THE NONLINEAR VOLTERRA INTEGRAL EQUATION WITH WEAKLY KERNELS AND TOEPLITZ MATRIX METHOD

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ABSTRACT

In this work, we investigate the existence and uniqueness of the solution of the NVIE with weakly kernels by using the Banach fixed point theorem. The Toeplitz matrix method is considered, to obtain a nonlinear system of algebraic equations, which can be solved numericall. Also, many important theorems related to derive the existence and uniqueness of the produced algebraic system are considered. Finally, we discuss some numerical examples, when the kernel takes a logarithmic and Carleman forms and the estimate error is calculated, in each case.

Keywords: *Singular nonlinear Volterra integral equation (SNVIE), Toeplitz matrix method, Nonlinear algebraic system (NAS), logarithmic kernel, carleman kernel .*

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1. INTRODUCTION:

Integral equations of various types and kinds play an important role in many branches of mathematics. Over the past thirty years substantial progress has been made in developing innovative approximate analytical and purely numerical solution techniques to a large class of singular integral equations. In recent years, singular integral equations arise in many problems of mathematical physics, such as the theory of elasticity, visco elasticity, or hydrodynamics. Also, fracture mechanics, aerodynamics, theory of porous filtering, antenna problems in electromagnetic theory, viscodynamics fluids, contact problems in the theory of elasticity, mixed boundary problems in mathematical physics, biology, chemistry and engineering can be formulated as integral equations of the first, second and third kind, see Green [1], Hochstadt [2], Kanwal [3], and Schiavone et al. [4].

The solutions of their applications can be obtained analytically; using the theory developed by Muskhelishvili [5] and [6]. At the same time, the scene of numerical methods takes an important place in solving integral equations. Abdalkhani [7] obtained a numerical solution of NVIE of the second kind, when the kernel took the form of Abel's function. Guoqiang et al. [8] obtained numerically the solution of two dimensional of the NVIE by collocation and iterated collocation methods.

In [9], Brunner et al., introduced a class of methods depending on some parameters to obtain numerically the solution of Able's integral equation of the second kind. In [10] Kauthen applied a linear multistep method to obtain numerically a SVIE. In [11] Kibas and Saigo used an asymptotic method to obtain the solution of NVIE of the second kind. In [12], Orsi used product Nystrom method as a numerical method to get the solution of NVIE when its kernel took a logarithmic and Carleman forms. A new quadrate method for solving nonlinear SVIE of the second kind could be found in [13]. In [14], a family of methods depending on a few parameters was introduced for the solution of NVIE of the second kind, with logarithmic kernel. Some books were edited by Linz [15], Kreyszig [16] containing many different numerical methods to obtain the solution of integral equation.

In this paper, we consider the NVIE with weakly kernels of the form

$$\mu\theta(t)=f(t)+\lambda\int_0^t k\left(\left|t-x\right|\right)\gamma\left(x,\theta(x)\right)dx . \tag{1.1}$$

where

$$k\left(\left|t-x\right|\right)=\begin{cases}\ln\left(\left|t-x\right|\right) \\ \left|t-x\right|^{-\upsilon}, \qquad (0<\upsilon<1)\end{cases} . \tag{1.2}$$

Here, $f(t)$ with its derivatives and $\gamma(t,\theta(t))$ are given functions belonging to the class in $C[0,T]$ continuous functions. The known function $k(|t-x|)$ is called the kernel of the integral equation which has a weak singularity, while $\theta(t)$ represents the unknown function to be found, for all $t\in[0,T]$. The constant μ defines the kind of the integral equation. Also, λ is a known constant, that may be complex, which has many physical meanings.

2. The Existence and Uniqueness of the Solution :

In this section, the Banach fixed point theorem in $C[0,T]$ is used to prove the existence and of the uniqueness solution of Eq. (1.1), which it satisfies the Lipschitz condition. The Banach fixed point theorem is very important to prove the existence and of the uniqueness solution of the SNVIE of the first kind, also in a homogeneous form. For this, we write it in the integral operator form

$$\bar{W}\theta(t)=\frac{1}{\mu}f(t)+W\theta(t)\ ,\ \mu\neq 0, \tag{2.1}$$

where

$$W\theta(t)=\frac{\lambda}{\mu}\int_0^t k(|t-x|)\gamma(x,\theta(x))dx\ ,\quad (0\le t\le T<\infty). \tag{2.2}$$

Now, we assume the following conditions:

(i) $f(t)$ is a given a continuous function in $0\le t\le T<\infty$, such that its norm is defined by

$$\|f(t)\|=\max_{0\le x\le T}|f(t)|\le H\ .$$

(ii) The discontinuous function $k(|t-x|)$ is absolutely integrable with respect to *x* for all $0\le t\le T<\infty$, and satisfies the following conditions

a) $\int_0^t |k(|t-x|)|dx\le L.$

b) For each continuous function $\gamma(x,\theta(x))$ and $0\le t_1\le t_2\le t$, the integrals

$$\int_{t_1}^{t_2} k(|t-x|)\gamma(x,\theta(x))dx\ ,\ \text{and}\ \int_0^t k(|t-x|)\gamma(x,\theta(x))dx\ ,$$

are continuous functions in [0,*T*].

(iii) The known continuous function $\gamma(t,\theta(t))$ in $0\le x\le t\le T<\infty$, satisfies for the constants $M>M_1,\ M>M_2$ the following conditions

a) $\|\gamma(t,\theta(t))\|\le M_1\|\theta(t)\|,$

b) $\|\gamma(x,\theta_1(x))-\gamma(x,\theta_2(x))\|\le M_2\|\theta_1(x)-\theta_2(x)\|,$

where $\|\theta(x)\|=\max_{0\le x\le t\le T}|\theta(x)|$..

Theorem 1 :

The solution of the NVIE (1.1) exists and unique under the condition

$$|\lambda|<\frac{|\mu|}{ML}. \tag{2.3}$$

The proof of Theorem (1) comes as a result of the following lemmas.

Lemma 1:

If the conditions (iii-a) are satisfied, then the integral operator $\bar{W}$ defined by (2.1) and (2.2), maps the space $C[0,T]$ into itself..

Proof :

In view of the formula (2.1) and (2.2), we obtain

$$\left|\bar{W}\theta(t)\right| \le \frac{1}{|\mu|}\left|f(t)\right| + \frac{|\lambda|}{|\mu|}\int_0^t \left|k(|t-x|)\right|\left|\gamma(x,\theta(x))\right|dx \,. \tag{2.4}$$

Using the conditions (i), (ii) and (iii-a), we get

$$\left\|\bar{W}\theta(t)\right\| \le \frac{H}{|\mu|} + \alpha\left\|\theta\right\|, \quad \alpha = \frac{|\lambda|}{|\mu|}LM \,, \quad \max_{0\le t\le T}|\theta| = \|\theta\| \,. \tag{2.5}$$

The last inequality (2.5) shows that the operator $\bar{W}$ maps the ball S_ρ into itself, where $\rho = \dfrac{H}{1-\alpha}$.

Since $\rho > 0$ and $H > 0$ we have $\alpha < 1$. Moreover, the inequality (2.5) involves the boundedness of the operator W of Eq. (2.2), where

$$\left\|W\theta(t)\right\| \le \alpha\left\|\theta\right\|, \tag{2.6}$$

Also, the inequalities (2.5) and (2.6) involve the boundedness of $\bar{W}$.

Lemma 2 :

Under the condition (2.3), $\bar{W}$ is a continuous and contraction mapping in $C[0,T]$.

Proof :

Let $\theta_1(t)$ and $\theta_2(t)$ any two functions in $C[0,T]$, then we get

$$\left|\bar{W}\theta_1 - \bar{W}\theta_2\right| \le \left|\frac{\lambda}{\mu}\right|\int_0^t \left|k(|t-x|)\right|\left[\gamma(x,\theta_1(x)) - \gamma(x,\theta_2(x))\right]dx \tag{2.7}$$

Using the conditions (ii), (iii-b), we get

$$\left\|\bar{W}\theta_1 - \bar{W}\theta_2\right\| \le \alpha\left\|\theta_1 - \theta_2\right\|, \quad \alpha = \frac{|\lambda|}{|\mu|}LM \,, \quad \max_{0\le t\le T}|\theta| = \|\theta\|. \tag{2.8}$$

The inequality (2.8) shows that the operator $\bar{W}$ is a continuous operator in the Banach space $C[0,T]$, and then, under the condition $\alpha < 1$, the operator $\bar{W}$ is a contracting.

Proof of Theorem 1:

The two previous lemmas (1) and (2) show that, the operator $\bar{W}$ defined by Eq. (2.1) is a contracting operator in the Banach space $C[0,T]$. Hence, by Banach fixed point theorem, $\bar{W}$ has a unique fixed point which is, of course, the unique solution of Eq. (1.1).

3. The Toeplitz Matrix Method (see Abdou et. al. [17]) :

Here, we discuss the solution of (1.1) numerically using Toeplitz matrix method. For this, we write the integral term of Eq. (1.1) by

$$\int_0^{t_i} k(|t_i - x|)\gamma(x,\theta(x))dx = \sum_{j=0}^{i-1}\int_{jh}^{jh+h} k(|t_i - x|)\gamma(x,\theta(x))dx \,. \tag{3.1}$$

where $t_i = ih$, $h = \dfrac{T}{N}$, $i = 1,2,...,N$ and $0 \le t \le T < \infty$, ($T = t_N$). The integral of the right hand side of Eq. (3.1) can be written in the form

$$\int_{jh}^{jh+h} k\left(|t_i - x|\right)\gamma\left(x,\theta(x)\right)dx = A_j(t_i)\gamma\left(jh,\theta(jh)\right)+B_j(t_i)\times$$
$$\gamma\left((j+1)h,\theta\left((j+1)h\right)\right)+R^{(T)}, \tag{3.2}$$

where, $A_j(t_i)$ and $B_j(t_i)$ are arbitrary which points will be determined, and $R^{(T)}$ is the estimate error. To determine the unknown points $A_j(t_i)$ and $B_j(t_i)$, in the light of Teoplitz matrix method, we put $\theta(x)=1,$ and x, respectively, in Eq. (3.2). In these cases, the error $R^{(T)}$ will be neglected. Then the values of $A_j(t_i)$ and $B_j(t_i)$ are obtained, in the forms

$$A_j(t_i)=\frac{1}{h_1}\left[\gamma(a+h,a+h)I_1(t_i)-\gamma(a+h,1)I_2(t_i)\right], \tag{3. 3}$$

$$B_j(t_i)=\frac{1}{h_1}\left[\gamma(a,1)I_2(t_i)-\gamma(a,a)I_1(t_i)\right], \tag{3. 4}$$

$$h_1=\gamma(a,1)\gamma(a+h,a+h)-\gamma(a,a)\gamma(a+h,1), h_1\neq 0. \tag{3.5}$$

where

$$I_1(t_i)=\int_a^{a+h} k\left(|t_i-x|\right)\gamma(x,1)dx = A_j(t_i)\gamma(a,1)+B_j(t_i)\gamma(a+h,1),\ a=jh, \tag{3.6}$$

and

$$I_2(t_i)=\int_a^{a+h} k\left(|t_i-x|\right)\gamma(x,x)dx = A_j(t_i)\gamma(a,a)+B_j(t_i)\gamma(a+h,a+h), \tag{3.7}$$

In view of these results, the integral term of the formula (3.1) becomes

$$\int_0^{t_i} k\left(|t_i-x|\right)\gamma\left(x,\theta(x)\right)dx = \sum_{j=0}^{i-1} D_j(t_i)\gamma\left(t_j,\theta(t_j)\right), \tag{3. 8}$$

where

$$D_j(t_i)=\begin{cases} A_0(t_i) & if \quad j=0 \\ A_j(t_i)+B_{j-1}(t_i) & if \quad 0<j<i \\ B_{i-1}(t_i) & if \quad j=i \end{cases} \tag{3.9}$$

Thus, the integral equation (3.1) takes the form

$$\mu\theta(t_i)-\lambda\sum_{j=0}^{i-1} D_j(t_i)\gamma\left(jh,\theta(jh)\right)=f(t_i), \qquad i=1,2,...,N. \tag{3.10}$$

Putting $t_i = ih$, and using the following notations

$$\theta_i=\theta(ih), f_i=f(ih), D_{j,i}=D_j(ih), \gamma_j(\theta_j)=\gamma\left(jh,\theta(jh)\right).$$

The formula (3.10) takes the form

$$\mu\theta_i = f_i+\lambda\left[D_{i,i}\gamma_i(\theta_i)+\sum_{\substack{j=0\\ j\neq i}}^{i-1} D_{j,i}\gamma_j(\theta_j)\right],\ \theta_0=f(t_0),\quad i=1,2,...,N, \tag{3.11}$$

Using Newton-Raphson method, the values of θ_N can be obtained after determining $\theta_1,\theta_2,...,\theta_{N-1}$.

The error term $R^{(T)}$ of the Toeplitz matrix method defined by

$$R^{(T)} = \left| \int_a^{a+h} k\left(|t_i - x|\right)\gamma\left(x, x^2\right)dx - \left[A_j(t_i)\gamma\left(a, a^2\right) + B_j(t_i)\gamma\left(a+h, (a+h)^2\right)\right]\right|. \quad (3.12)$$

4. The Existence and Uniqueness of the Solution of the NAS:

Here, the existence of a unique solution of the NAS (3.11), will be proved according to the Banach fixed point theorem, for this, we write it in the operator form

$$\bar{T}\theta_i = T\theta_i + \frac{1}{\mu} f_i \ , \quad (4.1)$$

where

$$T\theta_i = \frac{\lambda}{\mu}\sum_{j=0}^{i} D_{j,i}\ \gamma_j(\theta_j)\ ; \qquad (\ \mu \neq 0\ ,\ 1 \leq i \leq N\ ). \quad (4.2)$$

Consider the following lemma`.

Lemma 3 :

If the kernel of Eq. (1.1) satisfies the conditions

$$\int_a^{a+h} \left|k\left(|t - x|\right)\right| dx \leq L\ , \quad (4.3)$$

$$\lim_{t_i' \to t_i} \left\| k\left(|t_i' - x|\right) - k\left(|t_i - x|\right)\right\| = 0 \quad ; t_i', t_i \in C\,[0, T\,]\ , \quad (4.4)$$

then,

a) $\sup_j \sum_{j=0}^{i} \left|D_{i,j}(x)\right|$ is bounded,

b) $\lim_{i' \to i} \sup_j \sum_{j=0}^{i} \left|D_{i',j} - D_{i,j}\right| = 0.$

Proof :

To prove (a) , we use the formula (3.3), to obtain

$$\left|A_j(t_i)\right| \leq \frac{1}{|h_1|}\left[\left|\gamma(a+h,\ a+h)\right| \int_a^{a+h} \left|k\left(|t_i - x|\right)\right|\left|\gamma(x,1)\right|dx + \left|\gamma(a+h,\ 1)\right| \times \right.$$

$$\left. \int_a^{a+h} \left|k\left(|t_i - x|\right)\right|\left|\gamma(x,x)\right|dx\ \right], h_1 = \gamma(a,1)\gamma(a+h, a+h) - \gamma(a,a)\gamma(a+h,1). \quad (4.5)$$

By using the conditions (iii-a) and (ii) of Theorem (1) , then summing from $j = 0$ to i , we get

$$\sum_{j=0}^{i}\left|A_j(t_i)\right| \leq \frac{ML}{|h_1|}\sum_{j=0}^{i}\left[\left|\gamma(a+h,\ a+h)\right|\left\|\gamma(x,1)\right\| + \left|\gamma(a+h,\ 1)\right|\left\|\gamma(x,x)\right\|\right]. \quad (4.6)$$

Using the fact that each term of the inequality (4.6) is bounded for each j, and the continuity of the function $\gamma(x, \theta(x))$ in the interval [0,*T*], there exist a small constant z_1, such that

$$\sup_j \sum_{j=0}^{i}\left|A_j(ih)\right| \leq z_1 \quad , \forall\ j\ . \quad (4.7)$$

Similarly, from the formula (3.4), we have

$$\sup_j \sum_{j=0}^{i} \left| B_j\left(ih\right) \right| \le z_2 \quad \forall\ j\ , \qquad (z_2 \text{ is a small constant}). \tag{4.8}$$

Hence, from (4.7) and (4.8), the formula (3.9) can be written as

$$\sup_j \sum_{j=0}^{i} \left| D_{j,i} \right| \le z \quad , \quad z = z_1 + z_2 , \quad (z \text{ is small constant}). \tag{4.9}$$

Hence,

$$\sup_j \sum_{j=0}^{i} \left| D_{j,i} \right| \text{ is exists.}$$

For the second item (b), we use the formula (3.3), and apply the condition, (iii-a) of Theorem (1), then for $t_i', t_i \in C[0,T]$, we obtain

$$\sum_{j=0}^{i} \left| A_j\left(t_i'\right) - A_j\left(t_i\right) \right| \le \frac{M}{|h_1|} \int_a^{a+h} \left| k\left(|t_i - x|\right) - k\left(|t_i - x|\right) \right| dx \left[\sum_{j=0}^{i} \left| \gamma(a+h,\ a+h) \right| \left\| \gamma(x,1) \right\| + \sum_{j=0}^{i} \left| \gamma(a+h,\ 1) \right| \left\| \gamma(x,x) \right\| \right], \tag{4.10}$$

Since each term of the inequality (4.10) is bounded for each j, and from the continuity of $\gamma\left(x,\theta(x)\right)$ in the interval [*0*,*T*], we get

$$\sup_j \sum_{j=0}^{i} \left| A_j\left(t_i'\right) - A_j\left(t_i\right) \right| \le \frac{M}{|h_1|} \int_a^{a+h} \left| k\left(|t_i - x|\right) - k\left(|t_i - x|\right) \right| dx \left[\sup_j \sum_{j=0}^{i} \left| \gamma(a+h,\ a+h) \right| \times \left\| \gamma(x,1) \right\| + \sup_j \sum_{j=0}^{i} \left| \gamma(a+h,\ 1) \right| \left\| \gamma(x,x) \right\| \right]. \tag{4.11}$$

Using the condition (4.4) in (4.11), we have

$$\lim_{i' \to i} \sup_j \sum_{j=0}^{i} \left| A_j\left(i\hbar\right) - A_j\left(ih\right) \right| = 0 \qquad , t_i > t_i' \ , \ t_i = ih, t_i' = i\hbar \ . \tag{4.12}$$

Similarly, in view of the formula (3.4), we can prove

$$\lim_{i' \to i} \sup_j \sum_{j=0}^{i} \left| B_j\left(i\hbar\right) - B_j\left(ih\right) \right| = 0 . \tag{4.13}$$

Hence, from the relation (3.9), with (4.12) and (4.13), we get

$$\lim_{i' \to i} \sup_j \sum_{j=0}^{i} \left| D_{i',j} - D_{i,j} \right| = 0 . \tag{4.14}$$

Theorem 2 :

The NAS (3.11), when $i \to \infty$ is bounded and has a unique continuous solution in the Banah space ℓ^{∞}, under the following conditions

iv) $\sup_i \left| f_i \right| \le B < \infty$, (*B* is a constant).

v) $\sup_j \sum_{j=0}^{i} \left| D_{i,j} \right| \le E$, (*E* is a constant).

vi) The known elements $\gamma\left(jh, \theta(jh)\right)$, for the constants $P > P_1,\ P > P_2$ satisfies the conditions

(a) $\sup_j \left|\gamma\left(jh,\theta(jh)\right)\right| \le P_1 \|\Theta\|_{c^\infty}$,

(b) $\sup_j \left|\gamma\left(jh,\theta(jh)\right)-\gamma\left(jh,\psi(jh)\right)\right| \le P_2 \|\Theta-\Psi\|_{c^\infty}$,

where $\|\Theta\|_{\ell^\infty} = \sup_j \left|\theta_j\right|, \quad \forall$ j.

Now, the following lemmas must verify, to prove this Theorem.

Lemma 4 :

If the conditions (iv) – (vi-a) are verified, then the operator $\bar{T}$ defined by Eq.(4.1) maps the space ℓ^∞ into itself.

Proof :

Let U be the set of all functions $\Theta=\{\theta_i\}$ such that $\|\theta\|_{\ell^\infty} \le \beta$, β is a constant. Define the norm of operator $\bar{T}\Theta$ in the space ℓ^∞ by

$$\|\bar{T}\Theta\|_{\ell^\infty} = \sup_i \left|\bar{T}\theta_i\right|. \qquad \forall\ i. \tag{4.15}$$

From the formulas (4.1) and (4.2), we get

$$\left|\bar{T}\theta_i\right| \le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \left|D_{i,j}\right| \sup_j \left|\gamma(jh,\theta(jh))\right| + \frac{1}{|\mu|} \sup_i \left|f_i\right|.$$

In view of conditions (iv)-(vi-a), the above inequality can be adapted in the form

$$\|\bar{T}\Theta\| \le \omega\|\Theta\| + \frac{1}{|\mu|}B , \qquad \left(\omega = \left|\frac{\lambda}{\mu}\right| PE\right). \tag{4.16}$$

The inequality (4.16) shows that the operator $\bar{T}$ maps the set U into itself, where

$$\beta = \frac{B}{|\mu|(1-\omega)}. \tag{4.17}$$

Since $\beta > 0, B > 0$, therefore we have $\omega < 1$, Also, the inequality (4.16) involves the boundedness of the operator *T*, where

$$\|T\Theta\|_{\ell^\infty} \le \omega\|\Theta\|. \tag{4.18}$$

Furthermore, the inequalities (4.16) and (4.18) define the boundedness of the operator $\bar{T}$.

Lemma 5 :

The operator $\bar{T}$ is a continuous and contractive mapping in the space ℓ^∞ .

Proof :

For two sequences $\Theta=\{\theta_i\}$ and $\Psi=\{\psi_i\}$ in ℓ^∞, we will prove that the operator $\bar{T}$ defined by Eq.(4.1) is a contractive operator. For this, we write

$$\left|\bar{T}\theta_i - \bar{T}\psi_i\right| \le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \left|D_{i,j}\right| \sup_j \left|\gamma(jh,\theta(jh))-\gamma(jh,\psi(jh))\right|$$

Using the conditions (v)-(vi-b1), we obtain

$$\sup_i \left|\bar{T}\theta_i - \bar{T}\psi_i\right| \le \left|\frac{\lambda}{\mu}\right| PE \sup_j \left|\theta_j - \psi_j\right| , \quad \forall\ i . \tag{4.19}$$

Hence, in view of (4.15), we have

$$\left\|\bar{T}\Theta - \bar{T}\Psi\right\|_{\ell^\infty} \le \omega \left\|\Theta - \Psi\right\|_{\ell^\infty} \qquad , \omega = \left|\frac{\lambda}{\mu}\right| PE \ . \tag{4.20}$$

The inequality (4.20) shows that, the operator $\bar{T}$ is continuous in the in space ℓ^∞. Moreover, $\bar{T}$ is a contraction operator, under the condition $\omega < 1$.

Proof of Theorem 2

In the light of lemmas (4) and (5), the operator $\bar{T}$ is a continuous and contractive operator in the space ℓ^∞. Hence, by Banach fixed point theorem, The operator $\bar{T}$ has a unique fixed point which is the unique solution of the NAS (3.11) in the space ℓ^∞ .

It is convenient to consider the following theorem, which shows that the convergence of the sequences of approximate solutions to the exact solution $\theta(t)$ of Eq. (1.1) in space ℓ^∞.

Theorem 3 :

If the sequence of the continuous functions $\{f_i(t)\}$ converges uniformly to the function $f(t)$ as $i \to \infty$, then under the conditions of Theorem (2), the sequences of approximate solution $\{\theta_i(t)\}$ converges uniformly to the exact solution $\theta(t)$ of Eq. (1.1) in the space $C[0,T]$.

Proof :

The formula (1.1) with its approximate solution gives

$$\left\|\theta(t) - \theta_i(t)\right\| \le \left|\frac{\lambda}{\mu}\right| \left\| \int_0^t \left|k\left(|t-x|\right)\right| \left|\gamma\left(x, \theta(x)\right) - \gamma\left(x, \theta_i(x)\right)\right| dx \right\| + \frac{1}{|\mu|}\left\|f(t) - f_i(t)\right\| \tag{4.21}$$

Using the conditions (ii) and (iii-b) of Theorem (1), the above inequality yields

$$\left\|\theta(t) - \theta_i(t)\right\| < \left|\frac{\lambda}{\mu}\right| LM \left\|\theta(t) - \theta_i(t)\right\| + \frac{1}{|\mu|}\left\|f(t) - f_i(t)\right\|, \tag{4.22}$$

then, the previous inequality takes the form

$$\left\|\theta(t) - \theta_i(t)\right\| \le \frac{1}{\left(|\mu| - |\lambda| LM\right)} \left\|f - f_i\right\|. \tag{4.23}$$

Finally, we have

$\left\|\theta(t) - \theta_i(t)\right\| \to 0$, since $\left\|f - f_i\right\| \to 0$ as $i \to \infty$.

When $i \to \infty$, the sum $\sum_{j=0}^{i} D_{i,j}\,\gamma_j(\theta_j)$ becomes $\int_0^t k(t-x)\gamma(x, \theta(x))dx$, and also the solution of the algebraic system (3.11) equivalence to the solution of the nonlinear Volterra integral equation (1.1)..

Definition 1:

The estimate local error $R_N^{(T)}$ of Toeplitz matrix method can be determined by the following equation

$$\theta_i(t) - \left(\theta_i(t)\right)_N = \sum_{j=0}^{i} D_{i,j}\left[\gamma\left(jh, \theta(jh)\right) - \gamma\left(jh, \theta_N(jh)\right)\right] + R_N^{(T)}, \tag{4.24}$$

where $\left(\theta_i(t)\right)_N$ is the approximate solution of Eq. (3.11). Also, Eq. (4.24) gives

$$R_N^{(T)} = \left| \int_0^{t_i} k\left(t_i, x\right) \gamma\left(x, \theta(x)\right) dx - \sum_{j=0}^{i} D_{i,j} \gamma\left(jh, \theta_N\left(jh\right)\right) \right| . \tag{4.25}$$

Definition 2 :

The Toeplitz matrix method is said to be convergent of order r in the interval $\left[0, T\right]$, if and only if for sufficiently large N, there exists a constant $D > 0$ independent on N such that

$$\left\| \theta(x) - \theta_N(x) \right\| \leq D N^{-r} . \tag{4.26}$$

Theorem 4 :

If the conditions (v) and (vi-b) of Theorem (2) are satisfied, and the sequence of elements $\left\{F_N\right\} = \left\{\left(f_i\right)_N\right\}$ converges uniformly to the function $F = \left\{f_i\right\}$ in ℓ^{∞}. Then the sequence of approximate solutions $\left\{\Theta_N\right\} = \left\{\left(\theta_i\right)_N\right\}$ convergent uniformly to the solution $\Theta = \left\{\theta_i\right\}$ in ℓ^{∞} of the system (3.11).

Proof :

By virtue of Eq. (3.11), we have

$$\left|\theta_i - \left(\theta_i\right)_N\right| \leq \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \left|D_{j,i}\right| \sup_j \left|\gamma\left(jh, \theta(jh)\right) - \gamma\left(jh, \theta_N\left(jh\right)\right)\right| + \frac{1}{|\mu|} \left|f_i - \left(f_i\right)_N\right| . \tag{4.27}$$

After using condition (vi-b), the formula (4.27) holds for each integer i, hence from condition (v), we obtain

$$\left\|\Theta - \Theta_N\right\|_{\ell^{\infty}} \leq \left|\frac{\lambda}{\mu}\right| EP \left\|\Theta - \Theta_N\right\|_{\ell^{\infty}} + \frac{1}{\mu} \left\|F - F_N\right\|_{\ell^{\infty}} .$$

Finally, the previous inequality takes the form

$$\left\|\Theta - \Theta_N\right\|_{\ell^{\infty}} \leq \frac{1}{\left(|\mu| - |\lambda| EP\right)} \left\|F - F_N\right\|_{\ell^{\infty}} . \tag{4.28}$$

Since $\left\|F - F_N\right\| \to 0$ as $N \to \infty$, so that $\left\|\Theta - \Theta_N\right\| \to 0$ as $N \to \infty$.

Corollary 1 :

Assume that, the hypothesis of Theorem (2) are verified, then

$$\lim_{N \to \infty} R_N^{(T)} = 0. \tag{4.29}$$

Proof :

From the formula (4.24), we find

$$\left|R_N^{(T)}\right| \leq \left|\theta_i\left(t\right) - \left(\theta_i\left(t\right)\right)_N\right| + \sum_{j=0}^{i} \left|D_{i,j}\right| \sup_j \left|\gamma\left(jh, \theta(jh)\right) - \gamma\left(jh, \theta_N\left(jh\right)\right)\right| .$$

Therefore, form condition (vi-b), we get

$$\sup_N \left|R_N^{(T)}\right| \leq \sup_i \left|\theta_i\left(t\right) - \left(\theta_i\left(t\right)\right)_N\right| + P \sum_{j=0}^{i} \left|D_{i,j}\right| \sup_j \left|\theta(jh) - \theta_N\left(jh\right)\right| ,$$

The above inequality is true for each integer *N*, hence, from condition (v), we get

$$\left\|R_N^{(T)}\right\| \leq \left\|\Theta - \Theta_N\right\| + PE \left\|\Theta - \Theta_N\right\| , \qquad \forall\ N. \tag{4.30}$$

Since $\left\|\Theta - \Theta_N\right\| \to 0$ as $N \to \infty$, then $\left\|R_N^{(T)}\right\| \to 0$ and $R_N^{(T)} \to 0$.

5. The Numerical Examples and the Conclusions :

Here, we will consider the NVIE, with singular kernel (1.1), where the weak discontinuous kernel $k\left(\left|t-x\right|\right)$ takes a logarithmic and Carleman forms, and the given nonlinear function $\gamma\left(x,\theta(x)\right)=\theta^k\left(x\right),$ $;\left(1\le k\le N\,;N \text{ is finite number}\right)$. Then, we apply the Toeplitz matrix method to obtain the numerical solutions of Eq.(1.1), when we take the linear case *k* =1, and the nonlinear case *k* =3. Therefore, the parameter λ is computed from the relation $\lambda=\dfrac{2\mu\upsilon}{\left(1-2\upsilon\right)}$, that corresponding to the different values of Poison ratio υ, $\left(0<\upsilon<1\right)$ and μ as 0.389, 0.132*10^7 and 0.22 , 10^7 , respectively. Also, we assume that *T*= 0.1, 0.5 ,0.9, and $N=20,40$, where, we consider the exact solution $\gamma\left(t,\theta(t)\right)=t^2$.

Example 1:
Consider the NVIE (1.1) with Carleman kernel take the form

$$\mu\theta(t)-\lambda\int_0^t\left|t-x\right|^{-\upsilon}\theta^k\left(x\right)dx=f\left(t\right),\qquad\left(0<\upsilon<1\right). \tag{5.1}$$

The importance of Carleman kernel came from the work of Arutiunian [18], who has shown that the plane contact problem in the nonlinear theory of plasticity, in its first approximation can be reduced to Fredholm integral equation of the first kind with Carleman kernel.
In special case, if we set $k=1$, in (5.1), we have the LVIE of the second kind

$$\mu\theta(t)-\lambda\int_0^t\left|t-x\right|^{-\upsilon}\theta\left(x\right)dx=f\left(t\right),\quad\left(0<\upsilon<1\right), \tag{5.2}$$

Here, we will apply the Toeplitz matrix method to solve Eqs. (5.1) and (5.2), to get the following results.

The Numerical Results:

The program of Maple (10) is used to compute the exact and approximate solutions and errors E^T of Eqs.(5.1) , (5. 2) by using the Toeplitz matrix method for linear (*k* =1) and nonlinear (*k* = 3) cases, for different values of υ = 0.389 and 0.22 that corresponding to Polyurethane and Fibber materials at *T*= 0.1, 0.5 ,0.9, and $N=20,40$, which it represented in the following Table 1 and plot the error E^T for each case as shown in Figs. [(1-1) - (1-6)]

	N	t	Exact	nonlinear k=3				linear (k=1)			
				Polyurethane v=0.389		fiber v=0.22		Polyurethane v=0.389		fiber v=0.22	
				approximate so.	error	Approximate sol.	error	Approximate so.	error	Approximate sol.	error
0.1		2.00000E-02	4.00000E-04	4.00000E-04	1.40000E-12	4.00000E-04	1.00000E-13	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		4.00000E-02	1.60000E-03	1.60000E-03	2.30000E-11	1.60000E-03	2.00000E-12	6.43151E-04	1.81509E-05	6.25823E-04	8.22742E-07
		6.00000E-02	3.60000E-03	3.60000E-03	1.10600E-09	3.60000E-03	4.80000E-11	2.65230E-03	1.52297E-04	2.50584E-03	5.84435E-06
		8.00000E-02	6.40000E-03	6.40001E-03	8.54200E-09	6.40000E-03	3.92000E-10	6.08564E-03	4.60638E-04	5.64134E-03	1.63398E-05
	20	1.00000E-01	1.00000E-02	1.00000E-02	3.92300E-08	1.00000E-02	1.94000E-09	1.09898E-02	9.89804E-04	1.00329E-02	3.28827E-05
		0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		2.00000E-02	4.00000E-04	4.00000E-04	1.00000E-13	4.00000E-04	1.00000E-13	4.12232E-04	1.22317E-05	4.00466E-04	4.66028E-07
		4.00000E-02	1.60000E-03	1.60000E-03	8.30000E-11	1.60000E-03	4.00000E-12	1.67808E-03	7.80822E-05	1.60276E-03	2.75562E-06
		6.00000E-02	3.60000E-03	3.60000E-03	1.33200E-09	3.60000E-03	5.80000E-11	3.81669E-03	2.16690E-04	3.60000E-03	7.26792E-06
		8.00000E-02	6.40000E-03	6.40001E-03	8.85100E-09	6.40000E-03	3.86000E-10	6.84323E-03	4.43233E-04	6.41419E-03	1.41906E-05
	40	1.00000E-01	1.00000E-02	1.00000E-02	3.76100E-08	1.00000E-02	1.64000E-09	1.07728E-02	7.72808E-04	1.00237E-02	2.36560E-05
0.5		1.00000E-01	1.00000E-02	9.99999E-03	1.08140E-08	1.00000E-02	1.28000E-09	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		2.00000E-01	4.00000E-02	4.00019E-02	1.94868E-06	4.00002E-02	1.75930E-07	1.71113E-02	1.48629E-03	1.56989E-02	7.38860E-05
		3.00000E-01	9.00000E-02	9.00465E-02	4.65368E-05	9.00032E-02	3.22396E-06	7.98173E-02	1.73173E-02	6.30377E-02	5.37668E-04
		4.00000E-01	1.60000E-01	1.60367E-01	3.67100E-04	1.60024E-01	2.36063E-05	2.11889E-01	7.12640E-02	1.42160E-01	1.53499E-03
	20	5.00000E-01	2.50000E-01	2.51782E-01	1.78248E-03	2.50107E-01	1.07152E-04	4.61709E-01	2.11709E-01	2.53149E-01	3.14940E-03
		0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		1.00000E-01	1.00000E-02	1.00000E-02	1.99200E-08	1.00000E-02	1.60000E-09	1.10445E-02	1.04450E-03	1.00420E-02	4.17997E-05
		2.00000E-01	4.00000E-02	4.00037E-02	3.65491E-06	4.00002E-02	2.14220E-07	4.84977E-02	8.49774E-03	4.02520E-02	2.52000E-04
		3.00000E-01	9.00000E-02	9.00571E-02	5.71188E-05	9.00033E-02	3.25346E-06	1.19610E-01	2.96099E-02	9.06760E-02	6.76000E-04
		4.00000E-01	1.60000E-01	1.60385E-01	3.84744E-04	1.60021E-01	2.14806E-05	2.36522E-01	7.65216E-02	1.61340E-01	1.34000E-03
	40	5.00000E-01	2.50000E-01	2.51717E-01	1.71740E-03	2.50091E-01	9.14442E-05	4.20728E-01	4.20728E-01	2.52268E-01	2.26800E-03
0.9		1.80000E-01	3.24000E-02	3.23995E-02	5.27130E-07	3.24001E-02	6.86900E-08	3.53327E-02	2.93269E-03	3.25846E-02	1.84558E-04
		3.60000E-01	1.29600E-01	1.29696E-01	9.61040E-05	1.29609E-01	9.47900E-06	1.82022E-01	5.24217E-02	1.31154E-01	1.55373E-03
		5.40000E-01	2.91600E-01	2.94106E-01	2.50550E-03	2.91775E-01	1.75336E-04	5.68144E-01	2.76544E-01	2.96260E-01	4.66037E-03
		7.20000E-01	5.18400E-01	5.46812E-01	2.84121E-02	5.19728E-01	1.32796E-03	1.57689E+00	1.05849E+00	1.42160E-01	5.28244E-01
	20	9.00000E-01	8.10000E-01	2.93244E+00	3.74244E+00	8.16602E-01	6.60248E-03	4.38785E+00	3.57785E+00	8.27403E-01	1.74028E-02
		0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		1.80000E-01	3.24000E-02	3.24010E-02	9.70720E-07	3.24001E-02	8.61200E-08	3.82667E-02	5.86667E-03	3.26182E-02	2.18155E-04
		3.60000E-01	1.29600E-01	1.29781E-01	1.80935E-04	1.29612E-01	1.15427E-05	1.90888E-01	6.12878E-02	1.30937E-01	1.33660E-03
		5.40000E-01	2.91600E-01	2.94726E-01	3.12609E-03	2.91777E-01	1.77037E-04	5.75205E-01	2.83605E-01	2.95238E-01	3.63755E-03
		7.20000E-01	5.18400E-01	5.50016E-01	3.16164E-02	5.19610E-01	1.21019E-03	1.53742E+00	1.01902E+00	5.25714E-01	7.31450E-03
	40	9.00000E-01	8.10000E-01	3.84694E+00	4.65694E+00	8.15650E-01	5.65048E-03	4.11712E+00	3.30712E+00	8.22544E-01	1.25440E-02

Table (1)

Case1: linear (k=1) v=0.22

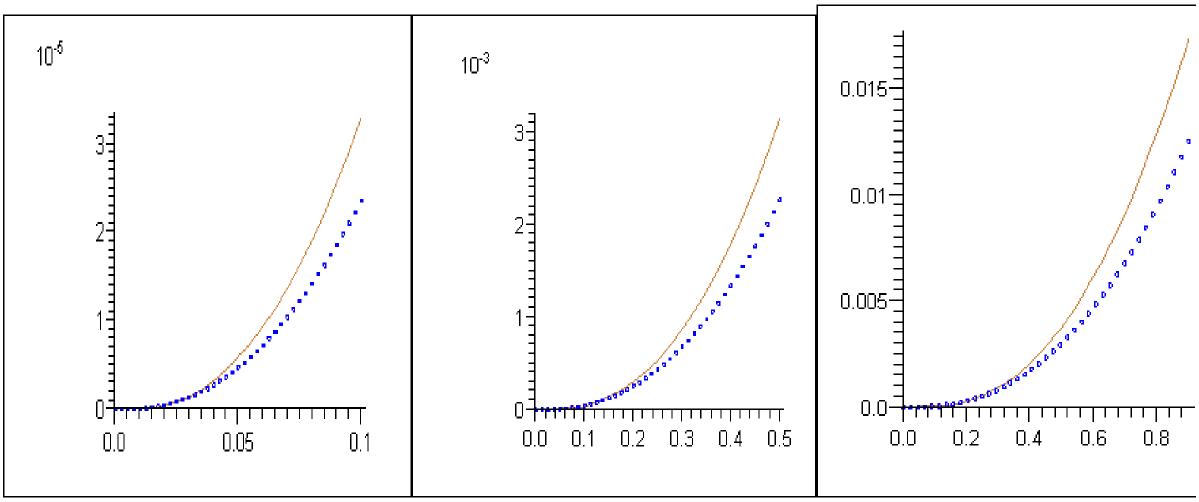

Fig.(1-1) :T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.22$

Case2: nonlinear (k=3) v=0.22

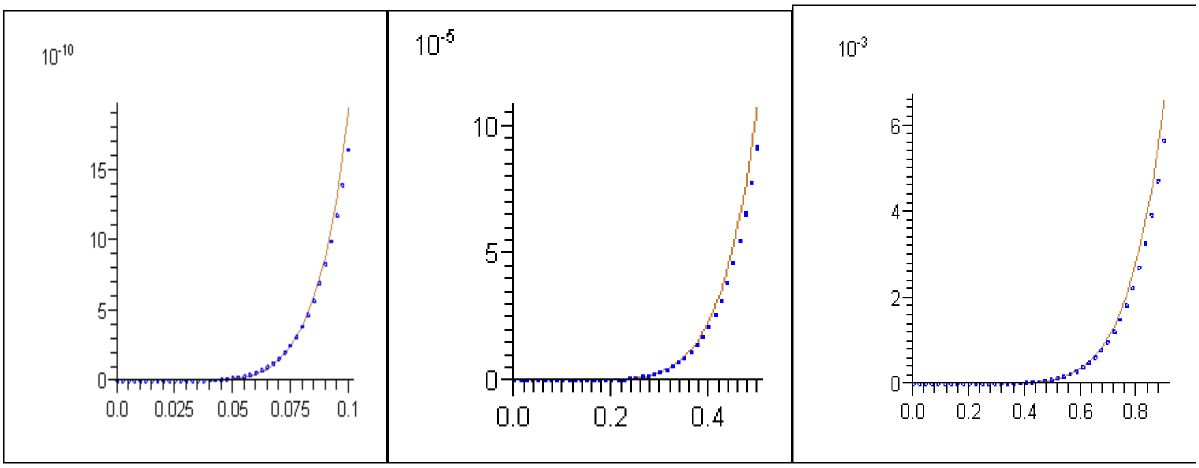

Fig.(1-2) :T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.22$,

Case1: linear (k=1) v=0.389

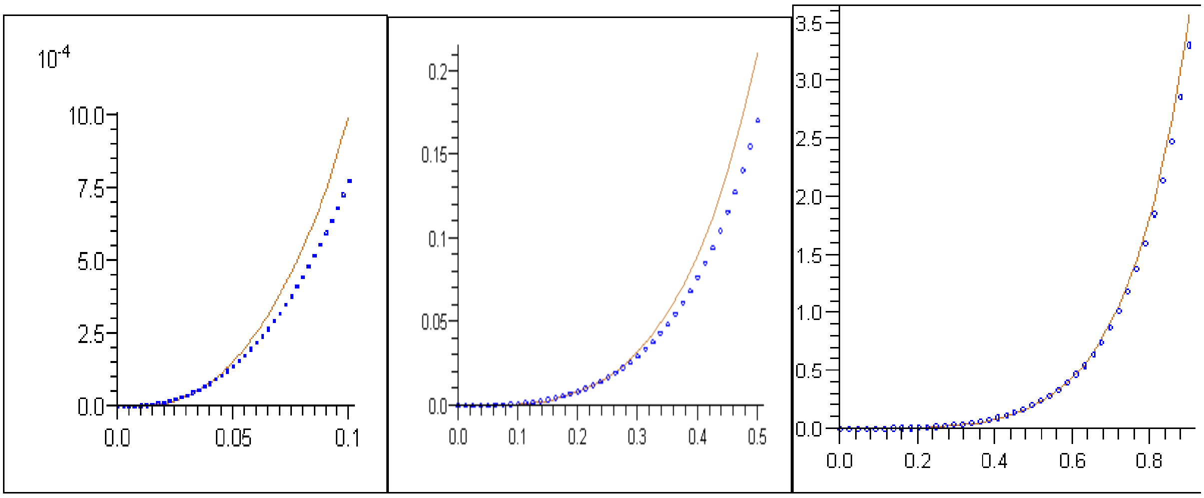

Fig.(1-3): T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.389$,

Case2: nonlinear (k=3) v=0.389

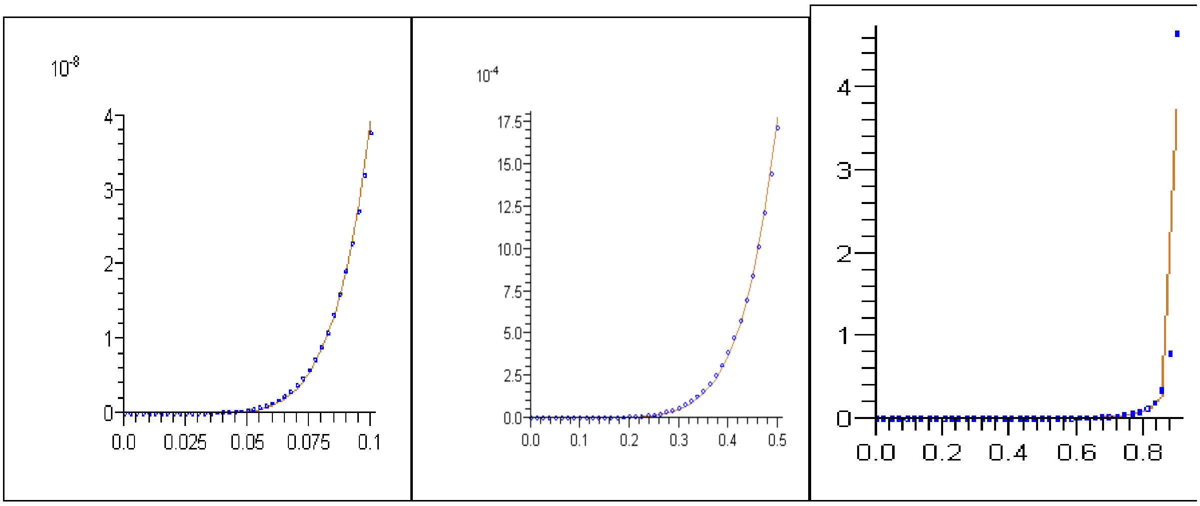

Fig.(1-4): T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.389$,

Example 2 :

If the kernel of the NVIE of the second kind can be represented in the logarithmic kernel $k(|t-x|) = \ln|t-x|$, we get

$$\mu\theta(t) - \lambda\int_0^t \ln|t-x|\,\theta^k(x)dx = f(t), \tag{5.3}$$

when k = 1, we have the linear Volterra integral equation of the second kind with logarithmic kernel as.

$$\mu\theta(t) - \lambda\int_0^t \ln|t-x|\,\theta(x)dx = f(t). \tag{5.4}$$

Also, by using the Toeplitz matrix method of Eqs .(5.3), and (5.4), we have the following results.

The Numerical Results:

Also, we use Maple programm to compute the exact and approximate solutions and the errors E^T of Eq.(5.3), (5.4) with logarithmic kernel, that corresponding to some point of t , by using the Toeplitz matrix method for linear (k =1) and nonlinear cases, (k = 3) for different values of υ = 0.389 and 0.22, that corresponding to Polyurethane and Fibber materials at T= 0.1, 0.5 ,0.9, and $N = 20, 40$, as shown in Table 2 , also, the error E^T is plotted as shown in Figs. (1-5) - (1-8).

T	N	t	Exact	nonlinear k=3				linear k=1			
				Polyurethane material v=0.389, $\lambda =$		Fibber material v=.28, $\lambda =$		Polyurethane material v=0.389, $\lambda =$		Nickle material v=.28, $\lambda =$	
				Approximate Sol.	error E^T	Approximate Sol.	error E^T	Approximate Sol.	error E^T	Approximate sol.	error E^T
0.1	20	2.00000E-02	4.00000E-04	4.00000E-04	4.00000E-13	4.00000E-04	0.00000E+00	3.97660E-04	2.33962E-06	3.99453E-04	5.47047E-07
		4.00000E-02	1.60000E-03	1.60000E-03	9.00000E-12	1.60000E-03	2.00000E-12	1.58262E-03	1.73804E-05	1.59579E-03	4.21273E-06
		6.00000E-02	3.60000E-03	3.60000E-03	2.90000E-10	3.60000E-03	6.60000E-11	3.55075E-03	4.92497E-05	3.58767E-03	1.23276E-05
		8.00000E-02	6.40000E-03	6.40000E-03	2.65600E-09	6.40000E-03	5.36000E-10	6.30111E-03	9.88877E-05	6.37454E-03	2.54551E-05
		1.00000E-01	1.00000E-02	9.99999E-03	1.20030E-08	1.00000E-02	2.69300E-09	9.83349E-03	1.66512E-04	9.95606E-03	4.39421E-05
	40	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		2.00000E-02	4.00000E-04	4.00000E-04	1.00000E-13	4.00000E-04	0.00000E+00	3.97708E-04	2.29238E-06	3.99462E-04	5.37541E-07
		4.00000E-02	1.60000E-03	1.60000E-03	1.90000E-11	1.60000E-03	5.00000E-12	1.58664E-03	1.33628E-05	1.59675E-03	3.25108E-06
		6.00000E-02	3.60000E-03	3.60000E-03	3.08000E-10	3.60000E-03	6.90000E-11	3.56521E-03	3.47915E-05	3.59127E-03	8.73021E-06
		8.00000E-02	6.40000E-03	6.40000E-03	2.09400E-09	6.40000E-03	4.70000E-10	6.33315E-03	6.68516E-05	6.38277E-03	1.72342E-05
		1.00000E-01	1.00000E-02	9.99999E-03	9.16800E-09	1.00000E-02	2.03600E-09	9.89049E-03	1.09514E-04	9.97108E-03	2.89222E-05
0.5	20	1.00000E-01	1.00000E-02	9.99999E-03	1.22990E-08	1.00000E-02	2.76000E-09	9.76514E-03	2.34863E-04	9.93987E-03	6.01260E-05
		2.00000E-01	4.00000E-02	3.99985E-02	1.47017E-06	3.99997E-02	3.29800E-07	3.83116E-02	1.68844E-03	3.95150E-02	4.84990E-04
		3.00000E-01	9.00000E-02	8.99733E-02	2.66634E-05	8.99940E-02	6.00135E-06	8.54732E-02	4.52679E-03	8.85934E-02	1.40656E-03
		4.00000E-01	1.60000E-01	1.59805E-01	1.94545E-04	1.59956E-01	4.43031E-05	1.51277E-01	8.72290E-03	1.57130E-01	2.87036E-03
		5.00000E-01	2.50000E-01	2.49142E-01	8.58280E-04	2.49799E-01	2.01289E-04	2.35746E-01	1.42538E-02	2.45098E-01	4.90220E-03
	40	0.00000E+0	0.00000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		1.00000E-01	1.00000E-02	9.99999E-03	1.24410E-08	1.00000E-02	2.80100E-09	9.75811E-03	2.41891E-04	9.93668E-03	6.33189E-05
		2.00000E-01	4.00000E-02	3.99984E-02	1.57996E-06	3.99996E-02	3.54440E-07	3.86924E-02	1.30758E-03	3.96205E-02	3.79452E-04
		3.00000E-01	9.00000E-02	8.99757E-02	2.43406E-05	8.99945E-02	5.47927E-06	8.67780E-02	3.22200E-03	8.89964E-02	1.00357E-03
		4.00000E-01	1.60000E-01	1.59839E-01	1.61190E-04	1.59963E-01	3.67188E-05	1.54046E-01	5.95353E-03	1.58045E-01	1.95519E-03
		5.00000E-01	2.50000E-01	2.49331E-01	6.68674E-04	2.49843E-01	1.56871E-04	2.40519E-01	9.48133E-03	2.46755E-01	3.24536E-03
	20	1.80000E-01	3.24000E-02	3.23994E-02	5.59550E-07	3.23999E-02	1.25510E-07	3.11840E-02	1.21595E-03	3.20695E-02	3.30454E-04
		3.60000E-01	1.29600E-01	1.29517E-01	8.28335E-05	1.29581E-01	1.87389E-05	1.21165E-01	8.43514E-03	1.26911E-01	2.68881E-03
		5.40000E-01	2.91600E-01	2.90130E-01	1.47023E-03	2.91249E-01	3.50635E-04	2.69434E-01	2.21657E-02	2.83860E-01	7.74045E-03
		7.20000E-01	5.18400E-01	5.09536E-01	8.86399E-03	5.15907E-01	2.49267E-03	4.76011E-01	4.23887E-02	5.02690E-01	1.57095E-02
		9.00000E-01	8.10000E-01	7.83621E-01	2.63793E-02	7.99905E-01	1.00955E-02	7.40782E-01	6.92181E-02	7.83258E-01	2.67423E-02
	40	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
		1.80000E-01	3.24000E-02	3.23993E-02	7.12160E-07	3.23998E-02	1.59710E-07	3.11394E-02	1.26064E-03	3.20437E-02	3.56266E-04
		3.60000E-01	1.29600E-01	1.29506E-01	9.40689E-05	1.29579E-01	2.12910E-05	1.23044E-01	6.55554E-03	1.27487E-01	2.11316E-03
		5.40000E-01	2.91600E-01	2.90234E-01	1.36647E-03	2.91273E-01	3.26830E-04	2.75709E-01	1.58909E-02	2.86056E-01	5.54374E-03
		7.20000E-01	5.18400E-01	5.11016E-01	7.38406E-03	5.16313E-01	2.08669E-03	4.89193E-01	2.92074E-02	5.07653E-01	1.07468E-02
		9.00000E-01	8.10000E-01	7.89346E-01	2.06539E-02	8.02084E-01	7.91580E-03	7.63443E-01	4.65570E-02	7.92209E-01	1.77910E-02

Table (2)

Case1: linear (k=1) v=0.22

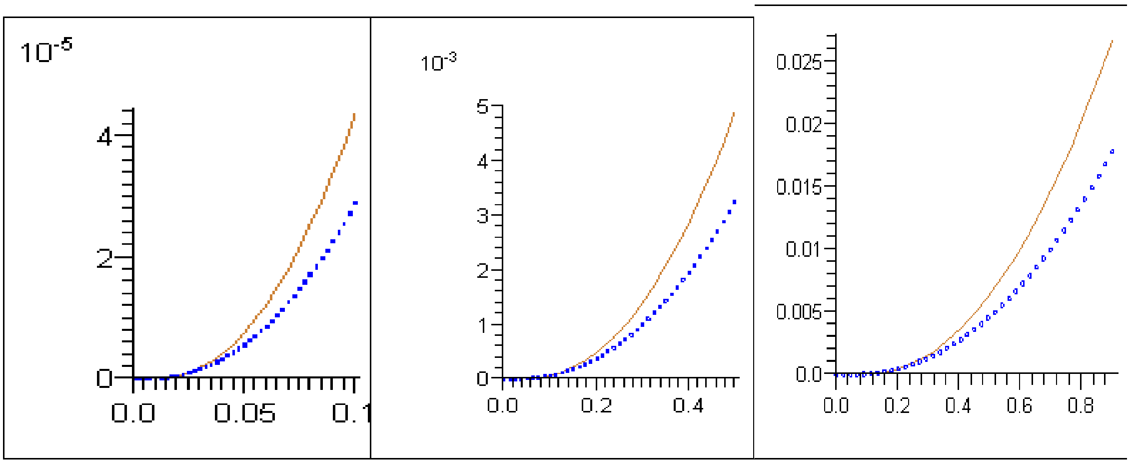

Fig.(1-5) : T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.22$,

Case 2: nonlinear (k=3), v=0.22

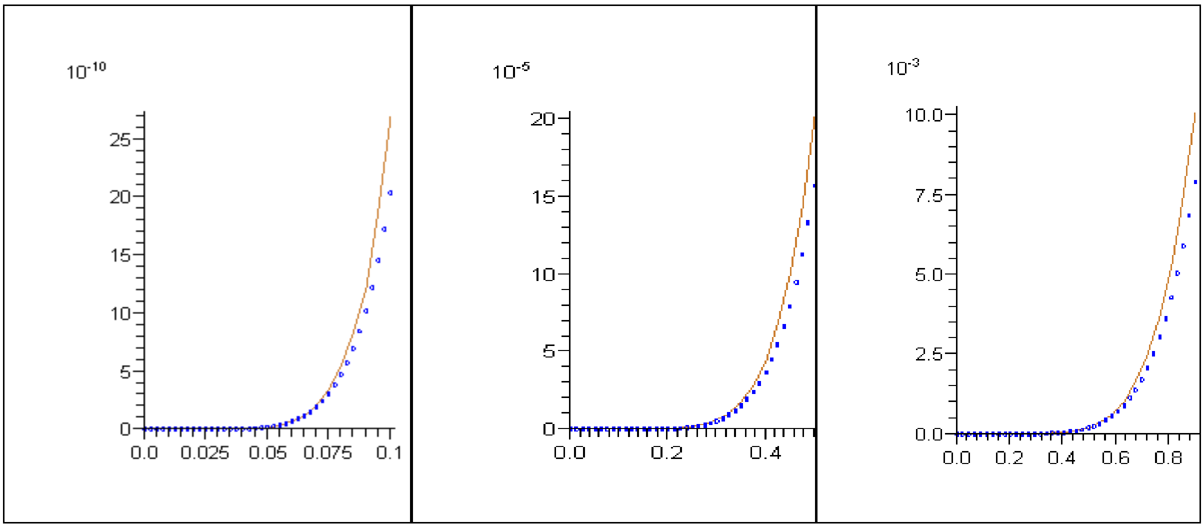

Fig.(1-6) : T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.22$,

Case1: linear (k=1) v=0.389

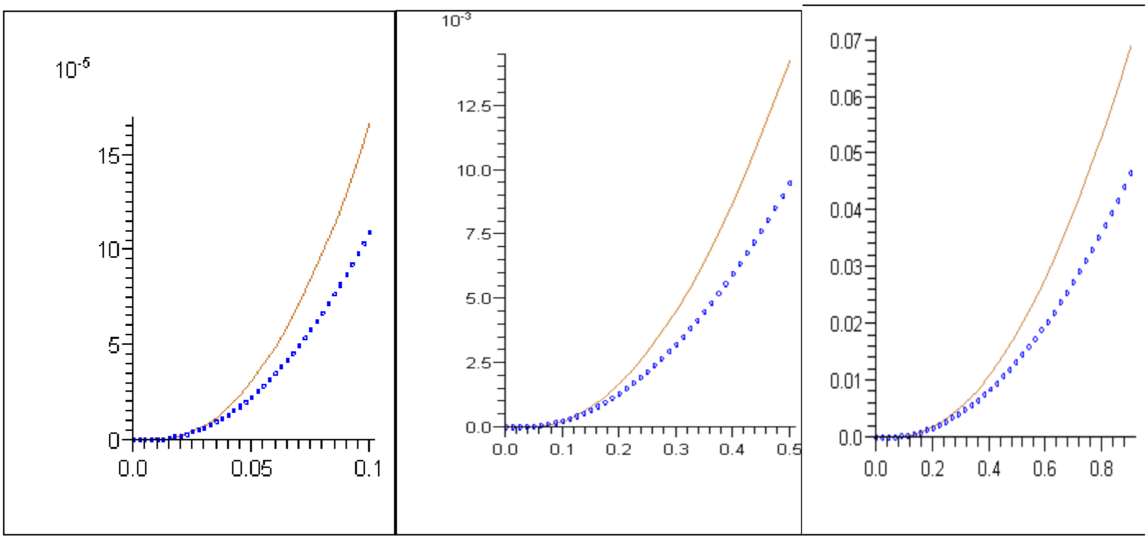

Fig.(1-7): T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.389$,

Case 2: nonlinear (k=3), v=0.389

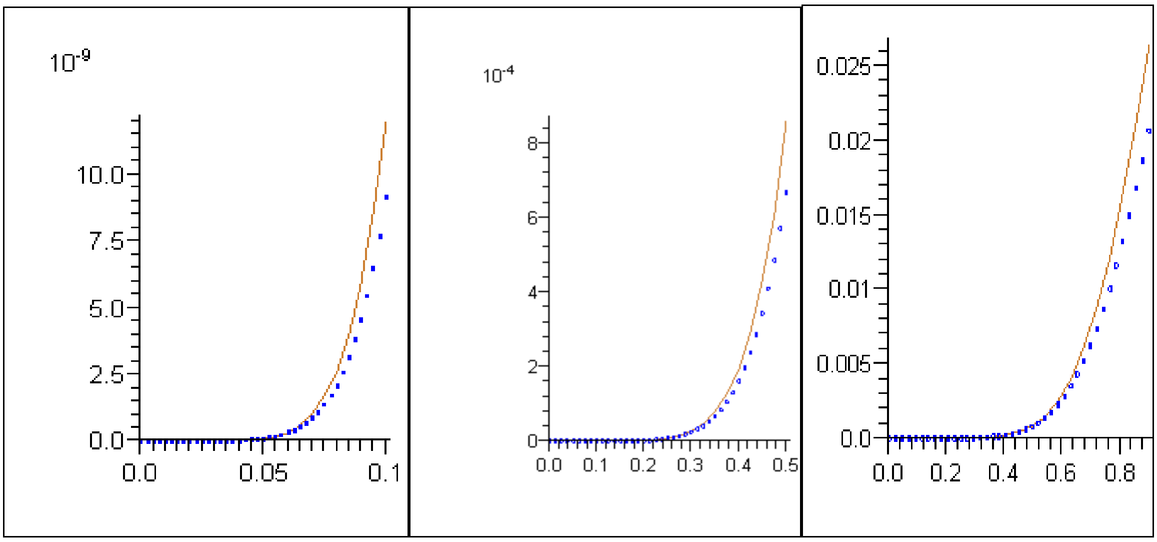

Fig.(1-8): T=0.1,0.5,0.9, $N = 20, 40$, $\upsilon = 0.389$,

The Conclusion:

In general, from the previous results in Tables (1)-(2), we discuss the error results E^T of Eqs (1.5) - (4.5) by using the Toeplitz matrix method for different value of υ, where N=20, 40 , and T = 0.1 0.5, 0.9, where the value of λ, μ are computed from some relations of the elasticity, as the following discussion:

1) For linear (k =1) and nonlinear (k =3) cases of SVIE , the error E^T decrease as well as the values of N increases for different values of υ, λ

2) The error E^T is equal 0.0 for the point $t = 0$ corresponding to different values of υ and λ, μ.

3) For different values of υ, λ, in the linear (k =1) and the nonlinear (k =3) cases, the result error E^T of Toeplitz matrix with a logarithmic kernel is the best from that products with the Carleman kernel, as shown in the Tables 1,2.

4) The error E^T is increases as well as the value of time T tends to unity for linear and nonlinear cases, and for all different values of υ λ, μ.

5) The error E^T for linear case are larger than nonlinear case of SVIE, for different values of υ and λ, k .

6) Due to the kernel of Carleman , as the values of υ is increasing i.e. $(0 < \upsilon < 0.4)$, the values of errors E^T are also increasing for T = 0.1 0.5, 0.9.

7) Due to the logarithmic kernel, as the value of λ is increasing $(0 < \upsilon < 0.5)$, the value of error E^T is also increasing for T = 0.1 0.5, 0.9.

8) Due to the kernel of Carleman, if the value of parameter υ tends to one, i.e. $(0.4 \le \upsilon < 1)$ and $(\upsilon \ne \frac{1}{2})$ (i.e. the materials is solid). and the time T also tends to unity i.e. T= 0.9, then the resultant errors E^T are greater than 1 (i.e. the numerical solutions farmer than the exact solutions) .

9) Due to the logarithmic kernel, if the value of parameter λ increasing ,i.e. $(0.5 < \upsilon < 1)$ (i.e. the materials is solid), and the time T also tends to unity i.e.T=0.9, then, the resultant errors E^T are greater than 1 (i.e. the numerical solutions farmer than the exact solutions).

In our results, we deduce that the numerical solution o f Eqs (5.1) and (5.3), (exact solution $\theta = t^2$), depends on the kernel of the integral equation and the constants λ, μ, Hence, the value of υ of different elastic materials play an important role for obtaining the numerical solution.
In general, we can apply this method for other different value of υ corresponding to (elastic materials) and deduce that the numerical solution depends on the type of kernel of the integral equation and the constants λ, μ.

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