SIMPSON'S RULE AND ROMBERG INTEGRATION FOR SOLVING NON-LINEAR FREDHOLM INTEGRAL EQUATION

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ABSTRACT
In this paper, we discussed Simpson's rule (**S.R**) and Romberg integration (**R.I**) for solving non-linear Fredholm integral equation of the second kind with continuous kernel (**NFIE**). Numerical examples are presented and results are compared with the analytical solution to demonstrate the validity and applicability of this methods.

Keywords: *Nonlinear Fredholm Integral Equation; Simpson's Rule; Romberg Integration.*

1. Introduction

When we talk about Fredholm integral equations we are talking about equations of paramount importance for solving the problems of applied sciences and study of natural phenomena: physics, engineering and biological, whether these equations linear or nonlinear, homogeneous or nonhomogeneous, with the continuous kernel or singular kernel. There are many different numerical methods to solve these equations. Many of the references discussed many numerical methods to solve linear Fredholm integral equations of which (Kress [1], Kanwal [2], Atkinson [3], and Mirzaee [4]). In [5], Guoqiang and Jiong discussed the extrapolation of Nystrom solution for two dimensional nonlinear Fredholm integral equations. In [6], Emamzadeh and Kajani studied the nonlinear Fredholm integral equation of the second kind with quadrature methods. Allahviranloo and Ghanbari in [7], discussed the Discrete homotopy analysis method for the nonlinear Fredholm integral equations. In [8], Shahsavaran solved the nonlinear integral equations of the second kind using Lagrange Functions. Matoog in [9], discussed the numerical treatment for solving nonlinear integral equation. In [10], AL-Bayati and others, used Romberg Algorithm to solve a system of nonlinear Fredholm integral equations of the second kind.

In this paper, we use (**S.R**) and (**R.I**) to discuss numerically the solution of the (**NFIE**) of the second kind with continuous kernel of the form

$$\mu\phi(x) = f(x) + \lambda\int_a^b k(x,t)\gamma(t,\phi(t))\,\mathrm{dt} \tag{1}$$

where μ is a constant defines the kind of the integral equation, $\phi(x)$ is an unknown function, will be determined, the functions $f(x)$ and $k(x,t)$ are given analytical functions defined, respectively, $\{a \le x \le b, \quad a \le t \le b\}$, μ and λ are constants that have many physical meanings.

2. The existence and uniqueness of the solution

The existence and uniqueness solution of equation (1), under certain conditions, will be discussed and proved using Banach fixed point theorem. For this, we state the following theorem.

Theorem1:
Consider a metric space $X = (X, d)$, where $X \ne \Phi$ Suppose that X is complete and let $T : X \to X$ be a contraction on X . Then T has precisely are fixed point.

In the light of Theorem (1), we write equation (1) in the integral operator from

$$\overline{w}\,\phi(x) = \frac{1}{\mu}f(x) + w\,\phi(x), \qquad (\mu \ne 0) \tag{2}$$

Where

$$w\,\phi(x)=\frac{\lambda}{\mu}\int_a^b k(x,t)\gamma(t,\phi(t))dt \tag{3}$$

Also, we assume the following conditions:

i) The kernel $k(x,t)$ satisfies the continuity condition

$$|k(x,t)|=c<\infty, \quad (c \text{ is a constant})$$

ii) The given function $f(x)$ is continuous in the space $L_2[a,b]$, and its norm is defined as

$$\|f\|_{L_2[a,b]}=\left\{\int_a^b |f(x)|^2 dx\right\}^{\frac{1}{2}}=\zeta, \quad (\zeta \text{ is a constant})$$

iii) The known continuous function $\gamma(x,\phi(x))$ satisfies for the constant $A>A_1, A>p,$ the following conditions

a) $\left\{\int_a^b |\gamma(x,\phi(x))|^2 dx\right\}^{\frac{1}{2}} \le A_1\|\phi(x)\|_{L_2[a,b]}$

b) $\|\gamma(x,\phi_1(x))-\gamma(x,\phi_2(x))\| \le M(x)|\phi_1(x)-\phi_2(x)|$

Where $\|M(x)\|_{L_2[a,b]}=p$

Theorem 2

If the condition (i)-(iii) are verified, then equation (1) has a unique solution in the Banach space $L_2[a,b]$.

The proof of this theorem depends on the following two lemmas:

Lemma 1

Under the conditions (i)-(iii-a), the operator $\overline{W}$ defined by (2), maps the space $L_2[a,b]$ into itself.

Proof

In view of the formula (2) and (3), we get

$$\|\overline{w}\phi(x)\|_{L_2[a,b]} \le \frac{1}{|\mu|}\|f(x)\|_{L_2[a,b]}+\left|\frac{\lambda}{\mu}\right|\left\|\int_a^b |k(x,t)||\gamma(\mathrm{t},\phi(t))|dt\right\|_{L_2[a,b]}$$

Using the condition (ii), then applying Cauchy-Schwarz inequality, we have:

$$\|\overline{w}\phi(x)\|_{L_2[a,b]} \le \frac{\zeta}{|\mu|}+\left|\frac{\lambda}{\mu}\right|c\left\{\int_a^b |\gamma(\mathrm{t},\phi(t))|^2 dt\right\}^{\frac{1}{2}}$$

In the light of the conditions (i) and (iii-a), the above inequality takes the form:

$$\left\|\overline{w}\phi(x)\right\|_{L_2[a,b]} \le \frac{\zeta}{|\mu|} + \sigma\left\|\phi(x)\right\|_{L_2[a,b]}, \qquad \left(\sigma = \left|\frac{\lambda}{\mu}\right| cA\right) \tag{4}$$

The lost inequality (4) shows that, the operator $\overline{W}$ maps the ball S_r into itself, where

$$r = \frac{\zeta}{\left(|\mu| - |\lambda| cA\right)} \tag{5}$$

Since $r > 0, \zeta > 0,$ therefore we have $\sigma < 1$. Moreover, the inequality (4) involves the boundedness of the operator w of equation (2), where

$$\left\|w\,\phi(x)\right\|_{L_2[a,b]} \le \sigma\left\|\phi(x)\right\|_{L_2[a,b]} \tag{6}$$

Also, the inequalities (4) and (6) define the boundedness of the operator $\overline{W}$.

Lemma 2

If the two conditions (i) and (iii-b) are satisfied, then the operator $\overline{W}$ is contractive in the Banach space $L_2[a,b]$.

Proof

For two functions $\phi_1(x)$ and $\phi_2(x)$ in the space $L_2[a,b]$,the formulas (2), (3) lead to

$$\left\|\left(\overline{w}\phi_1 - \overline{w}\phi_2\right)(x)\right\|_{L_2[a,b]} \le \left|\frac{\lambda}{\mu}\right| \left\|\int_a^b |k(x,t)| \left|\gamma\left(t,\phi_1(t)\right) - \gamma\left(t,\phi_2(t)\right)\right| dt\right\|_{L_2[a,b]}$$

Using the condition (iii-b), then apply Cauchy-Schwarz inequality, we have:

$$\left\|\left(\overline{w}\phi_1 - \overline{w}\phi_2\right)(x)\right\|_{L_2[a,b]} \le \left|\frac{\lambda}{\mu}\right| |k(x,t)| \left(\int_a^b M^2(t)\left|\phi_1(t) - \phi_2(t)\right|^2 dt\right)^{\frac{1}{2}}$$

Finally, with the aid of conditions (i) and (iii-b), we obtain:

$$\left\|\left(\overline{w}\phi_1 - \overline{w}\phi_2\right)(x)\right\|_{L_2[a,b]} \le \sigma\left\|\phi_1(x) - \phi_2(x)\right\|_{L_2[a,b]} \tag{7}$$

In equality (7) shows that, the operator $\overline{W}$ is continuous in the space $L_2[a,b]$, then $\overline{W}$ is a contraction operator under the condition $\sigma < 1$.

3. The (S.R)

Consider the (**NFIE**) of the second kind

$$\phi(x) = f(x) + \lambda\int_a^b k(x,t,\phi(t))dt \qquad , \; (\mu = 1) \tag{8}$$

Where $k(x,t,\phi(t))$ is a known function represents the kernel, and the function $f(x)$ is given and continuous on the interval $[a,b]$,we shall assume that $|f(x)|<G$,and the function $k(x,t,\phi(t))$ is continuous and satisfies a uniform Lipschitz condition, (G is a constant).

We will subdivide the interval of integration $[a,b]$in to n equal subintervals of width $h=\frac{x_n}{n}; n\geq 1, x_n=b.$

We shall set $x_i=ih, 0\leq i\leq n,$ then we can rewrite the integral part in (8) as:

$$\int_a^{x_i} k(x_i,t,\phi(t))dt \approx h\sum_{j=0}^{i} w_{ij}k(x_i,t_j\phi(t_j))$$

$$=h\sum_{j=0}^{i} w_{ij}k_{ij}\phi(t_j)+E_{i,t}(k((x_i,t,\phi(t))) \quad (9)$$

Such that, $x=x_i=a+ih, h=\frac{(b-a)}{n}, x_i=t_i, i=2,3,\ldots,n$ and $\{w_{ij}\}$ is the weights function, $E_{i,t}$ is error.

The general form of (**S.R**) is:

$$A=\int_a^b g(x)dx=\frac{h}{3}\left[g(a)+2\sum_{i=1}^{(n/2)-1} g(x_{2i})+4\sum_{i=1}^{n/2} g(x_{2i-1})+g(b)\right] \quad (10)$$

We can apply the (**S.R**) on (**NFIE**) by using Day's starting procedure:

$$\begin{aligned}
\phi_{11}&=f_1+hk(h,0,f_0)\\
\phi_{12}&=f_1+\frac{h}{2}\left[k(h,0,f_0)+k(h,h,\phi_1)\right]\\
\phi_{13}&=f_1+\frac{h}{4}\left[k\left(\frac{h}{2},0,f_0\right)+k\left(\frac{h}{2},\frac{h}{2},\frac{f_0}{2},\frac{\phi_{12}}{12}\right)\right]
\end{aligned} \quad (11)$$

Using (11) we get:

$$\phi_1=f_1+\frac{h}{6}\left[k\left(h,0,f_0\right)+4k\left(h,\frac{h}{2},\phi_{13}\right)+k(h,h,\phi_{12})\right] \quad (12)$$

Next let:

$$\phi_{21}=f_2+2hk(2h,h,\phi_1)$$

Then:

$$\phi_2=f_2+\frac{h}{3}\left[k\left(2h,0,f_0\right)+4k\left(2h,h,\phi_1\right)+k(2h,2h,\phi_{21})\right] \quad (13)$$

Finally:

$$\phi_{31} = f_3 + \frac{3h}{2}\left[k(3h,h,\phi_1) + k(3h,2h,\phi_2)\right]$$

We obtain:

$$\phi_3 = f_3 + \frac{3h}{8}\left[k\left(3h,0,f_0\right) + 3k\left(3h,h,\phi_1\right) + 3k(3h,2h,\phi_2) + k(3h,3h,\phi_3)\right]$$

A convenient and simple continuation of Day's starting procedure can be based on (**S.R**) in the following manner (for only ϕ_0 and ϕ_1 are required) we can use (**S.R**) to give:

$$\phi_r = f_r + \frac{h}{3}\sum_{j=0}^{r} w_{rj}\, k(rh, jh, \phi_j), \qquad r = 2,3,\ldots \tag{14}$$

Where r is even, using (12) we get:

$$\begin{aligned}
\phi_0 &= f_0 \\
\phi_1 &= f_1 + \frac{h}{6}\left[k(h,0,f_0) + 4k\left(h,\frac{h}{2},\phi_{13}\right) + k(h,h,\phi_{12})\right] \\
\phi_{11} &= f_1 + hk(h,0,f_0) \\
\phi_{12} &= f_1 + \frac{h}{2}\left[k(h,0,f_0) + k(h,h,\phi_{11})\right] \\
\phi_{13} &= f_1 + \frac{h}{4}\left[k\left(\frac{h}{2},0,f_0\right) + k\left(\frac{h}{2},\frac{h}{2},\frac{f_0}{2},\frac{\phi_{12}}{2}\right)\right]
\end{aligned} \tag{15}$$

Using (13) we get $\phi_2,\ldots,\phi_n$, and can be written as the following system:

$$\begin{aligned}
\phi_0 &= f_0 \\
\phi_1 &= f_1 + \frac{h}{6}\left[k(h,0,f_0) + 4k\left(h,\frac{h}{2},\phi_{13}\right) + k(h,h,\phi_{12})\right] \\
\phi_2 &= f_2 + \frac{h}{3}\left[k(2h,0,\phi_0) + 4k(2h,h,\phi_1) + k(2h,2h,\phi_2)\right] \\
\phi_3 &= f_3 + \frac{h}{3}\left[k\left(3h,0,\phi_0\right) + 4k\left(3h,h,\phi_1\right) + 2k(3h,2h,\phi_2) + k(3h,3h,\phi_3)\right] \\
&\vdots \\
\phi_n &= f_n + \frac{h}{3}\left[k\left(nh,0,\phi_0\right) + 4k\left(nh,h,\phi_1\right) + 2k(nh,2h,\phi_2) + \ldots + k(nh,nh,\phi_n)\right]
\end{aligned}$$

Where the weights are given by:

$$w_{r0} = w_{rr} = 1, \qquad w_{rj} = 3-(-1)^j, \qquad 1 \le j \le r-1$$

4. The (R.I)

The (**R.I**) is depend on the Trapezoidal rule for integer the function $g(x)$ in the interval $[a,b]$.

Consider the intervals $[x_{i-1}, x_i]$ where $x_i - x_{i-1} = h, i = 1, 2, \ldots, n$ and put $x_0 = a, x_n = b,$ then we get:

$$\int_a^b g(x)dx = \frac{h}{2}\left[g(a) + g(b) + 2\sum_{i=1}^{n-1} g(x_i)\right] - \frac{b-a}{12} h^2 g''(\zeta) \tag{16}$$

$a < \zeta < b, h = \dfrac{b-a}{n}, x_i = a + ih, i = 0, 1, \ldots, n$ if $h_m = \dfrac{b-a}{n_m} = \dfrac{b-a}{2^{m-1}},$ the Trapeziodal rule is become:

$$\int_a^b g(x)dx = \frac{h_m}{2}\left[g(a) + g(b) + 2\sum_{i=1}^{2^{m-1}-1} g(n, ih_{m-1})\right] - \frac{b-a}{12} h_m{}^2 g''(\zeta_m)$$

Let:

$$R_{1,1} = \frac{1}{2}(b-a)\left[f(a) + f(b)\right]$$

$$R_{2,1} = \frac{1}{2}R_{1,1} + \frac{1}{2}(b-a)f\left[\frac{a+b}{2}\right]$$

In general, we get:

$$R_{m,1} = \frac{1}{2}R_{m-1,1} + \frac{1}{2}h_{m-1}\sum_{i=1}^{2^{m-2}} f\left(a + \left(i - \frac{1}{2}\right)h_{m-1}\right), \qquad m = 2, 3, \ldots, n$$

Consider the (**NFIE**) of the second kind

$$\phi(x) = f(x) + \int_a^b k\left(x, t, \phi(t)\right)dt, \qquad (\lambda = 1) \tag{17}$$

Using (16) in (17) we get:

$$\phi(x) = f(x) + \int_a^b k\left(x, t, \phi(t)\right)dt$$

$$\approx f(x) + \frac{h_m}{2}\left[k\left(x, t_0, \phi(t_0)\right) + 2\sum_{i=1}^{2^{m-1}} k\left(x, t_i, \phi(t_i)\right) + k\left(x, t_n, \phi(t_n)\right)\right] \tag{18}$$

Where equation (18) is the approximate solution for equation (17), and we can found the error if we solve $n+1$ of the values $\phi_i = \phi(x_i) = \phi(t_i),\ i = 0, 1, \ldots, n.$

Then, equation(18) becomes the system of $n+1$ of equations:

$$\phi(x_j) = f(x_j) + \frac{h_m}{2}\left[k\left(x_j, t_0, \phi(t_0)\right) + 2\sum_{i=1}^{2^{m-1}} k\left(x_j, t_i, \phi(t_i)\right) + k\left(x_j, t_n, \phi(t_n)\right)\right]$$

$$j = 0, 1, \ldots, n, \qquad m = 1, 2, \ldots, n$$

or:

$$\phi(x_j)=f(x_j)=\frac{h_m}{2}\Big[k\left(x_j,t_0,\phi(t_0)\right)+2\Big(k\left(x_j,t_1,\phi(t_1)\right)+k\left(x_j,t_2,\phi(t_2)\right)+\ldots$$

$$+k\left(x_j,t_{2^{m-1}-1},\phi(t_{2^{m-1}-1})\right)\Big)+k\left(x_j,t_n,\phi(t_n)\right)\Big]$$

5. Numerical Experiments and Discussions

Example 1:

Consider the (**NFIE**):

$$\phi(x)=\frac{2x}{3}+1+\int_0^1 (x-t)\left(\phi(t)\right)^2 dt$$

where the exact solution is $\phi(x)=2x$, here $\lambda=1,\ \mu=1$. In tables (5.1)-(5.2) we present the exact solution, the approximate numerical solutions and their corresponding errors for some points, we suppose that $N=20,50.$

Case 1: $N=20$,

x	Exact. sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.2	0.4000000	0.402671948	2.6719×10^{-3}	0.402503751	2.5037×10^{-3}
0.4	0.8000000	0.809335405	9.3354×10^{-3}	0.808999030	8.8999×10^{-3}
0.6	1.2000000	1.215337640	1.5337×10^{-2}	1.215253124	1.5253×10^{-2}
0.8	1.6000000	1.654404085	5.4404×10^{-2}	1.654114057	5.4114×10^{-2}
1	2.0000000	2.169875874	1.6987×10^{-1}	2.169346523	1.6934×10^{-1}

Table 5.1

Case 2: $N=50$,

x	Exact. sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.2	0.400000	0.400267200	2.67200×10^{-4}	0.400264507	2.6450×10^{-4}
0.4	0.800000	0.801094849	1.09484×10^{-3}	0.801074628	1.07462×10^{-3}
0.6	1.200000	1.203782112	3.78211×10^{-3}	1.203770467	3.77046×10^{-3}
0.8	1.600000	1.606975271	6.97527×10^{-3}	1.606039602	6.03960×10^{-3}
1	2.000000	2.02858857	2.85885×10^{-2}	2.028566645	2.85666×10^{-2}

Table 5.2

Example 2:

Consider the (**NFIE**):

$$\phi(x)=\frac{7}{8}x+\int_0^1 \frac{xt}{2}\left(\phi(t)\right)^2 dt$$

where the exact solution is $\phi(x)=x$.

Case 1: $N=20$,

x	Exact. sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.2	0.200000	0.2002752142	2.75214×10^{-4}	0.2002668458	2.66845×10^{-4}
0.4	0.400000	0.4043240068	4.32400×10^{-3}	0.4042895147	4.28951×10^{-3}
0.6	0.600000	0.6091852901	9.18529×10^{-3}	0.6091275660	9.12756×10^{-3}
0.8	0.800000	0.8533775864	5.33775×10^{-2}	0.8532663739	5.32663×10^{-2}
1	1.000000	1.157212934	1.572129×10^{-1}	1.156957254	1.569572×10^{-1}

Table 5.3

Case 2: $N = 50$,

x	Exact. sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.2	0.200000	0.2000268183	2.68183×10^{-5}	0.2000266844	2.66844×10^{-5}
0.4	0.400000	0.400357471	3.57471×10^{-4}	0.4003572584	3.572584×10^{-4}
0.6	0.600000	0.6043639624	4.36396×10^{-3}	0.6043588923	4.358892×10^{-3}
0.8	0.800000	0.807340939	7.34093×10^{-3}	0.8073252893	7.325289×10^{-3}
1	1.000000	1.077650028	7.76500×10^{-2}	1.077602229	7.760222×10^{-2}

Table 5.4

6. Conclusion

From the previous examples we note the following:

1) (**R.I**) and (**S.R**)are proved effective in solving the (**NFIE**), and the results showed that in the previous examples.
2) (**R.I**) better than (**S.R**) in solving examples because, best approximation of the solution and the values of error less, in all cases when $N = 20$ and $N = 50$.
3) As x is increasing in the interval [0,1], the errors due to (**S.R**) and (**R.I**) are also increasing.
4) As N is increasing, the errors are decreasing in the (**S.R**) and (**R.I**) method.

So, the stability of Romberg integration more than (**S.R**).

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