PERIOD DOUBLING ROUTE TO CHAOS IN A TWO PARAMETER INVERTIBLE MAP WITH CONSTANT JACOBIAN

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ABSTRACT

The universality discovered by M.J.Feigenbaum with non-linear models has successfully led to observe that large classes of non-linear systems exhibit transitions to chaos through period doubling route. In this paper, we consider a **two parameter** map of the plane viz. the Henon map, develop some useful numerical algorithms to obtain fixed points and bifurcation values of periods 2^n, $n = 0, 1, 2, ...$ We have shown how the ratio of three successive period doubling bifurcation points ultimately converge to the Feigenbaum constant. This ascertains that the Henon map follows the period doubling route to chaos.

Keywords: *Bifurcation/ Chaos / Fixed Points / Feigenbaum constant.*

1. INTRODUCTION

The universality discovered by the elementary particle theorist, Mitchell J. Feigenbaum in 1975 in one-dimensional iterations with the logistic map, $x_{n+1} = \lambda x_n (1 - x_n)$ has successfully led to discover that large classes of non-linear systems exhibit transitions to chaos which are universal and quantitatively measurable [4], [5].

One of his fascinating discoveries is that if a family f presents period doubling bifurcations then there is an infinite sequence $\{\mu_n\}$ of bifurcation values such that $\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta$, where δ is a universal number which is now termed as Feigenbaum constant.

In [7] Henon, motivated by computer studies of the Lorenz system performed by Pomeau, studied a transformation which maps the plane into itself. Henon was able to prove, among other things, that the transformation which he considered was the most general quadratic map which carries the plane into itself and has constant Jacobian determinant. In a remarkable sequence of computer graphics he gave strong numerical evidence that the transformation he studied has a strange attractor.

The Henon map leads from the one dimensional dynamics of the quadratic transformation to higher dimensional strange attractors. It is simple enough to allow an analysis similar to the analysis of chaos in the logistic transformation, yet it possesses features inherent in more complicated attractors such as the Lorenz attractor. Recently control and synchronization algorithms for the Henon map have been utilized for secure communications [3] and for control of pathological rhythms in some models of cardiac activity [1].

2. THE HENON MAP AND THE FEIGENBAUM UNIVERSALITY:

The Henon map is a map from R^2 to R^2 depending on two real parameters μ, b and is given by $H_{\mu, b}(x, y)\ (\equiv H(x, y)) = (1 - \mu x^2 + y,\ bx)$. In a geometric sense, stretching and folding in phase space often gives rise to chaotic behavior. Stretching results in nearby points diverging, folding results in distant points being mixed together. We can partition the Henon map into three steps to see its correspondence to the stretch and fold action [8] in the following way.

(a) Bend up: The first step consists of a nonlinear bending in the y coordinate given by

$$H_1(x,y)=\left(x,\ 1-\mu x^2+y\right)$$

For example, a horizontal line ($y=$constant) becomes a parabola with the vertex at $(0,\ y+1)$ and opening up at the bottom. In contrast, the remaining two steps are linear transformations.

(b) Contraction in x: Next a contraction in x-direction is applied

$$H_2(x,y)=\left(\ bx,\ 1-\mu x^2+y\ \right)$$

The contraction factor is given by the parameter b , which is 0.3 for the Henon attractor.

(c) Reflection: Finally a reflection at the diagonal

$$H_3(x,y)=\left(y,\ x\right)$$

The result of the compression is the same as applying the original transformation once, i.e.

$$H(x,y)=H_3\left(H_2\left(H_1(x,y)\right)\right).$$

If $b=0$, the Henon map reduces to the Logistic map which follows period doubling route to chaos [6], [8]. The automatic question which comes up is that if $b\neq 0$, is there still the Feigenbaum scenario present ?

If $b\neq 0$, $H_{\mu,\,b}(x,y)=\left(1-\mu x^2+y,\ bx\right)$ is a diffeomorphism of R^2 onto itself. The inverse of the map is given by $H^{-1}_{\mu,\,b}(x,y)=\left(b^{-1}y,\ x-1+\mu b^{-2}y^2\right)$. Note that the Jacobian of $H_{\mu,\,b}$ is the constant $-b$. So, $H_{\mu,\,b}$ is dissipative if $|\,b\,|<1$, area-preserving if $b=\pm 1$, and the area-expanding if $|\,b\,|>1$. Note also that the case $|\,b\,|>1$ can be effectively reduced to the case $|\,b\,|<1$, since $H_{b^{-2}\mu,\ b^{-1}}=T^{-1}H^{-1}_{\mu,b}T$ where the mapping T is given by $T(x,y)=(-y,-x)$ [2].

The Henon map has two fixed points, say (x_1,y_1) and (x_2,y_2) whose coordinates are given by the solutions of $H(x,y)=\left(1-\mu x^2+y,\ bx\right)=(x,y)$ and they are found to be

$$x=\frac{(b-1)\pm\sqrt{(1-b)^2+4\mu}}{2\mu},\qquad \mathrm{y=bx}$$

Thus, the fixed points of the Henon map are found to be

$$\left(x=\frac{-1+b-\sqrt{1+4\mu-2b+b^2}}{2\mu}\ ,\ y=\frac{-b+b^2-b\sqrt{1+4\mu-2b+b^2}}{2\mu}\right)\ \text{and}$$

$$\left(x=\frac{-1+b+\sqrt{1+4\mu-2b+b^2}}{2\mu}\ ,\ y=\frac{-b+b^2+b\sqrt{1+4\mu-2b+b^2}}{2\mu}\right)$$

From this one finds that $H_{\mu,\,b}$ has no fixed point if $\mu<-\frac{1}{4}(1-b)^2$.

In this context, we also wish to point out that the stability theory is intimately connected with the Jacobian matrix of the map, and that the trace of the Jacobian matrix is the sum of its eigenvalues and the product of the eigenvalues equals the Jacobian determinant. For a particular value of b in the closed interval $[-1, 1]$, the Henon map H depends on the real parameter μ, and so a fixed point $\underline{x_0} = (x_0, y_0)$ (or a periodic point $\underline{x_0}$) of this map depends on the parameter value μ, i.e. $\underline{x_0} = \underline{x_0}(\mu)$. Henon chose $b = 0.3$ to obtain the attractor named after him.

The Jacobian matrix J_1 (say) of the Henon map is given by

$$J_1 = \begin{bmatrix} -2\mu x & 1 \\ b & 0 \end{bmatrix}$$

If λ_1, λ_2 are the eigenvalues of J_1 then we have

$$\lambda_1 + \lambda_2 = -2\mu x \qquad \text{and} \qquad \lambda_1\lambda_2 = -b$$

Now, for period doubling bifurcation to take place, one of the eigenvalues must be -1. So, if we take $\lambda_2 = -1$, then the above equations reduces to

$$\lambda_1 = b \qquad \text{(from the second equation)}$$

$$\text{and} \qquad b - 1 = -2\mu x \qquad \text{(from the first equation)}$$

Now, putting $x = \dfrac{(b-1) + \sqrt{(1-b)^2 + 4\mu}}{2\mu}$ in the second equation and solving for μ, we get $\mu = \dfrac{3}{4}(1-b)^2$. This implies that the first period doubling bifurcation for the Henon map takes place at $\mu_1\,(say) = \dfrac{3}{4}(1-b)^2$ and hence this becomes the first bifurcation point.

Implication of this is that the fixed point $\underline{x_0}$ given by $\left(x = \dfrac{-1 + b + \sqrt{1 + 4\mu - 2b + b^2}}{2\mu},\ y = \dfrac{-b + b^2 + b\sqrt{1 + 4\mu - 2b + b^2}}{2\mu} \right)$ remains stable for all values of μ lying in the interval $I_1 = \left(-\dfrac{1}{4}(1-b)^2, \dfrac{3}{4}(1-b)^2 \right)$ and a stable periodic trajectory of period one appears around it. This means that the two eigenvalues of the Jacobian matrix

$$J_1 = \begin{bmatrix} -2\mu x & 1 \\ b & 0 \end{bmatrix}$$

at $\underline{x_0}$ remains less than one in absolute value. For e.g. if we choose $b = 0.4$ then $I_1 = (-.09,\ .27)$. Now, if we take $\mu = 0.02 \in I_1$ then the eigenvalues for the above case are found to be -0.66491 and 0.601585 which are less than 1 in absolute value. So, all the neighbouring points (the points in the domain of attraction) are attracted towards $\underline{x_0}(\mu)$, μ lying in I_1. Interestingly, for the same set of values of b and μ, if we calculate the eigenvalues for the other fixed point viz.

$$\left(x = \frac{-1+b-\sqrt{1+4\mu-2b+b^2}}{2\mu}\ ,\ y = \frac{-b+b^2-b\sqrt{1+4\mu-2b+b^2}}{2\mu} \right),$$ they are found to be 1.52553 and -0.262204. Thus, one of the eigenvalues is greater than one and this suggests that this fixed point is unstable.

For some negative values of b for which μ lies in the region between the boundary curves $\mu = -b \pm (1-b)\sqrt{-b}$ yield complex eigenvalues for the Jacobian J_1. For e.g. if we consider $b = -0.1$ and $\mu = 0.3$, then the eigenvalues for the first fixed point are found to be $-0.226209 + 0.220974i$ and $-0.226209 - 0.220974i$. This region where complex eigenvalues appear is exhibited in figure 1.

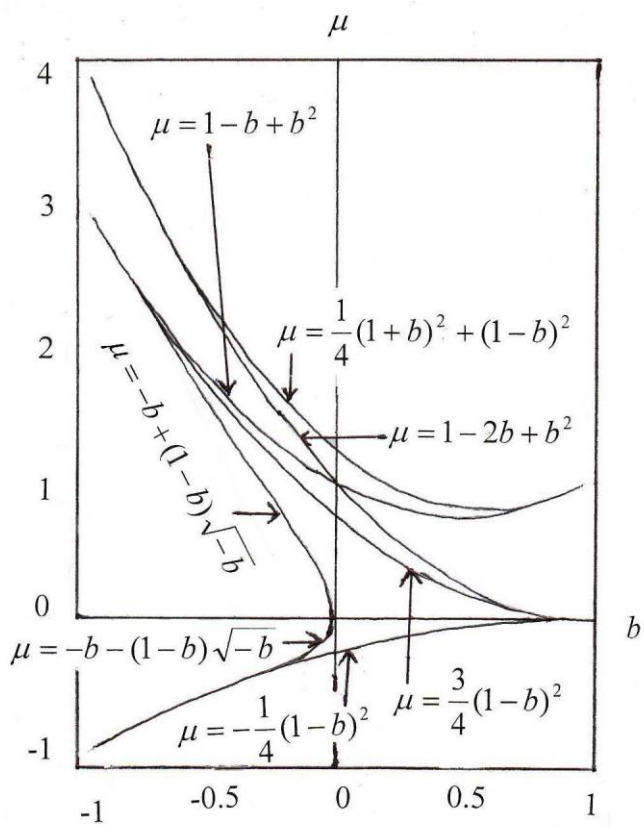

Fig.1: Regions of complex eigenvalues for periods 1 and 2

Significance of complex eigenvalues is that successive iterations of the map spiral into (or spiral out from) the stable (unstable) fixed point, and that of real eigenvalues is that consecutive iterations approach the stable fixed point along the direction of the eigenvector corresponding to the higher eigenvalues in modulus.

If we now begin to increase the value of μ, then it happens that one of the eigenvalues starts decreasing through -1 and the other remains less than one in modulus, because their product is always equal to $(-b)$ and we have taken $b \in [-1, 1]$. When μ equals $\frac{3}{4}(1-b)^2$, one of the eigenvalues becomes -1 and then $\underline{x_0}$ loses its stability, i.e. $\mu_1 = \frac{3}{4}(1-b)^2$ emerging as the first bifurcation value of μ.

Now, if we keep on increasing the value of μ the point $\underline{x_0}(\mu)$ becomes unstable (this can be verified by finding out the eigenvalues which will be greater than one in absolute value) and there arises around it two points, say, $\underline{x_{21}}(\mu)$ and $\underline{x_{22}}(\mu)$ forming a stable periodic trajectory of period 2.

So, at this point we have to shift our attention from the first iteration to the second iteration of the map which is given by $H^2(x,y) = H\big(H(x,y)\big)$.

The fixed points of $H^2(x,y)$ are the periodic points of period 2 for the Henon map $H(x,y)$ and they are found by solving the equation

$$H^2(x,y) = \big(1+bx-\mu(1-\mu x^2+y)^2,\, b(1-\mu x^2+y)\big) = (x,y).$$

The solutions of this fourth degree equation are found to be

$$\left(x = \frac{-1+b-\sqrt{1+4\mu-2b+b^2}}{2\mu},\; y = \frac{-b+b^2-b\sqrt{1+4\mu-2b+b^2}}{2\mu}\right),$$

$$\left(x = \frac{-1+b+\sqrt{1+4\mu-2b+b^2}}{2\mu},\; y = \frac{-b+b^2+b\sqrt{1+4\mu-2b+b^2}}{2\mu}\right),$$

$$\left(x = \frac{1-b+\sqrt{-3+4\mu+6b-3b^2}}{2\mu},\; y = \frac{b-b^2-b\sqrt{-3+4\mu+6b-3b^2}}{2\mu}\right) \text{ and}$$

$$\left(x = \frac{1-b-\sqrt{-3+4\mu+6b-3b^2}}{2\mu},\; y = \frac{b-b^2+b\sqrt{-3+4\mu+6b-3b^2}}{2\mu}\right)$$

Interestingly, the first two periodic points of period 2 are already fixed points of $H(x,y)$ which already become unstable once the parameter value $\mu = \frac{3}{4}(1-b)^2$ is attained where the first bifurcation took place. So, we are not concerned with those two points and we discuss only about the nature of stability of the other two periodic points. It can be easily seen that the two periodic points of period 2 do not exist if $\mu < \frac{3}{4}(1-b)^2$.

The Jacobian matrix J_2 (say) of the second iteration of the Henon map is given by

$$J_2 = \begin{bmatrix} b+4\mu^2 x(1-\mu x^2+y) & -2\mu(1-\mu x^2+y) \\ -2\mu bx & b \end{bmatrix}$$

If λ_1, λ_2 are the eigenvalues of J_2 then, as earlier, we have

$$\lambda_1 + \lambda_2 = 2b + 4\mu^2 x(1-\mu x^2+y) \quad \text{and} \quad \lambda_1\lambda_2 = b^2$$

If we take $\lambda_2 = -1,$ then the above equations lead to

$$b^2 + 2b + 1 + 4\mu^2 x(1-\mu x^2+y) = 0$$

Now, putting $\left(x = \frac{1-b+\sqrt{-3+4\mu+6b-3b^2}}{2\mu},\; y = \frac{b-b^2-b\sqrt{-3+4\mu+6b-3b^2}}{2\mu}\right)$ in the second equation and solving for μ, we get $\mu = \frac{1}{4}(1+b)^2 + (1-b)^2$.

The same can be verified for the other periodic point

$$\left(x = \frac{1-b-\sqrt{-3+4\mu+6b-3b^2}}{2\mu} \;,\; y = \frac{b-b^2+b\sqrt{-3+4\mu+6b-3b^2}}{2\mu} \right).$$

This implies that the second period doubling bifurcation for the Henon map takes place at $\mu_2\,(say) = \frac{1}{4}(1+b)^2 + (1-b)^2$. As discussed earlier, if we find the eigenvalues, this time we will find that eigenvalues in both the cases are same and will be less than one in absolute value. For e.g. if we consider $b = 0.4$ and $\mu = 0.3 \in (0.27,\, 0.85) = I_2$, then the eigenvalues for both the periodic points of period 2 are found to be 0.852265 and 0.187735. This implies that both the periodic points of period 2 are stable in between the parameter values $\mu_1 = \frac{3}{4}(1-b)^2$ and $\mu_2 = \frac{1}{4}(1+b)^2 + (1-b)^2$. If we increase the parameter value still further, the periodic points of period 2 become unstable. This means that all the neighbouring points except the stable manifold of $\underline{x_0}(\mu)$ are attracted towards these two points and this phenomenon continues for all μ lying in the open interval $\left(\frac{3}{4}(1-b)^2,\; \frac{1}{4}(1+b)^2 + (1-b)^2\right)$. Since the period emerged becomes double, the previous eigenvalue which was -1 becomes $+1$ and as we keep increasing μ, one of the eigenvalues starts decreasing from $+1$ to -1.

As earlier, it can be shown that some negative values of b for which μ lies in the region between the curves $\mu = 1-2b+b^2$ and $\mu = 1-b+b^2$ yield complex eigenvalues for the Jacobian J_2. For e.g. if we consider $b = -0.1$ and $\mu = 1.2 \in (1.11,\, 1.21) = \left(1-b+b^2,\, 1-2b+b^2\right)$, then the eigenvalues for both the periodic points of period 2 are found to be $-0.08 \pm 0.06i$. This region is also shown in fig. 1. Since the trace is always real, when eigenvalues are complex, they are conjugate to each other moving along the circle of radius $\sqrt{b_e}$, where $b_e = b^{2^n}$ is the effective Jacobian, in the opposite directions. When we reach $\mu_2 = \left(\frac{1}{4}(1+b)^2 + (1-b)^2\right)$, we find that one of the eigenvalues of the Jacobian of H^2 becomes -1 indicating the loss of stability of the periodic trajectory of period two. At this point, it is important to mention that because of the chain rule of differentiation, it does not matter at which periodic point we evaluate the eigenvalues,. Thus, the second bifurcation takes place at this value μ_2 of μ.

If we increase the parameter beyond $\mu_2 = \left(\frac{1}{4}(1+b)^2 + (1-b)^2\right)$, then all the fixed points and periodic points we found out earlier become unstable and new periodic points of period 4 appear into the scene. The study regarding their nature of stability can be done with the help of the fourth order iteration of the Henon map viz. $H^4(x, y)$.

Increasing the value μ further and further, and repeating the same arguments we obtain a sequence $\{\mu_n(b)\}$ as bifurcation values for the parameter μ such that at $\mu = \mu_n(b)$ a periodic trajectory of period 2^n arises and all periodic trajectories of period $2^m (m < n)$ remain unstable. The sequence $\{\mu_n(b)\}$ behaves in a universal manner such that $\mu_\infty(b) - \mu_n(b) \approx c(b)\, \delta^{-n}$, where $c(b)$ is independent of n and δ and is the Feigenbaum Universal constant. Since the Henon map has constant Jacobian $-b,\ |b| < 1$ gives the dissipative case, that is, contraction of area and in this case δ equals 4.6692016091029… . For $|b|=1$ we have the conservative case, i.e. the preservation of area and in this case δ equals 8.721097200… .

Already we have seen the enormous computational difficulty associated with the analytical discussion in case of the periodic points of period 2. So, analytical discussion beyond this stage is practically impossible and we have to take recourse to some numerical technique. Our effort in this paper is to put forward a numerical scheme which is simple and straightforward, by which we can find the periodic points of higher period, corresponding bifurcation points with the help of the first two bifurcation points which we found analytically and then show how the Feigenbaum constant is achieved via a series of period doubling bifurcations in case of the two parameter Henon map.

3. NUMERICAL METHOD FOR OBTAINING PERIODIC POINT:

Although there are so many sophisticated numerical algorithims to find a periodic fixed point, we have found that the Newton Recurrence formula is one of the best numerical methods with negligible error for our purpose. Moreover, it gives fast convergence of a periodic fixed point.

The Newton Recurrence formula is

$\underline{x_{n+1}} = \underline{x_n} - Df(\underline{x_n})^{-1} f(\underline{x_n})$, where $n = 0, 1, 2, ...$ and $(Df)(\underline{x})$ is the Jacobian of the map f at the vector $\underline{x}$. We see that this map f is equal to $H^k - I$ in our case, where k is the appropriate period. The Newton formula actually gives the zero(es) of a map, and to apply this numerical tool in the Henon map one needs a number of recurrence formulae which are given below.

Let the initial point be (x_0, y_0) ,

Then,
$$H(x_0, y_0) = \left(1 - \mu x_0^{\ 2} + y_0, bx_0\right) = (x_1, y_1)$$
$$H^2(x_0, y_0) = H\left(H(x_0, y_0)\right) = H(x_1, y_1) = \left(1 - \mu x_1^{\ 2} + y_1, bx_1\right) = (x_2, y_2)$$

Proceeding in this manner the following recurrence formula for the Henon map can be established.

$$x_n = 1 - \mu x_{n-1}^{\ \ 2} + y_{n-1} \quad \text{and} \quad y_n = bx_{n-1} \quad \text{where } n = 1, 2, 3, ...$$

Since the Jacobian of H^k (k times iteration of the Henon map) is the product of the Jacobian of each iteration of the map, we proceed as follows to describe our recurrence mechanism for the Jacobian Matrix.

The Jacobian J_1 for the transformation

$$H(x_0, y_0) = \left(1 - \mu x_0^{\ 2} + y_0,\ bx_0\right) \text{ is}$$

$$J_1 = \begin{pmatrix} -2\mu x_0 & 1 \\ b & 0 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{where } A_1 = -2\mu x_0,\ B_1 = 1,\ C_1 = b,\ D_1 = 0.$$

Next the Jacobian J_2 for the transformation

$H^2(x_0, y_0) = (x_2, y_2)$ where x_2 and y_2 are as mentioned above, is the product of the Jacobians for the transformations

$$H(x_1, y_1) = \left(1 - \mu x_1^{\ 2} + y_1,\ bx_1\right) \text{ and } H(x_0, y_0) = \left(1 - \mu x_0^{\ 2} + y_0,\ bx_0\right).$$

So we obtain

$$J_2 = \begin{pmatrix} -2\mu x_1 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} -2\mu x_1 A_1 + C_1 & -2\mu x_1 B_1 + D_1 \\ bA_1 & bB_1 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

where $A_2 = -2\mu x_1 A_1 + C_1, \quad B_2 = -2\mu x_1 B_1 + D_1, \quad C_2 = bA_1, \quad D_2 = bB_1.$

Continuing this process in this way we have the Jacobian for H^m as

$$J_m = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}$$ with a set of recursive formula as

$$A_m = -2\mu x_{m-1} A_{m-1} + C_{m-1}, \quad B_m = -2\mu x_{m-1} B_{m-1} + D_{m-1}, \quad C_m = bA_{m-1}, \quad D_m = bB_{m-1}.$$
$$m = 2, 3, 4, \ldots.$$

Since the fixed point of this map H is a zero of the map

$H^{/}(x,y) = H(x,y) - (x,y)$, the Jacobian of $H^{/(k)}$ is given by

$J_k - I = \begin{pmatrix} A_k - 1 & B_k \\ C_k & D_k - 1 \end{pmatrix}$. Its inverse is $\left(J_k - I\right)^{-1} = \frac{1}{\Delta}\begin{pmatrix} D_k - 1 & -B_k \\ -C_k & A_k - 1 \end{pmatrix}$

where $\Delta = (A_k - 1)(D_k - 1) - B_k C_k$, the Jacobian determinant. Therefore, Newton's method gives the following recurrence formula in order to yield a periodic point of H^k

$$x_{n+1} = x_n - \frac{(D_k - 1)(\overline{x_n} - x_n) - B_k(\overline{y_n} - y_n)}{\Delta}$$

$$y_{n+1} = y_n - \frac{(-C_k)(\overline{x_n} - x_n) + (A_k - 1)(\overline{y_n} - y_n)}{\Delta}$$

where $\mathrm{H}^{k}(\underline{x_n}) = (\overline{x_n}, \overline{y_n})$

4. NUMERICAL METHODS FOR FINDING BIFURCATION VALUES:

First of all, we recall our recurrence relations for the Jacobian Matrix of the map H^k described in the Newton's method and then the eigenvalue theory gives the relation $A_k + D_k = -1 - (-b)^k$ at the bifurcation value. Again the Feigenbaum theory says that

$$\mu_{n+2} = \mu_{n+1} + \frac{\mu_{n+1} - \mu_n}{\delta} \qquad (*)$$

where $n = 1, 2, 3, \ldots$ and δ is the Feigenbaum Universal constant.

In the case of the Henon map, the first two bifurcation values μ_1 and μ_2 can be evaluated by their explicit formulae, viz.

$$\mu_1 = \frac{3}{4}(1-b)^2 \text{ and } \mu_2 = \frac{1}{4}(1+b)^2 + (1-b)^2$$

Furthermore, it is easy to find the periodic points for these μ_1 and μ_2 for any value of b. We note that if we put $I = A_k + D_k + 1 + (-b)^k$, then I turns out to be a function of the parameter μ. The bifurcation value of μ of the period k occurs when $I(\mu)$ equals zero. This means, in order to find a bifurcation value of period k, one needs the zero of the function $I(\mu)$, which is given by the Secant method,

$$\mu_{n+1} = \mu_n - \frac{I(\mu_n)(\mu_n - \mu_{n-1})}{I(\mu_n) - I(\mu_{n-1})}.$$

Then using the relation (*), an approximate value $\mu_3^{/}$ of μ_3 is obtained. Since the Secant method needs two initial values, we use $\mu_3^{/}$ and a slightly larger value, say, $\mu_3^{/} + 10^{-4}$ as the two initial values to apply this method and ultimately obtain μ_3. In like manner, the same procedure is employed to obtain the successive bifurcation values $\mu_4,\ \mu_5,\ ...$etc. to our requirement. For $b = 0.4$, we enlist in table 1 some fixed/ periodic points, bifurcation values, the value of δ for the periods, $k = 2^2,\ 2^3,\ 2^4,\ 2^5,\ 2^6$. We have given below the related c programs which were used in deriving them and they can be used for more higher iterations also, with slight modifications in the given programs.

Table 1 (For b=0.4)

Value of k (period)	**Corresponding Fixed/Periodic point**		**Bif. value μ**	**Value of δ**
$2^2 = 4$	x = 0.918282704031	y = -0.321894581383	0.9604256555251	5.25240
$2^3 = 8$	x = 0.826868693798	y = -0.338715771759	0.9846927066042	4.55044
$2^4 = 16$	x = 0.807283361091	y = -0.343278299162	0.9899196678604	4.64267
$2^5 = 32$	x = 1.010391757182	y = -0.289199527902	0.9910420559598	4.657
$2^6 = 64$	x = 1.011434848026	y = -0.288831698131	0.9912825840017	4.66635

We have furnished the programs below through which we have found the above periodic points and bifurcation values.

Program 1 for finding approx. periodic and bifurcation points by trial and error method which utilizes Newton's method.

```
#include<stdio.h>
#include<math.h>
main()
{
   int k, c1, c2, c3, c4;
   double m, a, c, x1, x2, y1, y2, z1, z2, t1, t2, n1, n2, b, e1, e2, l1, l2, q, d, i, p, h1, h2, f1, f2;
   k=32;
   m=.992301;
   b=0.4;
   printf("\n enter the values of x1 and y1\n");
   scanf("%lf%lf",&x1,&y1);
   for(c1=1;c1<=100;c1++)
       {
          a=x1;c=y1;
             for(c2=1;c2<=k;c2++)
                 {
                     z1=1-m*x1*x1+y1;f1=b*x1; x1=z1;y1=f1;
                 }
    x2 = x1;    y2 = y1;    x1= a;      y1= c;
    l1= -2*m*x1;    e1=1;    n1=b;    t1=0;
            for(c3=1;c3<=(k-1);c3++)
                  {
                      z2 = 1-m*x1*x1+y1;   f2 = b*x1;    d = -2*m*z2;    l2 = d*l1+n1;
                     e2 =d*e1+t1;   n2 = b*l1;   t2 = b*e1;   x1= z2;
                       y1=f2;   n1= n2;    e1= e2;   t1= t2;   l1= l2;
                   }
```

```
        i = l2 + t2 +1+ pow((-b),k);
        x1=a;    y1=c;
        h1= x1-((t2-1)*(x2-x1)-e2*(y2-y1))/((t2-1)*(l2-1)-e2*n2);
        h2 = y1-((-n2)*(x2-x1)+(l2-1)*(y2-y1))/((t2-1)*(l2-1)-e2*n2);
        x1= h1;   y1=h2;
       printf("\n the fixed pts and i values are\n");
       printf("%lf %lf %lf\n",x1,y1,i);
    }
  return 1;
 }
```

Program 2 for finding final periodic and bifurcation points by Secant method.

```
#include<stdio.h>
#include<math.h>
main()
{
    int k, c1, c2, c3, c4, c5, c6;
    double m, a, c, x1, x2, y1, y2, z1, z2, t1, t2, n1, n2, b, e1, e2, l1, l2, q, d, i, p, h1, h2, f1, f2;
    double m1, m2, m3, ao, co, zo1, zo2, fo1, fo2, lo1, lo2, eo1, eo2, no1, no2, to1, to2, i1, i2,
    ho1, ho2;
    k = 4;
    m1=.96042563;            m2=.9604256300001;
    b=0.4;
    printf("\n enter the values of x1 and y1\n");
    scanf("%lf%lf",&x1,&y1);
    for(c1=1;c1<=100;c1++)
      {
         a=x1;     c=y1;
             for(c2=1;c2<=k;c2++)
                  {
                        z1=1-m1*x1*x1+y1;     f1=b*x1;    x1=z1;    y1=f1;
                   }
          x2=x1;     y2=y1;    x1=a;     y1=c;
          l1=-2*m1*x1;        e1=1;      n1=b;     t1=0;
              for(c3=1;c3<=(k-1);c3++)
                  {
                       z2=1-m1*x1*x1+y1;     f2=b*x1;    d=-2*m1*z2;     l2=d*l1+n1;   e2=d*e1+t1;
                      n2=b*l1;        t2=b*e1;      x1=z2;      y1=f2;   n1=n2;   e1=e2;   t1=t2;  l1=l2;
                   }
          x1=a;    y1=c;
        h1 = x1-((t2-1)*(x2-x1)-e2*(y2-y1))/((t2-1)*(l2-1)-e2*n2);
        h2 = y1-((-n2)*(x2-x1)+(l2-1)*(y2-y1))/((t2-1)*(l2-1)-e2*n2);
        x1=h1;    y1=h2;
     }
    i1=l2+t2+1+pow((-b),k);
            for(c4=1;c4<=100;c4++)
              {
                 ao=x1;      co=y1;
                     for(c5=1;c5<=k;c5++)
                           {
                              zo1=1-m2*x1*x1+y1;      fo1=b*x1;      x1=zo1;     y1=fo1;
                            }
                  x2=x1;    y2=y1;   x1=ao;   y1=co;
                  lo1= -2*m2*x1;    eo1=1;   no1=b;     to1=0;
          for(c6=1;c6<=(k-1);c6++)
              {
```

```
              zo2=1-m2*x1*x1+y1;   fo2=b*x1;    d= -2*m2*zo2;    lo2=d*lo1+no1;
             eo2=d*eo1+to1;   no2=b*lo1;     to2=b*eo1;
              x1=zo2;     y1=fo2;    no1=no2;    eo1=eo2;   to1=to2;  lo1=lo2;
         }
   x1=ao;    y1=co;
   ho1=x1-((to2-1)*(x2-x1)-eo2*(y2-y1))/((to2-1)*(lo2-1)-eo2*no2);
   ho2=y1-((-no2)*(x2-x1)+(lo2-1)*(y2-y1))/((to2-1)*(lo2-1)-eo2*no2);
   x1=ho1;y1=ho2;
  }
  i2=lo2+to2+1+pow((-b),k);
  m3=m2-(i2*(m2-m1))/(i2-i1);
  m1=m2;m2=m3;
  printf("\n the values of x1 y1 and m2 are");
  printf("%15.12f %15.12f %15.20f",x1,y1,m2);
  return 1;
}
```

5. REFERENCES:

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