon. We have derived the following equation:

$$\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial^2 u}{\partial x^2}+\alpha^2\frac{\partial^2 u}{\partial x\partial t}-\beta^2\frac{\partial u}{\partial t}e^{-\eta u}=0,\quad u=u(x,t),\ -\infty<x<\infty, \tag{10a}$$

where a concentrations gradient is taken into account. Such types of nPDEs can be analyzed by methods derived by the author, e.g. [23], [24], [25] and [26].

**Note**: The correct terminus for solution(s) of PDE(s) is (are) always 'classes of solutions'. We here arrange to write simply 'solutions'.

### 1.5. An example of solid state physics – electrons in a semiconductor (SC)

Classically the motion of electrons through a SC can be described by a diffusion current. Such a current is originated by a change of a particle density otherwise a drift current acts due to the electrical field. Let $n$ be the particle density and $\varphi$ the electrical field, the total current density is

$$J=\nabla n-n\nabla\varphi. \tag{11}$$

The evolution of the electron density can be expressed by mass conservation so that one has

$$n_t=\frac{\partial n}{\partial t}-divJ=0 \tag{12}$$

The electrical potential is a solution of the Poisson Equation

$$\Delta\varphi=n-f(x), \tag{12a}$$

where the function $f(x)$ describes charges in the SC and $n-f$ is the total charge density. The pair $(n,f)$ is a solution of the coupled equations

$$div(\nabla n-n\nabla\varphi)=0 \text{ and } \Delta\varphi=n-f(x). \tag{12b}$$

We assume that $(n,f)$ is known upon the boundary, so that $n=g$ and $\varphi=\psi$ upon $\partial\Omega$.

Since electrons are quantum-like particles they must be described by a complex-valued function, the wave function $\psi(\vec{r},t)$ and this function satisfies the following PDE, the Schrödinger Equation [27]

$$i\psi_t+\Delta\psi-V(\vec{r},t)\psi=0 \text{ in } R^n \text{ and } \psi(.,0)=\psi_0. \tag{12c}$$

It can be shown that $\partial_t |\psi|^2 - divJ = 0$ holds; this can be interpreted as a conservation equation for the particle density so that $n = |\psi|^2$, (the square of this function describes physical reality).

**1.6. A mathematical example I: A fourth-order nPDE:**
The scaled equation in (1+1) dimension under consideration is given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial^3 x \partial t} + 3\,\frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 u}{\partial x \partial t} = 0\ ,\ u = u(x,t)\ ,\ u \in (-\infty,\infty)\,,\ t > 0\,. \tag{13}$$

A physical meaning is not known up to now but intensive studies were done by the author [28], [29], [30] and [31]. The author has shown that the equation possesses the Painlevé property that means solutions can be expressed in terms of Laurent series. Also the nPDE possesses traveling wave solutions and further solutions could derived by classical Lie Group Analysis [28].

**1.7. A mathematical example II: A nPDE of the sixth-order – or Ramani's Equation:**
The following less studied nPDE is given by

$$u_{6x} + 15\,u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} - 5u_{xxxt} - 15u_{xx}u_t - 15u_x u_{xt} - 5u_{tt} = 0\,,$$
$$u = u(x,t)\ ,\ u \in (-\infty,\infty)\,,\ t > 0\,. \tag{14}$$

A physical meaning is unknown up to now however travelling waves exist (there are several difficulties due to the different mixed nonlinear terms) and moreover, a similarity study using classical and nonclassical symmetry methods were done also by the author [32] and [33].

**2. Some functional analytical basics**
We abstain from definitions like Hilbert-, Banach-, and Sobolev spaces however some facts should bear in mind for the following. $B$ is always a Banach space with norm $\|.\|$.

A dual space is defined by $B' = \{f : B \to R,\ linear,\ continous\}$. The norm in $B'$ is defined as $\|f\|_{B'} = \sup\limits_{\|u\| \le 1} |f(u)|$.

**Definition:** A series $(u_k) \subset B$ is called weakly convergent as $u \in B$ if for all $F \in B'$ follows $\langle F, u_k \rangle_{B'} \to \langle F, u \rangle_{B'}$ as $k \to \infty$. In this case we write $u_k \to u$ as $k \to \infty$. By $\Omega$ we mean an open set so that $\Omega \subset R^n,\ n \ge 1.$ Remember that each Hilbert space is a special Banach space!
A Sobolev space $W_0^{m,p}(\Omega)$ is the ending of $C_0^\infty(\Omega)$ with norm $\|.\|_{W^{m,p}(\Omega)}$ and $m \in N$ and $1 \le p \le \infty$. The space $W_0^{m,p}(\Omega)$ is also a Banach space (we refer to the imbedding theorems from Sobolev).

**Proposition:** (properties of weak convergence)
(i) strong convergence implies weak convergence, (ii) if $\dim B < \infty$ then weak convergence implies strong convergence, (iii) theorem of Banach-Steinhaus: For $u_k \to u$ ($k \to \infty$) for a series $(u)_k \subset B$ then $(u)_k$ is bounded and $\|u\| \le \liminf \|u_k\|$ as $k \to \infty$.

**Definition:** A series $(F_k) \in B'$ is weak convergent against $F \in B'$ if for all $u \in B$

$$\langle F_k, u \rangle_{B'} \to \langle F, u \rangle_{B'} \text{ as } k \to \infty\,. \tag{15}$$

In this case we write $F_k \xrightarrow{k \to \infty} {}^*F$.

**Theorem:** Let $B$ a separable Banach space. Each bounded series in $B'$ has a weak convergent subseries.
**Proof:** For a detailed proof see [34]: Let $(u_k)$ be a bounded series in $L^\infty(\Omega)$ and $B = L^1(\Omega)$. Since

$L^1(\Omega)' = L^\infty(\Omega)$ holds, a partial series $(u_{k'})$ exists so that $u_k \to {*u}$ in $L^\infty(\Omega)$, or otherwise

$$\int u_{k'} v\, dx \to \int u\, v\, dx \text{ for all } v \in L^1(\Omega) \ \blacksquare \tag{16}$$

**Proposition:** Properties of weak convergence; (i) strong convergence implies weak * convergence,
(ii) in reflexive Banach spaces strong and weak *convergence are equal, (iii) the theorem of Banach-Steinhaus holds, (iv) if $u_k \to {*u}$ $u_k \to {*u}$ in $B'$ and $v_k \to {*v}$ in $B$ and $u_k \to u$ in $B'$ and $v_k \to v$ in $B$, it follows that: $\langle u_k, v_k \rangle_{B'} \to \langle u, v \rangle_{B'}$ as $k \to \infty$.

**Example:** Consider the following IVP

$$\rho_t + q_x = 0,\ x \in R,\ t > 0, \tag{17}$$

$$\rho(x,0) = f(x),\ x \in R, \tag{17a}$$

where $q(x,t)$ is a continuously differentiable function on $R$. A classical solution of the PDE, eq.(17) is a smooth function $\rho = \rho(x,t)$ which satisfies eq.(17a). Now we assume that $\rho(x,t)$ is a classical solution of (17a). We consider a class of test functions $\varphi = \varphi(x,t)$, such that $\varphi \in C^\infty(\Omega)$ and has a support upon the $(x,t)$-plane. We can choose an arbitrary rectangle $D = \{(x,t) : a \le x \le b,\ 0 \le t \le T\}$ where $\varphi = 0$ outside $D$ and on the boundary lines $x = a,\quad x = b$ and $t = T$. Eq.(17) is multiplied with $\varphi(x,t)$ and integrated over $D$ to obtain the two-fold integral $\iint_D (\rho_t + q_x)\varphi\, dx\, dt = 0$. Integrating both terms by parts one ends up at the final form

$$\iint_{D U t \ge 0} (\rho \varphi_t + q\varphi_x)\, dx\, dt + \int_R \varphi(x,0) f(x)\, dx = 0\,. \tag{17b}$$

**Definition:** A bounded measurable function $\rho(x,t)$ is called a weak solution of the IVP (17), (17a) with bounded and measurable initial data $f(x)$ provided that (17b) holds.

**Theorem:** If (17b) holds for all test functions $\varphi$ with compact support for $t > 0$ and if $u$ is smooth then $\rho$ is a classical solution of the IVP (17), (17a).

**Proof:** Since $\varphi$ has a compact support for $t > 0$ then we integrate (17b) by parts to obtain

$$\iint_{R U t>0} (\rho_t + q_x)\, \varphi\, dx\, dt = 0\ . \tag{17c}$$

This is true for all test functions $\varphi$ and hence (because the bracket expression vanishes)

$$\rho_t + q_x = 0 \text{ for } x \in R,\ t > 0\,. \tag{17d}$$

We next multiply (17d) by $\varphi$ and integrate the resulting equation by parts over the domain $D$ and finally, by subtracting the result from (17b) one derives at

$$\int_R [\rho(x,0) - f(x)]\varphi(x,0) = 0\,. \tag{17e}$$

This is true for all test functions $\varphi(x,0)$. Since $f(x)$ is continuous, (17e) leads to $\rho(x,0) = f(x)$. This shows that $\rho(x,t)$ is the classical solution of the given IVP ■

## 2. Some special equations

### 2.1. Semi-linear equations, from now precisely differential equations (DEs)

Consider the following DE

$$L(u) = f(x,u) \text{ in the domain } \Omega \text{ and } u = g \text{ on } \partial\Omega\,. \tag{18}$$

We arrange some basics: The set $\Omega \in R^n$ is a bounded set with $\partial\Omega \in C^1$. The differential operator $L(u)$ is defined by $L(u) = -div\left(A\nabla u\right) + cu$ where $A(x) = (a_{i\,ij}(x))$ is a $(n \times n)$-matrix and $c(x)$ is a function and moreover $a_{i\,j} \in L^\infty(\Omega)$ and $c \in L^\infty(\Omega)$. The goal should be to prove existence as well as uniqueness. Therefore, in short, we define a fix point operator so that

$$L(u) = f(x,v) \text{ in the domain } \Omega \text{ and } u = g \text{ on } \partial\Omega. \tag{18a}$$

For a given $v$ this is a linear DE. Considering the lemma of Lax-Milgram, e.g. [35]:
A unique solution $u$ exists and this defines the mapping $S(u^*) = u^*$. Can we show this we then have proven the existence of the solution since the DE for the fix point is $L(u^*) = f(x,u^*)$. We refer to the fix point theorems especially from Banach, Schauder and Brouwer [36], [37], [38].

**Theorem (Existence theorem for semi-linear DEs):** The assumptions from above hold and further, let $f$ be a Carathéodory function [39] so that

$$\left| f(x,u) \right| \le h(x) \text{ for } x \in \Omega,\ u \in R \text{ where } h \in L^q(\Omega),\ q \in N_+ . \tag{18b}$$

Then a (weak) solution of eq.(18) exists in $u \in H^1(\Omega)$ with a constant $C > 0$ and the estimation

$$\| u \|_{H^1(\Omega)} \le C\Big( \| g \|_{H^1(\Omega)} + \| h \|_{L^q(\Omega)} \Big). \tag{18c}$$

**2.2. Equations which are quasi-linear**

The goal should be to handle DEs of the form

$$-div(a\nabla u) = f, \text{ in } \Omega,\ u = 0 \text{ upon } \partial\Omega \tag{19}$$

Further we meet the following restrictions: The set $\Omega \in R^n$, $(n \ge 0)$ is a bounded domain with $\partial\Omega \in C^1$, $a = (a_1,.....,a_n): R^n \to R^n$ is a bounded vector field and $f \in L^2(\Omega)$. We have to prove the existence of a solution of eq.(19). As usual: Find a $u \in H_0^1(\Omega)$ with $a(\nabla u) \in L^2(\Omega)$ so that

$$\int_\Omega a(\nabla u)v\,dx = \int f v\,dx \text{ for all } v \in H_0^1(\Omega) \tag{19a}$$

**Theorem (existence theorem for quasi-linear DEs):** The assumptions from above are valid, further we assume that $a$ growths linearly, that means, constants $C > 0, \alpha > 0$ and $\beta \ge 0$ exist, so

$$\left|\alpha(p)\right| \le C\left(1 + |p|\right) \text{ and } \alpha(p).p \ge \alpha(p)^2 - \beta \text{ for all } p \in R^n . \tag{19b}$$

Then a solution $u \in H_0^1(\Omega)$ exists (for a complete proof see e.g. [40]).

**Theorem (uniqueness theorem for quasi-linear DEs):** The assumptions from above are valid, further, $a$ is string monotone. Then just a solution $u \in H_0^1(\Omega)$ exists.

**Proof:** Let $u_1$ and $u_2$ be two solutions of eq.(19). Then for all $v \in H_0^1(\Omega)$ we have from (19a)

$$\int_\Omega a(\nabla u_i)v\,dx = \int_\Omega f v\,dx,\ i = 1,2. \tag{19c}$$

Subtraction of both equations for $i = 1$ and $i = 2$ and using $v = u_1 - u_2$ as a test function we have

$$0 \le \int_\Omega \left\{a\left(\nabla u_1\right) - a\left(\nabla u_2\right)\right\}\left(\nabla u_1 - \nabla u_2\right)dx \ge \gamma \int_\Omega \left|\nabla u_1 - \nabla u_2\right|^2 dx, \tag{19d}$$

and this implies $\nabla(u_1 - u_2) = 0$ in $\Omega$ and due to $(u_1 - u_2) = 0$ upon $\partial\Omega$ in $\Omega$ ∎

**2.3. Drift–Diffusion Equations**

Here we consider equations of the type

$$\nabla(u - u\nabla\varphi) = 0, \;\; \Delta\varphi = u - f(x) \text{ in } \Omega \tag{20}$$

with the boundary conditions $u = g, \, \varphi = \psi$ upon $\partial\Omega$. Such types of equations are physically important, e.g. considering semi conductors in solid state physics. Here $u$ is a particle density and $\psi$ is an electric potential. This is a set of nonlinear equations since the term $u\nabla\varphi$ is nonlinear. One can prove the existence of solutions by applying a fix point theorem. The main result is:

**Theorem (existence theorem for drift–diffusion equations):** Let $\Omega \in R^n, \, (n \geq 0)$ a restricted domain in $\Omega$ with $\partial\Omega \in C^1$, $f \in L^\infty(\Omega)$ with $0 \leq f_\bullet \leq f(x)$, $f(x) \leq f_\bullet$ for $x \in \Omega$, $g, \psi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $0 \leq g_\bullet \leq g(x)$, $g(x) \leq g_\bullet$ for $x \in \partial\Omega$. Then a solution of eq.(20) exists with $(u, \psi) \in \left(H^1(\Omega) \cap L^\infty(\Omega)\right)^2$ and

$$\frac{1}{M e^K} \leq u \leq M e^K, \;\; -K \leq \varphi \leq K \text{ in } \Omega, \tag{20a}$$

where $K$ and $M$ are suitable chosen, e.g. $M = e^{\|\varphi\|_L} \max\left\{g^\bullet, 1/g_\bullet\right\}$.

A weak formulation of (20) is the following: Look for a solution $(u, \varphi) \in H^1(\Omega)^2$ with $u = g$ and $\varphi = \psi$ upon $\partial\Omega$. so that for all $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega u \nabla\varphi \nabla v dx, \;\; \int_\Omega \nabla\varphi \nabla v \, dx = -\int_\Omega (u - f(x) v dx. \tag{20b}$$

All integrals are well-defined and the drift term $\int_\Omega u \nabla\varphi . \nabla v dx$ for all test functions $v \in H_0^1(\Omega)$ must be well-defined, therefore one has to claim $u \in L^\infty(\Omega)$. The drift term is difficult to handle, so a good choice is to introduce a variable transformation of the form $v = e^{-\varphi} u$. Then we have the transformed system

$$\nabla v = e^{-\varphi}(\nabla u - u\nabla\varphi) \text{ and } 0 = div(\nabla u - u\nabla\varphi) = div\left(e^{\varphi}\nabla v\right), \tag{20c}$$

and thus, one has to solve the system

$$div\left(e^{\varphi}\nabla v\right) = 0, \; \Delta\varphi = e^{\varphi} v - f(x) \text{ in } \Omega \tag{20d}$$

with the boundary conditions $v := v_D = e^{-\psi} g$, $g = \psi$ upon $\partial\Omega$.

Any solution of the system $(v, \psi) = \left(H^1(\Omega) \cap L^\infty(\Omega)\right)^2$ defines a weak solution of the original system (20) due to the transformation $v = e^{-\varphi} u$.

**3. Some parabolic equations of physical relevance**

Here we consider types of equations of the form

$$u_t - \Delta u = f(x, u) \tag{21}$$

or similarly

$$u_t = \left(div(au)\nabla u\right) = f(x, t). \tag{21a}$$

Since parabolic equations are functions of space and time one needs Sobolev spaces to differ between space and time.

**3.1. A typical example**

As a simple model of turbulence, introduce a function $u(x,t)$ for the fluid velocity, then one has

$$u_t + u u_x = \delta u_{xx}, \tag{21b}$$

where $\delta$ represents the kinematic viscosity. This equation is known as the Burgers Equation (BE, for the original work see [41] and [42]). The BE is a balance between time evolution, nonlinearity and diffusion. Burgers first developed this equation primarily to shed light on the study of turbulence, described by the interaction of the two opposite effects of convection and diffusion. The BE also models sound waves (and shock waves) in viscous medium, waves in fluid-filled viscous elastic tubes, MHD, and waves in a medium with finite electrical conductivity. It is important to mention that the nonlinear equation can be linearized by the Cole-Hopf transformation [43]. An interesting way to derive new classes of solutions can be found in [44]. For properties of Sobolev spaces see e.g. [45].

**Definition:** Let $B$ a Banach space and $T > 0$. We define
**(i)** The space $C^k[0,T,B]$ is the set of all functions $u:[0,T] \rightarrow B$, they are bounded differentiable. The norm is given by

$$\| u \|_{C^k[0,T,B]} = \sum_i^k \max \| u \|_B^i . \tag{21c}$$

**(ii)** The space $L[0,T,B]$ is the set of all equivalent classes of measuring functions $u: L[0,T] \rightarrow B$ valid for

$$\| u \|_{P^P[0,T,B]} = \left( \int_0^T \| u(t) \|_B^P dt \right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty \text{, and} \tag{21d}$$

$$\| u \|_{P^\infty[0,T,B]} = \underset{0<T<\infty}{ess \sup} \| u(t) \|_B < \infty . \tag{21e}$$

The defined spaces are all Banach spaces. If $H$ is a Hilbert space, it is $L^2[0,T,H]$ with the scalar product $(u,v)_{L^2[0,T,H]} = \int_0^T (u(t),v(t),)_H \, dt$ with $u,v \in L^2[0,T,H]$. Remember in what sense we have defined weak solutions for parabolic equations: Let $u$ be a weak solution of $u_t - \Delta u = f(x,u)$ in $\Omega$, $u = 0$ upon $\partial\Omega$ and $u(.,0) = u_0$ in $\Omega$ where $f(x,u)$ is a regular function. Multiplying the DE with a test function $w \in C_0^\infty(\Omega \times (0,T)$ and integrating over $\Omega \times (0,T)$ partially, one has

$$\int_0^T (w,u_t)_{L^2} dt + \int_0^T \int_\Omega \nabla u \nabla w dxdt = \int_0^T \int_\Omega w dxdt . \tag{21f}$$

A weak solution should be differentiable weakly w.r.t. x and the function $t \rightarrow \nabla u(t)$ should be square-integrable. Time derivation fulfill the equation $u_t = \Delta u(t) + f(t) \in H^{-1}(\Omega)$.
We therefore restrict $u_t \in L^2(0,T.H^{-1}(\Omega))$, that means $u(t)$ and $u_t(t)$ appears in different Banach spaces. We have already seen that the property of weak convergence for nonlinear equations does not suffice in the nonlinearities to run over the limit. A way out can be by applying special imbedding theorems, especially by using the lemma from Showalter, [46]; we abstain from this proof.
The following results are valid for time-depending functions $A$ and $c$ and inhomogeneous Dirichlet boundary conditions under certain restrictions.

**Theorem (existence theorem for parabolic DEs):** Let $f \in L^2(\Omega \times (L,T))$ and $T > 0$ and the assumptions from above are valid. Then a unique weak solution $u \in W^{1,2}(0,T;V,H)$ of the equation $u_t - L[u] = f(x,t)$ in $\Omega$, $t > 0$, $u = 0$ upon $\partial\Omega$ and $u(0) = 0$ in $\Omega$ exists in the sense of definition fro above. Remark: Is the nonlinearity $f(x,t,u)$ Lippschitz-continously one can prove the existence of solutions of the eq.(21).

**Theorem (global existence theorem for parabolic DEs):** The assumptions from above are valid. Further, let $T > 0$, $f(.,.,0) \in L^2(0,T.L^2(\Omega))$ and $f$ is continously in the sense of Lipschitz and smooth in $(x,t)$. That means a constant $L > 0$ exists (the Lipschitz constant) so that for all $x \in \Omega$ and $T \in (0,T)$ and $(u,v) \in R$

$$\left| f(x,t,u - f(x,t,v) \right| \le L[u - v]. \tag{21g}$$

Then a unique weak solution of the parabolic equation, eq.(21) exists.

**Note:** A proof can be performed with the fix point theorem of Banach, we do not follow this long calculations. Semi-linear parabolic equations play an important role in chemical reaction kinetic problems. The requirements of the Continuity of Lipschitz mean a strong restriction.

**Example:** Since the mapping $u \to u^2$ is locally continous, a solution in time locally exists. Following [40] we have the special equation

$$u_t - \Delta u = u^2 \text{ in } \Omega\text{, and } t > 0,\ u(0) = u_0 > 0. \tag{21h}$$

A solution is derived to give

$$u(t) = \frac{1}{1/u_0 - t},\ t > 0, \tag{21i}$$

and exists only for $t > 1/u_0$, the diffusions term acts smooth so that the reaction term and the diffusion term $\Delta u$ are in competition. Since the term $u^2$ has a linear dependence and reactions can take more faster than the diffusion term can smooth. Finally we show the uniqueness of solutions.

**Theorem (the uniqueness theorem of solutions for parabolic DEs):** Let $u_1$ and $u_2$ two different solutions of the eq.(20). We use the test function $u_1 - u_2$. Due to the ellipticy of the operator $L$ and the positivity of $c$ we have

$$\frac{1}{2}\int_\Omega (u_1 - u_2)\,\tau^2 dx \le \int_0^\tau \int_\Omega \left( f(x,t,u_1 - f(x,t,u_2) \right)\left( u_1 - u_2 \right) dxdt \le 0, \tag{22}$$

where we have used the monotony of $f$. Therefore $u_1 = u_2$ and the uniqueness result is proven.■

**4. Some notes about the positivity and the long-term behaviour**

In Chapter (1.3) we introduced the following equation

$$u_t - \Delta u = R_{0+} - u^2 \text{ in } \Omega,\ t > 0 \text{ and } u(0) = u_0 \text{ in } \Omega \tag{23}$$

with boundary conditions $u = 0$ upon $\partial\Omega$. Considering chemical applications it is naturally to assume Neumann conditions so that $\frac{\partial u}{\partial n} = 0$ upon $\partial\Omega$. For positive initial values $u_0$ we expect that the concentration is positive and at time $t \to \infty$ converges to the stationary value $\sqrt{R_0}$.

Now we assume more general equations of the form

$$u_t + L(u) = f(u) \text{ on } \Omega,\ t > 0 \text{ upon } \partial\Omega \text{ and } u(0) = u_0 \text{ in } \Omega. \tag{23a}$$

The existence theorems from above care not applicable here since we have Neumann conditions.

Note: Without any proof we remark that the solution of eq.(23a) remains positive for $u_0 > 0$.

**Theorem:** The function $R - u^2$ satisfies the assumption with $m_0 = \sqrt{R_0}$. The solution of eq.(23) satisfies $u \ge \min\left(R_0, u_\bullet\right)$.

**Proof:** A test function $(u. - m)^-$ can not be used since problems arise during integrations. Therefore we use test functions of the form $(u - me^{-\lambda t})^-$ with a suitable constant $\lambda > 0$. Then we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u - me^{-\lambda t})^- dx \le \int_\Omega (\lambda - c)e^{-\lambda t}(u - me^{-\lambda t})^- dx, \tag{23b}$$

choosing $\lambda = \sup_\Omega c$ then the integral on the r.h.s. is non-positive and the argument is in agreement. Finally one can show that solutions of eq.(23a) converges to the weak solution as $t \to \infty$.

The solution of the problem is given if the constant $c$ is strictly positive providing an estimation for $u_\infty$ in $L^2$.

The assumptions for $\Omega$ and $L(u)$ are the same as before. The function $f: R \to R$ is continously and monotonic falling.

The weak formulation is: Let $V \in H^1(\Omega)$ and $H \in L^2(\Omega)$, then $V \mapsto H \mapsto V'$ is an evolution triple. We are looking for solutions $u \in W^{1,2}(0,T;V,H)$ so that for all $v \in L^2(0,T;V)$

$$\int_0^T (u_t v)_{V'} dt + \int_0^T \int_\Omega \left(\nabla u^T A \nabla v + cuv\right) dx dt = \int_0^T \int_\Omega f(u) v dx dt. \tag{23c}$$

**Theorem:** Let $-f$ a monotonic function with $\gamma \geq 0$

$$-(f(u) - f(v))(u - v) \geq \gamma (u - v)^2 \text{ for all } (u, v) \in R. \tag{23d}$$

Further let $u \in W^{1,2}(0,T;V,H)$ a weak solution of eq.(23a). Then

$$\| u_t - u_\infty \| \leq \| u_0 - u_\infty \|_{l^\infty(\Omega)} e^{-\lambda t}, \ t \geq 0, \tag{23e}$$

whereby $\lambda = \inf_\Omega c + \gamma$. The proof is performed as follow: First one has to show the estimation for $u_t - u_\infty$ in $L^p$ and then goes to the limit as $p \to \infty$. (Hint: One can use test functions of the form $(u_t - u_\infty)^p$ in the weak formulation). Thus one has

$$\left\langle ((u - u_\infty)_t (u - u_\infty)^p \right\rangle_{V'} + p\int_\Omega \left((u - u_\infty)^{p-1} \nabla (u - u_\infty)\right)^T A \nabla (u - u_\infty) dx + c\int_\Omega (u - u_\infty)^{p-1} dx = \\ = \int (f(u) - f(u_\infty) u - u_\infty)^p dx. \tag{23f}$$

The second integral of the l.h.s. of eq.(23f) can be estimated by

$$p\int_\Omega \left((u - u_\infty)^{p-1} \nabla (u - u_\infty)\right)^T A \nabla (u - u_\infty) dx = \\ = \frac{4p}{p+1} \int_\Omega \left(\nabla (u - u_\infty)^{p+1/2}\right)^T A \nabla (u - u_\infty)^{(p+1)/2} dx \geq \\ \geq \frac{4p\alpha}{p+1} \int_\Omega \left| (u - u_\infty)^{(p+1)/2} \right|^2 dx. \tag{23g}$$

We did not utilize the last integral since for Neumann problems no Poincaré inequality exists. But for Dirichlet problems an estimation for the upper region is possible.

Finally one can show that the solution of eq.(23a) converges to the weak solution of stationary problem as $t \to \infty$ so that one has

$$L(u_\infty) = f(u_\infty) \text{ in } \Omega \text{ and } \frac{\partial u_\infty}{\partial v} = 0 \text{ upon } \partial\Omega. \tag{23h}$$

The function $f$ is strictly monotonous upon the interval $[m, \infty)$ and the solution of the boundary value problem $-\Delta u_\infty = R_0 - u_\infty^2$ in $\Omega$ and $\partial u_\infty / \partial v$ upon $\partial\Omega$ is constant and $u_\infty = \sqrt{R_0}$ represents the unique solution. The consequences of the convergence implies that

$$\| u(t) - R_0 \|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \tag{23i}$$

**5. Some short notes for quasi-linear diffusion equations**

For the following equations of physical relevance we give here only the existence and uniqueness theorems. Let us study equations of the form

$$u_t - div\left(a(u) \nabla u\right) = 0. \tag{24}$$

Such equations can describe the evolution of concentrations $u$ with locally dependence of the diffusion coefficient $a(u)$ such like electrons, ions in a medium. A DE is said to be 'quasi-linear' if it is linear in the first-partial derivatives of the unknown function $u(x_1,...x_n,t)$.

**Theorem: (Existence and uniqueness theorem for quasi-linear DEs):** Let $a \in C^0(R)$ with $0 < a_\bullet \le a(u) \ge a^\bullet$ for all $u \in R$ and $u_0 \in L^2(\Omega)$. Then a weak solution $u \in W^{1,2}(0,T;V,H)$ exists exactly for $u_t - div(a(u)\nabla u) = 0$ in $\Omega$, $t > 0$, $u = 0$ upon $\partial\Omega$, $u(.,0) = u_0$ in $\Omega$ where $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. For a proof we refer to the literature, e.g. [47], [48] and [49]. One can say that the problem of existence (and uniqueness) of solutions of quasi-linear evolution equations in Banach spaces has been studied by many authors, e. g. [50], [51] and [52]. Here it is only possible to cite a few collection of relevant intensions. A deeper insight by considering integrodifferential equations gives [53].

**6. The porous media equation (PME)**

We analyze quasi-linear equations of the type

$$u_t - \Delta(u^m) = 0 \text{ in } \Omega,\ t > 0,\ u^m = 0 \text{ upon } \partial\Omega \text{ and } u_0 = u(0) \text{ in } \Omega, \qquad (25)$$

where $\Omega \subset R^n$ is a restricted domain with $\partial\Omega \in C^1$ and $m > 1$. So that $u^m$ is defined for all $m > 1$. we assume $u_0 \ge 1$. Thus, following the maximum principle one has $u(t) \ge 1$ for $t > 1$. Eq.(25) is similar to eq.(24) if we use the form

$$u_t - div\left(m u^{m-1}\nabla u\right) = 0. \qquad (25a)$$

Since $u = 0$ is degenerated one has to use another term of solutions. We can not expect $u(t) \in H^1(\Omega)$, so we have to use $u(t)^m \in H^1(\Omega)$. This leads to the following

**Definition:** A function $u_0 \ge 0$ is called weak solution of eq.(25) if $u^m \in L^2(0,T;H_0^1(\Omega))$ and $u_t \in L^2(0,T;H^{-1}(\Omega))$ for all $v \in L^2(0,T;H_0^1(\Omega))$ so that

$$\int_0^T \langle u_t, v\rangle_{H\ 1} dt + \int_0^T \int_\Omega \nabla(u^m)\nabla v\, dx\, dt\ . \qquad (25b)$$

Then we have the theorem

**Theorem:** Let $T > 0$ and $m > 1$ and $u_0 \in L^{m+1}(\Omega)$. Under these circumstances a solution of eq.(25) exists exactly in the sense of the definition given below. A proof can be found in literature, e.g. [54], [55], and [56]. For more information and applications relating heat problems, blow-up solutions and boundary value problems see [57], [58], and [59]. An example of latest research in this domain is [60].

**7. The stationary Navier-Stokes Equation (NSE)**

Here one has to consider the combined equations

$$-\Delta u + (u\nabla)u + \nabla p = f\ ,\ div\, u = 0 \text{ in } \Omega \text{ and } u = 0 \text{ upon } \partial\Omega, \qquad (26)$$

where $\Omega \subset R^3$ is s restricted domain with $\partial\Omega \in C^1$. The vector-valued function $u$ describes the velocity of a liquid, $p$ is the pressure and $f$ is an outer force. The difficulty is that both of the terms are related with the velocity, therefore we eliminate the pressure in the first equation. If we multiply the first equation with a suitable test function $v$ we have

$$\int \nabla u : \nabla v\, dx + \int (u\nabla)uv dx - \int p\, div\, v\, dx = \int f.v\, dx. \qquad (26a)$$

If $div\, v$ vanishes so the integral containing the pressure disappears and we have to solve only one equation. We define a space with exactly this properties, so that $X = \left\{ v \in H_0^1(\Omega)^3 : div\, v = 0 \; in\; \Omega \right\}$. This space has the norm

$$\|v\|_X^2 = \|\nabla v\|_{L^2(\Omega)^{3x3}}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\Omega)}^2 . \qquad (26b)$$

Then, $X$ is a reflexive and separable Banach space.

**Theorem: (Existence theorem for stationary NSE):** Let $f \in L^2(\Omega)^3$. Then a weak solution $(u,p) \in X \times L^2(\Omega)$ of eq.(26) exists. We left the proof to consult the literature, e.g. [62], [63].
An interesting contribution discussing this object can be found in [64].
If a vector field $F$ is orthogonal to the space $X$ , so $F$ is a gradient, e.g. $F = \nabla p$ for the function $p$ . This statement leads to the following

**Lemma (de Rham):** Let $f \in \left(H_0^1(\Omega)^3\right)'$ so that $\langle F, v\rangle = 0 \quad \forall\; v \in X$ . Then a $p \in L^2(\Omega)$ exists with $\int_\Omega p dx = 0$ so that

$$\langle F, v\rangle = \int_\Omega p\, div\, v dx \quad \forall \quad v \in H_0^1(\Omega)^3 . \qquad (26c)$$

If we apply the de Rham lemma onto the functional

$$\langle F, v\rangle = \int_\Omega \nabla u : \nabla v\, dx + \int_\Omega (u\nabla)\, u\, v\, dx - \int_\Omega F.v\, dx , \qquad (26d)$$

which, after construction of the space $X$ , satisfies the lemma from de Rham.

## 8. Some short notes to the Schrödinger Equation

Consider a quantum mechanical particle in a restricted domain $\Omega$ which probability vanishes omto $\partial\Omega$ . Then the particle can be described by the Schrödinger equation (SE):

$$i\psi_t + \Delta\psi = 0 \text{ in } \Omega,\; t > 0,\; u = 0 \text{ on } \partial\Omega \text{ and } u(0) = u_0 \text{ upon } \Omega . \qquad (27)$$

Let $u_0$ be an eigenfunction of $\Delta v = \lambda v$ in $\Omega$ , $v = 0$ upon $\partial\Omega$ with eigenvalue $\lambda$ . Then, by

$$u(x,t) = u_0(x)e^{i\lambda t} \qquad (27a)$$

a special solution of the eq.(27) is given.

**Theorem: (Existence and uniqueness theorem for linear Schrödinger Equations):** Let $f \in L^2(0,T;H)$ with $f_t \in L^2(0,T;V')$ and $u_0 \in V$ . Then, exactly one solution $u \in W^{1,2}(0,T;V,H)$ of eq.(27) exists. It fulfils the conditions $u \in L^\infty(0,T;V)$ and $u_t \in L^\infty(0,T;V')$ . This can be generalized in the sense that the SE is not only solvable onto $[\infty,0)$ but also onto the whole domain $R$ . This can be seen by introducing a transformation $t \to -t$ and analyzing the equation $i\psi_t - \Delta\psi = 0$ . It is sufficient to assume initial data of the form $u_0 \in L^2(\Omega)$ . Then, a solution $u \in C^0(R, L^2(\Omega))$ with $u_t \in C^0(R, D')$ exists with $D = H^2(\Omega) \cap H_0^1(\Omega)$ for a proof see, e.g. [64] or [65].
The SE has some remarkable properties: It is not a diffusion equation, it is rather similar to the wave equation. To see this, consider the whole $R^n$ and imagine

$$i\psi_t + \Delta\psi = 0 \text{ in } R^n,\; t \in R^n,\; u(0) = u_0 . \qquad (27b)$$

For smooth initial data one can solve the problem exactly.

**Proposition:** Let $u_0 \in C_0^\infty(R^n)$. Then a unique solution of the eq.(27b) is

$$u(x,t) = \left(\frac{1}{4\pi i t}\right)^{n/2} e^{i|x|^2} \int_{R^n} e^{-ixy} e^{i|y|^2/4t} u_0(y)\,dy\,, \tag{27c}$$

The proposition follows from a Fourier transformation of eq.(27b) fulfilling the inequality

$$\| u(t) \|_{L^\infty(R^n)} \le \left(4\pi |t|\right)^{-n/2} \| u_0 \|_{L^1(R^n)}\,. \tag{27d}$$

We note that solutions of the SE in $L^2$ do not fade away; they still preserve so that

$$\| u(t) \|_{L^2(R^n)} = \| u_0 \|_{L^2(R^n)} \quad \text{for all } t \in R^n\,. \tag{27e}$$

Also, for semi-linear SE existence and uniqueness can be proven under certain restrictions for the nonlinearity, e.g.

$$i\psi_t + \Delta\psi - f(u) = 0 \text{ in } R^n,\ t \in R,\ u(0) = u_0\,. \tag{27f}$$

In physical applications the following case is of interest:

$$f(u) = g|u|^2 u \quad \text{with } g: R \to R\,. \tag{27g}$$

A typical example is the cubic function $f(u) = \pm|u|^2 u$. The real-valued function $g\left(|u|^2\right)$ can be interpreted as a self-interaction potential (the cubic SE plays an important role in nonlinear optic).
Finally a possibility to distinguish between existence and non-existence for nonlinear SE is given.
In principle the error term can have powers of arbitrary order and maybe of fractional order.

**Theorem: Existence and non-existence of nonlinear SEs:**
Let $f(u) = \lambda|u|^\alpha u$, $\lambda \in R$, $0 < \alpha < 4/(n-2)$ for $\alpha < \infty$ if $n \le 2$, and the following statements hold:

(i) if $\lambda > 0$ all solutions of eq.(27f) are global in time,
(ii) if $\lambda < 0$ and $\alpha < 4/n$ all solutions of eq.(27f) are called global,
(iii) if $\lambda < 0$ and $\alpha > 4/n$ solutions of eq.(27f) are only global if $\|u_0\|_{H^1(R^n)}$ is small enough,

for sufficient large $\|u_0\|_{H^1(R^n)}$ solutions exist only local in time, e.g. [66], [67], and [68].

**9. Conclusion and outstanding problems**

We have considered here some special nPDEs importantly in physical, chemical and technical applications. The main part treats the terminus ‘solution’ from an analytical view. We refer to some original studies which, we believe, have not lost their originality and therefore, it is worth to remember the great substantial progress during the last century. Starting by chapter 2 we discussed semi- and quasi-linear DEs. Parabolic DEs, their properties and behaviour of solutions are given in short.
Sometimes equations are parabolic or hyperbolic "modulo the action of some group": for example, the “Ricci flow Equation” is not quite parabolic, but is "parabolic modulo the action of the diffeomorphism group", which implies that it has most of the good properties of parabolic equations (the Ricci flow Equation is $\partial_t g_{ij} = -2R_{ij}$, $g_{ij}$ is the metric tensor and $R_{ij}$ the Riemann tensor).
Let us take some notes relating to the nPDE eq.(13), e.g. [28], [69] where ‘classical’ solutions have been studied using algebraic methods as well as similarity methods. It can be shown that other classical solutions can expressed in terms of elliptic functions. One has, after suitable transformations, to solve the nODE:

$$\left(\frac{dz}{df}\right)^2 - \lambda z^2 + 3z^3 = const\,, \tag{28}$$

with $u(x,t) \to f(\xi)$, $\xi = x - \lambda t$ and $f''(\xi) = z(f)$, $z = z(f(\xi))$. From eq.(28) we have

$$\int \frac{dz}{\sqrt{const - 3z^3 + \lambda z}} = (f - f_0)\,, \tag{28a}$$

where $f_0$ is a further constant of integration. The polynomial in (28a) has a real root $a$ and two complex conjugated roots $b$ and $c$. It follows with $\infty < y < a$, $a \in R$, $(b,c) \in C$ and considering the case $\lambda = const = 1$ (the case $const = 0$ leads to rational functions in terms of arctan-functions):

$$\int_y^a \frac{dz}{\sqrt{const - 3z^3 + z^2}} = g\int_0^{u_1} du = g.u_1 = g.cn^{-1}(\cos\varphi, k) = g.F(\varphi, k), \qquad (28b)$$

**Note:** To enlarge the solution manifold one can introduce a real factor in front of the similarity transformation.

with the factors $g = \left(\sqrt{A}\right)^{-1}$, $A^2 = (b_1 - a)^2 + a_1$, $a_1 = \sqrt{-\frac{(b-\bar{b})^2}{4}}$, $b_1 = \frac{(b-\bar{b})}{2}$ and the modulus of the elliptic functions of the first kind $k^2 = \frac{A + b_1 - a}{2A}$, where $\bar{b}$ is the complex conjugated zero.

The eq.(14), Ramani's equation is hard to solve but travelling wave solutions are known derived by the author, e.g. [32], [33].

**Note:** In future works, the electrochemical model equations, eq.(10) and (10a) will be analyzed in detail especially to consider both electric and/or magnetic fields.

## 10. Acknowledgment

The author wishes to dedicate his scientific work to the beloved and wonderful Mrs. Yunxu.
Also, the author wishes a lot of strength and joy of life which hopefully will be increased by this work.

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