THE EXTENDED TANH METHOD AND ITS APPLICATIONS FOR SOLVING NONLINEAR PHYSICAL MODELS

A. Hendi

Department of Physics,Faculty of science,King Saud University,
Riyadh 11321,P.O.Box:1846,Saudi Arabia.

ABSTRACT

The tanh method is a powerful solution method; various extension forms of the tanh method have been developed with a computerized symbolic computation and is used for constructing the exact travelling wave solutions, of coupled nonlinear equations arising in physics. The obtained solutions include solitons, kinks and plane periodic solutions. First a power series in tanh was used as an ansatz to obtain analytical solutions of traveling wave type of certain nonlinear evolution equations .The main properties of the method will be explained and then applied to particular and well-chosen examples in further works to establish more entirely new solutions for other kinds of nonlinear evolution equations arising in physics

Keywords: *Extended tanh method, nonlinear physical models, Solitons, kinks solution,plane periodic solutions*

1. INTRODUCTION

Since the discovery of the soliton 1965 by Zabusky and Kruscal[1].The large class of nonlinear wave phenomena plays a major role in science ,widely applied in various branches of physics and applied mathematics like condensed matter ,fluid dynamics[2],plasma physics [3],theory of turbulence ,ocean dynamics,biophysics and star formation [4] , solid state physics[5],optical fibers chemical kinetics[6] and geochemistry,mathematical biology[7] ,the phenomena of dispertion ,dissipation ,diffusion , reaction and convection are very important in nonlinear wave equations(nonlinear wave phenomena frequently appear) .

The method introduces a unifying method that one can find exact [8],[9],[10] as well as approximate solutions in a straightforward and systematic way [4]. The tanh method has been subjected to many modifications that mainly depend on the Riccati equation and the solutions of well-known equations. The standard tanh method and the proposed modifications all depend on the balance method, where the linear terms of highest order are balanced with the highest order nonlinear terms of the reduced equation. The exact analytical solutions for these (NLEE's)including the soliton solutions , periodic solutions and rational solutions ,peakons , breathers,casps and compactons solutions [11-18]

Because of the complexity of the nonlinear wave equations there is no unified method to find all solutions of there equations .Several methods ,analytical and numerical such as Backlund transformation ,the inverse scattering ,bilinear transformation ,the homogeneous balance method,the sine – cosine method ,the Riccati expansion method with constant coefficients ,variational iteration methods ,collocation method Hirota's bilinear technique ,Painleve analysis ,Jacobi elliptic function ,multilinear variable separation approach and tanh method [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31] are used to treat these topics.

A search for exactly solutions of nonlinear equations has been more interest in recent years because of the availability of symbolic computation Mathematica or Maple .These programs allow us to perform some complicated and differential calculations on a computer .

Physical structures of exact solutions are important to give more insight into the physical aspects of the nonlinear wave equations .

Applicable for a considerable number of nonlinear systems. The tanh function method provides a straight forward and effective way to construct the kink- and ball shape – solitary – wave solutions in terms of a finite series of tanh functions .

One of the excellent improvements is the extended tanh function method which achieves a uniform construction of several different types of travelling wave solutions by replacing the tanh function with the solutions of a Riccati equation involving an arbitrary parameter .Furthermore ,the extended tanh function method has been modified by proposing a generalized ansatz in [6] .Beyond the restriction of travelling –wave solutions ,the further extended tanh function method [7] generalized hyperbolic – function method [11],improved tanh function methods [12] and generalized extended tanh function method [13] have also been developed to seek the non travelling solitary – wave

and periodicals wave solutions for variable – coefficient and higher – dimensional NLEE's.
One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods to look for exact solutions for nonlinear partial differential equations. A search of directly seeking for exactly solutions of nonlinear equations has been more interest in recent years because of the availability of symbolic computation Mathematica or Maple. These computer systems allow us to perform some complicated and tedious algebraic and differential calculations on a computer.
The useful tanh method is widely used by different authers such as in [20], [21], [22], [23], [24], [25], [26], [27], [28],[29],[30] and [31], and by the references therein. The method introduces a unifying method that one can find exact as well as approximate solutions in a straightforward and systematic way .

2. EXTENDED TANH METHOD

In this section, we give a brief description of the extended tanh method as follows:
For the given system of nonlinear evolution equations, say, in two variables

$$N(u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx}\ldots)=0, \quad (1)$$

$$M(u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx}\ldots)=0. \quad (2)$$

We seek the following travelling wave solutions:

$$u(x,t)=u(\xi), \quad v(x,t)=v(\xi), \quad \xi=x\pm ct,$$

which are of important physical significance, k and c are constants to be determined later. Then system (1) and (2) reduces to a system of nonlinear ordinary differential equations

$$N_0(u,v,u_\xi,v_\xi,u_\xi,v_\xi,u_{\xi\xi},v_{\xi\xi},u_\xi,v_{\xi\xi}\ldots)=0, \quad (3)$$

$$M_0(u,v,u_\xi,v_\xi,u_\xi,v_\xi,u_{\xi\xi},v_{\xi\xi},u_\xi,v_{\xi\xi}\ldots)=0. \quad (4)$$

Introducing a new independent variables in the form

$$Y=\tanh(\mu\xi), \quad \xi=x\pm ct, \quad (5)$$

that leads to the change of derivatives

$$\frac{d}{d\xi}=\mu(1-Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\xi^2}=-2\mu^2(1-Y^2)\frac{d}{dY}+\mu^2(1-Y^2)^2\frac{d^2}{dY^2},$$

$$\frac{d^3}{d\xi^3}=2\mu^3(1-Y^2)(3Y^2-1)\frac{d}{dY}-6\mu^3Y(1-Y^2)^2\frac{d^2}{dY^2}+\mu^3(1-Y^2)^3\frac{d^3}{dY^3}. \quad (6)$$

In the context of tanh function method, many authors [1], [2], [3], [4], [9] and [10] used the ansatz

$$u(\xi)=\sum_{i=0}^{M}a_iY^i(\xi),$$

$$v(\xi)=\sum_{i=0}^{N}c_iY^i(\xi). \quad (7)$$

In order to construct more general, it is reasonable to introduce the following ansatz [5]:

$$u(\xi)=\sum_{i=0}^{M}a_iY^i(\xi)+\sum_{i=1}^{M}b_iY^{-i}(\xi),$$

$$v(\xi)=\sum_{i=0}^{N}c_iY^i(\xi)+\sum_{i=1}^{N}d_iY^{-i}(\xi), \quad (8)$$

in which $a_i,\ b_i\ (i=0,1,\ldots,M)$ and $c_i,\ d_i\ (i=0,1,\ldots,N)$ are all real constants to be determined later, the balancing numbers M and N are positive integers which can be determined by balancing the highest order derivative terms with highest power nonlinear terms in Eqs. (3) and (4). We substitute ansatz (7) into Eqs. (3) and (4) with computerized symbolic computation, equating to zero the coefficients of all power $Y^{\pm i}$ yields a set of algebraic equations for a_i , b_i , c_i , d_i and μ .
Our aim is to give an overview of the possibilities this tanh method offers together with some selected examples. It

is based on the research we have undertaken during the last decade and the progress made by other researchers in the field. More details of the basics can be found back in [14, 15 and 16], where numerous other examples are treated. For the sake of completeness, we should mention that this technique is restricted to the search for stationary waves and we thus essentially deal with shock and/or solitary type of solutions. Solitary waves are waves packets or pulses ,which propagate ,in nonlinear dispersive media .Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. A soliton is a very special type of solitary wave .which also keeps its waveform after collision with other solitons.

Moreover, we are inherently limited to one dimension (or direction of propagation). Nevertheless, numerous equations have been already solved in this way and, due to some generalizations of the method, this number is still increasing.

3. APPLICATIONS

In this section, we will demonstrate the proposed method on three nonlinear evolution equations of special interest in physics.

3-1 The Regularized Long Wave (RLW)equation of the form:

$$u_t+u_x-\beta u_{xxt}+\frac{\gamma}{2}(u^2)x=0 \qquad (9)$$

where β and γ are positive constants , was first put forward as a model for small-amplitude long-waves on the surface of water in a channel by Peregrine[32] and later by Benjamin [33].For q = 1, this equation is called the regularized long-wave which has exactly three conservation laws and it has been studied extensively in the past.

This equation is very important in physics media since it describes phenomena with weak nonlinearity and dispersion waves including hydrodynamic waves in plasma and phonon packets in non linear transverse waves in shallow water, ion – acoustic and magneto crystals. The solutions of this equation are kinds of solitary waves named solitons whose shape are not affected by a collision .RLW equation is special case of the Generalized Long Waves(GRLW)equation which has the form

$$ut + ux - \beta\ uxxt + \gamma u^{q}\ ux = 0, \qquad (10)$$

where q is a positive integer, β and γ are positive constants .

$$u(x,t)=u(\eta), \qquad (11)$$

In physical situations such as unidirectional waves propagation in a water channel, long crested waves in near shore zones, and many other, the generalized regularized long wave (RLW) equation serves an alternative model to the KdV equations. Using the transformation

$$\eta = kx + wt, \qquad (12)$$

then ,Eq. (10)becomes

$$(w+k)u'+\alpha kuu'-\beta k^2uu'''=0 \qquad (13)$$

and by using

$$u=a_0+a_{-2}F^{-2}+a_{-1}F^{-1}+a_1F^1+a_2F^2 \qquad (14)$$

we find

$$case1 \rightarrow a$$

$$d_2=0\,,d_1=0\,,c_1=0\,,c_0=-\tfrac{1}{6}\frac{3k+3w+4\alpha kc_2}{\alpha k}\,,\mu:=\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}} \qquad (15)$$

According to case(1-a),we have a new solitary wave solution as follows

$$u=\frac{c_0\tanh(\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^2+c_1\tanh(\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^3+d_1\tanh(\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)+c_2\tanh((\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^4+d_2}{c_1\tanh((\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^2}$$

(16)

$case1-b$

$$d_2=0,\mu:=-\frac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}},d_1=0,c_1=0,c_0=-\frac{1}{6}\frac{3k+3w+4\alpha kc_2}{\alpha k} \quad (17)$$

According to case(1-b),we have a new solitary wave solution as follows

$$u=\frac{c_0\tanh(\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^2-c_1\tanh(\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^3-d_1\tanh(\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)+c_2\tanh((\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^4+d_2}{c_1\tanh((\tfrac{1}{6}\sqrt{6}\sqrt{\frac{\alpha c_2}{\beta mk}}\xi)^2}$$

(18)

$Case2$

$$c_2=0,d_1=0,c_1=0,d_2=0,c_0=c_0,\mu=\mu \quad (19)$$

In view of case(2),we have a new solitary wave solution as follows

$$u=\frac{c_0\tanh(\mu\xi)^2+d_1\tanh(\mu\xi)+c_2\tanh(\mu\xi)^4+d_2}{\tanh(\mu\xi)^2} \quad (20)$$

$Case3:$

$$d_1=0,d_2=0,c_2=\frac{6\beta mk\mu^2}{\alpha},c_1=0,c_0=-\frac{1}{2}\frac{k+w+8\beta mk^2\mu^2}{\alpha k},\mu=\mu \quad (21)$$

From case(3),we obtain solitary wave solution as

$$u=\frac{1}{2}\frac{1}{\alpha k\tanh(\mu\xi)^2}(-\tanh(\mu\xi)^2k-\tanh(\mu\xi)^2w-8\tanh(\mu\xi)^2\beta mk^2\mu^2+2c_1\tanh(\mu\xi)^3\alpha k$$
$$+12\beta mk^2\eta^2\tanh(\mu\xi)^4+2\alpha kd_2)$$

(22)

Case 4:

$$c_0=-\frac{1}{2}\frac{k+w+8\beta mk^2\mu^2}{\alpha k},d_1=0,c_1=0,c_2=\frac{6\beta mk\mu^2}{\alpha},d_2=\frac{6\beta mk\mu^2}{\alpha},\mu=\mu$$

By means of case(4), admits to solitary wave solution as follows

(23)

$$u=\frac{1}{2}\frac{1}{\alpha k\tanh(\mu\xi)^2}(-\tanh(\mu\xi)^2k-\tanh(\mu\xi)^2w-8\beta mk^2\mu^2\tanh(\mu\xi)^2$$
$$+12\beta mk^2\mu^2\tanh(\mu\xi)^4+12\beta mk^2\mu^2$$

(24)

3-2 <u>Ear Drum Vibrations equation:</u>

the nonlinear eardrum-type oscillations are studied. The nonlinear eardrum-type oscillation is defined such that an even function for the displacement may contain the nonlinear restoring force . For the nonlinear eardrum oscillation, an iteration procedure for determining the motion and period of the oscillation was suggested. In the study, the results up to second round iteration were presented.

Earlier, it was suggested that one could evaluate the eigen values of the ordinary differential equation (ODE) by the iteration of solving the ODE .

In following analysis, the nonlinear eardrum oscillation is taken as an example. The oscillation is defined by:

$$\frac{d^2u}{dt^2}+\omega_o^2u(1+\varepsilon u)=0, \tag{25}$$

where u is the displacement, ω_o is the circular frequency given before hand, ε is a constant which may not be a small value. The imposed boundary conditions take the form

$$u|_{t=0}=A,\quad \left.\frac{du}{dt}\right|_{t=0}=0, \tag{26}$$

where A is a positive value.

Case1

$$\mu=0.5k,d_1=0,c_1=0,c_0=\frac{3}{2}\frac{k^2}{\beta},d_2=0,c_2=-\frac{3}{2}\frac{k^2}{\beta} \tag{27}$$

According to case(1-a),we have a new solitary wave solution as follows

$$u=\frac{3}{2}\frac{k^2}{\beta}-\frac{3}{2}\frac{k^2\tanh(0.5k\xi)^2}{\beta} \tag{28}$$

Case2

$$\mu=\frac{1}{2}Ik,-\frac{1}{2}Ik,c_1=0,c_0=-\frac{1}{2}\frac{k^2}{\beta},d_2=0,c_2=\frac{3}{2}\frac{k^2}{\beta} \tag{29}$$

In view of case(2),we have a new solitary wave solution as follows

$$u=-\frac{1}{2}\frac{k^2}{\beta}-\frac{3}{2}\frac{k^2\tan(\frac{1}{2}k\xi)^2}{\beta} \tag{30}$$

Case3

$$\mu=\frac{1}{2}k,d_1=0,c_1=0,c_0=\frac{3}{2}\frac{k^2}{\beta},d_2=-\frac{3}{2}\frac{k^2}{\beta},c_2=0 \quad (31)$$

From case(3),we obtain solitary wave solution as

$$u=\frac{3}{2}\frac{k^2}{\beta}+\frac{3}{2}\frac{k^2}{\beta\tan(\frac{1}{2}k\xi)^2} \quad (32)$$

Case4

$$\mu=\frac{1}{4}\mathrm{I},-\frac{1}{4}\mathrm{I},d_1=0,c_1=0,c_0=\frac{1}{4}\frac{k^2}{\beta},d_2=\frac{3}{8}\frac{k^2}{\beta},c_2=\frac{3}{8}\frac{k^2}{\beta} \quad (33)$$

By means of case(4), admits to solitary wave solution as follows

$$u=\frac{1}{4}\frac{k^2}{\beta}-\frac{3}{8}\frac{k^2\tan(\frac{1}{4}\xi)^2}{\beta}-\frac{3}{8}\frac{k^3}{\beta\tan(\frac{1}{4}\xi)^2} \quad (34)$$

Case5

$$\mu=-\frac{1}{4}k,d_1=0,c_1=0,c_0=\frac{3}{4}\frac{k^2}{\beta},d_2=-\frac{3}{8}\frac{k^2}{\beta},c_2=-\frac{3}{8}\frac{k^2}{\beta} \quad (35)$$

According to case(5),we have a new solitary wave solution as follows

$$u=\frac{3}{4}\frac{k^2}{\beta}-\frac{3}{8}\frac{k^2\tanh(\frac{1}{4}k\xi)^2}{\beta}-\frac{3}{8}\frac{k^2}{\beta\tanh(\frac{1}{4}k\xi)^2} \quad (36)$$

3-3 <u>Jimbo Miwa Equation</u>

uxxxy + 3ux uxy + 3uy uxx + 2uyt − 3uxz = 0

the (3 + 1)- dimensional Jimbo–Miwa equation,travelling wave solutions

u = u(x, y, z, t) = U (ξ), ξ = kx + ly + mz + ωt, (37)

where k, l, m and ω are arbitrary constants.

Substituting (37) into (36) gives rise to ODE

$$u''+3u^2-(2\omega+3)u=0 \quad (38)$$

We next introduce a new independent variable Φ as follows:

u(x, y, z, t) = U (ξ) = $\sum_{i=0}^{\infty} a_i\varphi_i$ (39)

where ai are constatnts.

Case1

$$d_1 = 0, c_1 = 0, c_0 = \beta + \frac{3}{2}, d_2 = 0, c_2 = -\beta - \frac{3}{2}, \mu = \frac{1}{2}\sqrt{2\beta + 3} \tag{40}$$

According to case(1),we have a new solitary wave solution as follows

$$u = \beta + \frac{3}{2} + (-\beta - \frac{3}{2})\tanh(\frac{1}{2}\sqrt{2\beta + 3}\xi)^2 \tag{41}$$

Case2

$$\mu = \frac{1}{2}\sqrt{-2\beta - 3}, -\frac{1}{2}\sqrt{-2\beta - 3}, d_1 = 0, c_1 = 0, c_0 = -\frac{1}{3}\beta - \frac{1}{2}, d_2 = 0 \tag{42}$$

In view of case(2),we have a new solitary wave solution as follows

$$u = -\frac{1}{3}\beta - \frac{1}{2} + (\beta + \frac{3}{2})\tanh(\frac{1}{2}\sqrt{-2\beta - 3}\xi)^2 \tag{43}$$

Case3

$$\mu = \frac{1}{2}\sqrt{2\beta + 3}, -\frac{1}{2}\sqrt{2\beta + 3}, d_1 = 0, c_1 = 0, c_0 = \beta + \frac{3}{2}, d_2 = -\beta - \frac{3}{2}, c_2 = 0 \tag{44}$$

From case(3),we obtain solitary wave solution as

$$u = \beta + \frac{3}{2} + \frac{-\beta - \frac{3}{2}}{\tanh(\frac{1}{2}\sqrt{2\beta + 3}\xi)^2} \tag{45}$$

Case 4

$$\mu = \frac{1}{2}\sqrt{-2\beta - 3}, -\frac{1}{2}\sqrt{-2\beta - 3}, d_1 = 0, c_1 = 0, c_0 = -\frac{1}{3}\beta - \frac{1}{2}, d_2 = \beta + \frac{3}{2}, c_2 = 0 \tag{46}$$

According case(3),we obtain solitary wave solution as

$$u = -\frac{1}{3}\beta - \frac{1}{2} + \frac{\beta + \frac{3}{2}}{\tanh(\frac{1}{2}\sqrt{-2\beta - 3}\xi)^2} \tag{47}$$

Case 5

$$\mu = \frac{1}{4}\sqrt{2\beta + 3}, -\frac{1}{4}\sqrt{2\beta + 3}, d_1 = 0, c_1 = 0, c_0 = -\frac{1}{3}\beta - \frac{1}{2}, d_2 = -\frac{1}{4}\beta - \frac{3}{8}, c_2 = 0 \tag{48}$$

From case(5),we obtain solitary wave solution as

$$u=-\frac{1}{3}\beta-\frac{1}{2}+\frac{-\frac{1}{4}\beta-\frac{3}{8}}{\tanh(\frac{1}{4}\sqrt{2\beta+3}\xi)^2} \tag{49}$$

Case 6

$$\mu=\frac{1}{4}\sqrt{-2\beta-3},-\frac{1}{4}\sqrt{-2\beta-3},d_1=0,c_1=0,c_0=-\frac{1}{3}\beta-\frac{1}{2},d_2=\frac{1}{4}\beta+\frac{3}{8},c_2=\frac{1}{4}\beta+\frac{3}{8} \tag{50}$$

In view of case(6),we have a new solitary wave solution as follows

$$u=-\frac{1}{3}\beta-\frac{1}{2}+(\frac{1}{4}\beta+\frac{3}{8})\tanh(\frac{1}{4}\sqrt{-2\beta-3}\xi)^2+\frac{\frac{1}{4}\beta+\frac{3}{8}}{\tanh(\frac{1}{4}\sqrt{-2\beta-3}\xi)^2} \tag{51}$$

4.CONCLUSIONS

In this paper, the extended tanh method with a computerized symbolic computation is used for constructing wide classes of periodic travelling wave solutions of three nonlinear equations arising in nonlinear physics namely , the generalized regularized long-wave equation, eardrum vibrations equation ,the Jimbo –Miwa equation.
In this work, we presented an extended tanh method based on the general ansatz leads to the conclusion that it is improving the tanh method.
Finally, it is worth while to mention that the proposed method is reliable and effective and gives more solutions .The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear equations.

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