AN M/M/1 RETRIAL QUEUE WITH WORKING VACATION AND COLLISION

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ABSTRACT
In this paper, We considered an M/M/1 retrial queue with working vacation and Collision. In order to obtain the necessary and sufficient condition for system to be stable, the matrix-analytic method is used. The stationary probability distribution and some performance measures were also derived. Then we give the conditional stochastic decomposition for the queue length in the orbit when the server is busy. Finally, the model parameters on the system's characteristics are presented by some numerical examples.

Key words: *Retrial, Working Vacation, Collision, Conditional Stochastic Decomposition.*

1 Introduction

Queueing systems with server's vacations have been well studied extensively in many real world. Some general models can be found in Tian and Zhang [1]. On the basis of ordinary vacation, Servi and Finn [2] first introduced a new policy as working vacation, and studied an M/M/1 queue, rather than completely stopping the service, the server commits a lower service rate during the working vacation. Do [3] discussed an M/M/1 retrial queue with working vacations. The combination of retrial and working vacation have also been well studied by many scholars. Many other results can be referenced by Li et al. [4]. On the other hand, Collision is also innovated. Jailaxmi et al. [5] analyzed of an M/G/1 retrial queue with general retrial time, modified M-vacations and collision. Nazarov et al [6] considered of closed Markov retrial queuing system with collision. Moreover, working vacation combination with collision has been also investigated extensively. Li et al. [7] studied an M/M/1 retrial queue with collisions and working vacation interruption under N-policy. Some other results can be find in [8, 9].

To the authors' best knowledge, there is no research work investigating an M/M/1 retrial queue with working vacation and collision. This motivates us to deal with such a queueing model in this paper. And if we let parameters in this paper take proper values, many M/M/1 retrial queues will be special cases of our model.

This paper is organized as follows. In Section 2, we establish the model and obtain the infinitesimal generator. In Section 3, we obtained the stationary probability distribution. In Section 4, a conditional stochastic decomposition was given. In Section 5, some numerical examples are presented to illustrate the effect of some parameters on the expected queue length. Finally, Section 6 concludes this paper.

2 Model Formulation

We consider an M/M/1 retrial queue with working vacation and collision. The detailed description of this model is given as follows:

The interarrival times of customers are exponentially distributed with parameter λ. Upon the arrival of customers, if the server is free, service will begin immediately. If the server is busy, on the other hand, customers will go to the orbit to wait for service. Request retrials from the orbit follow a Poisson process with rate α. And, if the server is busy that moment, the retrial customer collides with the customer in service resulting in both being shifted to the orbit. Each time when the system becomes empty the server will begin a working vacation, and the vacation time follows an exponential distribution with parameter θ. Furthermore, when a working vacation ends, if the system still empty, a new busy period will start, if the system is non-empty at that moment, on the other hand, the server will start another working vacation. The service in a regular busy period is governed by an exponential distribution with parameter μ, and in working vacation period follows an exponential distribution with parameter $\eta(\eta < \mu)$.

We assume that interarrival times, interretrial times, service times and vacation times are mutually independent.

Let Q(t) be the number of customers in the orbit at time t, and let J(t) be the state of server at time t. There are four possible states of the server as follows:

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$$J(t)=\begin{cases}1, theserverisinaworkingvacationperiodattimetandtheserverisbusy.\\ 2, theserverisinaworkingvacationperiodattimetandtheserverisfree.\\ 3, theserverisduringanormalserviceperiodattimetandtheserverisbusy.\\ 4, theserverisduringanormalserviceperiodattimetandtheserverisfree.\end{cases}$$

Obviously, {Q(t),J(t)} is a Markov process with state space,

$$\Omega=\{(0,j),j=1,2,3\}\cup\{(n,j),n\geq 1,j=1,2,3,4\}.$$

Using the lexicographical sequence for the states, the infinitesimal generator can be written as

$$\tilde{Q}=\begin{pmatrix}B_1B_0 & & \\ A_2A_1A_0 & & \\ A_2A_1A_0 & & \\ \ddots\ddots\ddots & & \end{pmatrix},$$

where

$$B_1=\begin{pmatrix}-(\lambda+\eta+\theta) & \eta & \theta & 0\\ \lambda & -\lambda & 0 & 0\\ 0 & \mu & -(\lambda+\mu) & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$B_0=\begin{pmatrix}\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, A_0=\begin{pmatrix}\lambda & \alpha & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \lambda & \alpha\\ 0 & 0 & 0 & 0\end{pmatrix}, A_2=\begin{pmatrix}0 & 0 & 0 & 0\\ \alpha & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \alpha & 0\end{pmatrix},$$

$$A_1=\begin{pmatrix}-(\lambda+\alpha+\eta+\theta) & \eta & \theta & 0\\ \lambda & -(\lambda+\theta+\alpha) & 0 & \theta\\ 0 & 0 & -(\lambda+\mu+\alpha) & \mu\\ 0 & 0 & \lambda & -(\lambda+\alpha)\end{pmatrix}.$$

Due to the block structure of matrix $\tilde{Q}$, {Q(t),J(t)} is called a quasi birth and death (QBD) process.

3 Stability Condition and Stationary Distribution

Theorem 3.1 *The QBD process {Q(T),J(t)} is positive recurrent if and only if* $(\mu+\alpha)\alpha>(\lambda+\alpha)^2$.

Proof: First, we assume

$$A=A_0+A_1+A_2=\begin{pmatrix}-(\alpha+\eta+\theta) & \alpha+\eta & \theta & 0\\ \alpha+\lambda & -(\lambda+\alpha+\theta) & 0 & \theta\\ 0 & 0 & -(\mu+\alpha) & \mu+\alpha\\ 0 & 0 & \alpha+\lambda & -(\lambda+\alpha)\end{pmatrix}.$$

Since matrix A is reducible, the Theorem 7.3.1 in [10] gives the condition for positive recurrence of the QBD. After permutation of rows and columns, the Theorem 7.3.1 states that the QBD is positive recurrent if and only if

$$\pi^*\begin{pmatrix}0 & 0\\ \alpha & 0\end{pmatrix}e>\pi^*\begin{pmatrix}\lambda & \alpha\\ 0 & 0\end{pmatrix}e,$$

where e is a column vector with all elements equal to one, and π^* is the unique solution of the system $\pi^*\begin{pmatrix}-(\mu+\alpha) & \mu+\alpha\\ \alpha+\lambda & -(\lambda+\alpha)\end{pmatrix}=0$, π^*e=1. After some algebraic manipulation, the QBD process is positive recurrent if and only if $\frac{\mu+\alpha}{\lambda+\alpha}\cdot\alpha>\lambda+\alpha$, ie. $(\mu+\alpha)\alpha>(\lambda+\alpha)^2$.

Theorem 3.2 *If* $(\mu+\alpha)\alpha>(\lambda+\alpha)^2$, *the matrix equation* $R^2A_2+RA_1+A_0=0$ *has the minimal non-negative solution*

$$R = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 & r_6 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$r_1 = \frac{t-\sqrt{t^2-4\eta\alpha[\alpha\lambda+\lambda(\lambda+\alpha+\theta)]}}{2\eta\alpha},$$
$$r_2 = \frac{r_1\eta+\alpha}{\lambda+\alpha+\theta}, r_3 = \frac{r_1 r_2\theta\alpha+r_1\theta(\lambda+\alpha)+r_2\theta\lambda}{(\lambda+\mu+\alpha)(\lambda+\alpha)-r_1\alpha\mu-r_6\alpha(\lambda+\alpha)-\mu\alpha},$$
$$r_4 = \frac{r_2\theta+r_3\mu}{\lambda+\alpha}, r_5 = \lambda(\lambda+2\alpha), r_6 = \mu(\lambda+\alpha).$$

where

where

$$t = (\lambda+\alpha+\theta+\eta)(\lambda+\alpha+\theta) - \eta\lambda - \alpha^2,$$

Proof: From the structure of A_0, A_1 and A_2, we can assume $R = \begin{pmatrix} R_{11} & R_{12} \\ \mathbf{0} & R_{22} \end{pmatrix}$, where R_{11},$R_{12}$and R_{22} are all 2×2 matrices. Taking R into $R^2A_2 + RA_1 + A_0 = \mathbf{0}$, we have

$$\begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = R_{11}\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} + R_{11}\begin{pmatrix} -(\alpha+\lambda+\theta+\eta) & \eta \\ \lambda & -(\lambda+\alpha+\theta) \end{pmatrix} + \begin{pmatrix} \lambda & \alpha \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = R_{11}R_{12}\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} + R_{11}\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} + R_{12}\begin{pmatrix} -(\lambda+\mu+\alpha) & \mu \\ \lambda & -(\lambda+\alpha) \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = {R_{22}}^2\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} + R_{22}\begin{pmatrix} -(\lambda+\mu+\theta) & \mu \\ \lambda & -(\lambda+\alpha) \end{pmatrix} + \begin{pmatrix} \lambda & \alpha \\ 0 & 0 \end{pmatrix}. \end{cases}$$

From the first equation, we get $R_{11} = \begin{pmatrix} r_1 & r_2 \\ 0 & 0 \end{pmatrix}$. Similarly, $R_{22} = \begin{pmatrix} r_5 & r_6 \\ 0 & 0 \end{pmatrix}$ can be derived from the third equation. Taking R_{11} and R_{22} into the second equation, we finally obtain $R_{12} = \begin{pmatrix} r_3 & r_4 \\ 0 & 0 \end{pmatrix}$ by some computation.

Under the stability condition, let (Q,J) be the stationary limit of the process {Q(t), J(t)}, and denote

$$\pi_n = (\pi_{n1}, \pi_{n2}, \pi_{n3}, \pi_{n4}), n \ge 0,$$

$$\pi_{nj} = P\{Q=n, J=j\} = \lim_{t\to\infty} P\{Q(t)=n, J(t)=j\}, (n,j) \in \Omega.$$

Note that when where is no customer in the orbit, the probability that the server is in a busy period and does not serve a customer is zero. Thus, $\pi_{04} = 0$.

Theorem 3.3 *If* $(\mu+\alpha)\alpha > (\lambda+\alpha)^2$, *the stationary probability distribution of (Q,J) is given by*

$$\begin{cases} \pi_{n1} = \pi_{11} r_1^{n-1}, \quad n \ge 1, \\ \pi_{n2} = \pi_{11} r_1^{n-2} r_2, \quad n \ge 1, \\ \pi_{n3} = \pi_{11}\left[\frac{r_3}{r_5-r_1}(r_5^{n-1} - r_1^{n-1})\right] + \pi_{13} r_5^{n-1}, \quad n \ge 1, \\ \pi_{n4} = \pi_{11}\left[r_4 r_1^{n-2} + \frac{r_3 r_6}{r_5-r_1}(r_5^{n-2} - r_1^{n-2})\right] + \pi_{13} r_5^{n-2} r_6, \quad n \ge 1, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \pi_{11} = a\pi_{01}, \\ \pi_{12} = ab\pi_{01}, \\ \pi_{13} = c\pi_{01}, \\ \pi_{14} = \frac{\lambda+2\eta+\theta-ab-\mu\theta}{\mu\alpha}\cdot\pi_{01}, \\ \pi_{02} = \frac{\lambda+\eta+\theta-ab}{\lambda}\cdot\pi_{01}, \\ \pi_{03} = \frac{\lambda+2\eta+\theta-ab}{\mu}\cdot\pi_{01}. \end{cases} \tag{3.2}$$

where

$$\begin{cases} a = \frac{\lambda(\lambda+\theta+\alpha)}{(\lambda+\alpha+\eta+\theta+r_2\alpha)(\lambda+\alpha+\theta)+\eta\lambda}, \\ b = \frac{\eta}{\lambda+\theta+\alpha}, \\ c = \frac{(\lambda+2\eta-ab-\mu\theta+\theta)(\lambda+\alpha)-ab\theta\mu\alpha}{\mu^2\alpha}. \end{cases} \quad (3.3)$$

Andπ_{01} can be determined by the normalization condition.

Proof: Using the matrix-geometric solution method, we have

$$\pi_n = (\pi_{n1}, \pi_{n2}, \pi_{n3}, \pi_{n4}) = \pi_1 R^{n-1} = (\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}) R^{n-1}, n \geq 1.$$

Since

$$R^{n-1} = \begin{pmatrix} r_1^{n-1} & r_1^{n-2} r_2 & \frac{r_3}{r_5-r_1}(r_5^{n-1} - r_1^{n-1}) & r_4 r_1^{n-2} + \frac{r_3 r_6}{r_5-r_1}(r_5^{n-2} - r_1^{n-2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5^{n-1} & r_5^{n-2} r_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}, n \geq 1,$$

substituting R^{n-1} into the above equation, we can get (3.1).

Moreover, π_0 satisfies the next equation

$$\pi_0(B_1 + RA_2) = 0. \quad (3.4)$$

Using Equation (3.4), we can obtain (3.2) by some caculations. Since

$$\sum_{j=1}^{4} \sum_{n=0}^{\infty} \pi_{nj} = 1,$$

we can get

$$\pi_{01} = (1 + x + y + z)^{-1},$$

where

$$x = \frac{\mu(\lambda+\eta+\theta-ab)+\lambda(\lambda+2\eta+\theta-ab)}{\lambda\mu}, \qquad y = \frac{a(1-r_5)+c(1-r_1)+ar_3}{(1-r_1)(1-r_5)},$$

$$z = \frac{a(r_2+r_4)(1-r_5)r_5+cr_6(1-r_1)r_1+(r_1+r_5-1)ar_3r_6}{(1-r_5)(1-r_1)r_1r_5}.$$

Clearly, the state probability of the server is given by

$$\begin{aligned} P_1 &= P\{J=1\} = \sum_{n=0}^{\infty} \pi_{n1} = \sum_{n=0}^{\infty} \pi_{11} r_1^n + \pi_{01} = \frac{1}{1-r_1}\pi_{11} + \pi_{01}, \\ P_2 &= P\{J=2\} = \sum_{n=0}^{\infty} \pi_{n2} = \sum_{n=0}^{\infty} \pi_{11} r_1^{n-2} r_2 + \pi_{02} = \frac{r_2}{(1-r_1)r_1^2} \cdot \pi_{11} + \pi_{02}, \\ P_3 &= P\{J=3\} = \sum_{n=0}^{\infty} \pi_{n3} = \sum_{n=0}^{\infty} \left\{\pi_{11}\left[\frac{r_3}{r_5-r_1}(r_5^{n-1} - r_1^{n-1})\right] + \pi_{13} r_5^{n-1}\right\} + \pi_{03} \\ &= \frac{r_3(r_1+r_5-1)}{(1-r_1)(1-r_5)r_1r_5} \cdot \pi_{11} + \frac{1}{(1-r_5)r_5} \cdot \pi_{13} + \pi_{03}, \\ P_4 &= P\{J=4\} = \sum_{n=1}^{\infty} \pi_{n4} = \sum_{n=1}^{\infty} \pi_{11}\left[r_4 r_1^{n-2} + \frac{r_3 r_6}{r_5-r_1}(r_5^{n-2} - r_1^{n-2})\right] + \sum_{n=1}^{\infty} \pi_{13} r_5^{n-2} r_6 \\ &= \frac{r_4}{(1-r_1)r_1}\pi_{11} + \frac{r_3 r_6}{r_5-r_1}\left[\frac{1}{(1-r_5)r_5^2} - \frac{1}{(1-r_1)r_1^2}\right]\pi_{01} + \frac{r_6}{(1-r_5)r_5^2}\pi_{13}. \end{aligned}$$

The probability that the server is busy is

$$P_b = P\{J=1\} + P\{J=3\} = P_1 + P_3.$$

The probability that the server is free is

$$P_c = P\{J=2\} + P\{J=4\} = P_2 + P_4 = 1 - P_b.$$

Let L be the number of customers in the orbit, we can obtain

$$\begin{aligned} E[L] &= \sum_{n=1}^{\infty} n(\pi_{n1} + \pi_{n2} + \pi_{n3} + \pi_{n4}) + \pi_{01} + \pi_{02} + \pi_{03} \\ &= \left\{\frac{r_1+r_2+r_4}{(1-r_1)^2 r_1} + \frac{r_3}{r_5-r_1}\left[\frac{1}{(1-r_5)^2 r_5} - \frac{1}{(1-r_1)^2 r_1}\right]\right\} \cdot \pi_{11} \\ &+ \left\{\frac{r_5+r_6}{(1-r_5)^2 r_5} + \frac{r_3 r_6}{r_5-r_1}\left[\frac{1}{(1-r_5)^2 r_5} - \frac{1}{(1-r_1)^2 r_1}\right]\right\} \cdot \pi_{13} + \pi_{01} + \pi_{02} + \pi_{03}. \end{aligned}$$

Let $\tilde{L}$ be the number of customers in the system, we have

$$\begin{aligned} E[\tilde{L}] &= \sum_{n=1}^{\infty} n(\pi_{n2} + \pi_{n4}) + \sum_{n=0}^{\infty} (n+1)(\pi_{n1} + \pi_{n3}) \\ &= E[L] + P_1 + P_3 = E[L] + P_b. \end{aligned}$$

4 Conditional Stochastic Decomposition

Lemma 4.1 *If* $(\mu+\alpha)\alpha > (\lambda+\alpha)^2$, *let* Q_0 *be the conditional queue length of the retrial M/M/1 queue with collision in the orbit given that the server is busy, then* Q_0 *has a probability generating fuction*

$$G_{Q_0}(z) = \frac{1-r_5}{1-r_5 z}.$$

Proof: The proof of this lemma is similar to the proof of Lemma 1 in Li et al. [11], so we omit it here.

Probability generating function of Q_b can be derived as follows: Additional queue length Q_b has a distribution

$$P\{Q_b = n\} = \frac{[r_1^{n-1}(r_5-r_1)+r_3(r_5^{n-1}-r_1^{n-1})]\pi_{11}+r_5^{n-1}(r_5-r_1)\pi_{13}}{r_5-r_1}\cdot\frac{1}{P_b}$$

.

Likewise, the probability generating function of Q_b can be derived as follows:

$$G_{Q_b}(z) = \sum_{n=0}^{\infty} P\{Q_b = n\}\cdot z^n$$
$$= \frac{\pi_{n1}+\pi_{n3}}{P_b}\cdot z^n$$
$$= \frac{1-r_5}{1-r_5 z}\cdot\frac{[(1-r_5 z)^2 r_5 + r_3(1-r_5 z)(1+r_5 z+r_1 z)]\pi_{11}+(1-r_5 z)(1-r_1 z)r_1\pi_{13}}{(1-r_5 z)(1-r_1 z)(1-r_5)P_b}$$
$$= G_{Q_0}(z)G_{Q_c}(z)$$

If $(\mu+\alpha)\alpha > (\lambda+\alpha)^2$, the condition queue length in the orbit when the server is busy Q_b can be decomposed into the sum of two independent random variables: Q_b=Q_0+Q_c, where Q_0 is defined in Lemma 1, and follows a geometric distribution with parameter $1-r_5$, Q_c is an additional queue length, the probability distribution of Q_c can be derived directly from $G_{Q_c}(z)$.

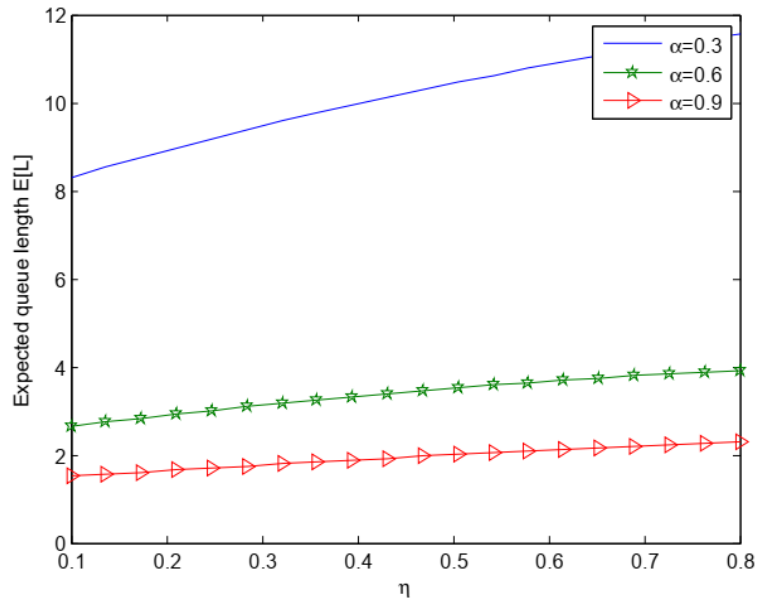

Figure 1: The expected queue length in the orbit with the change of eta.

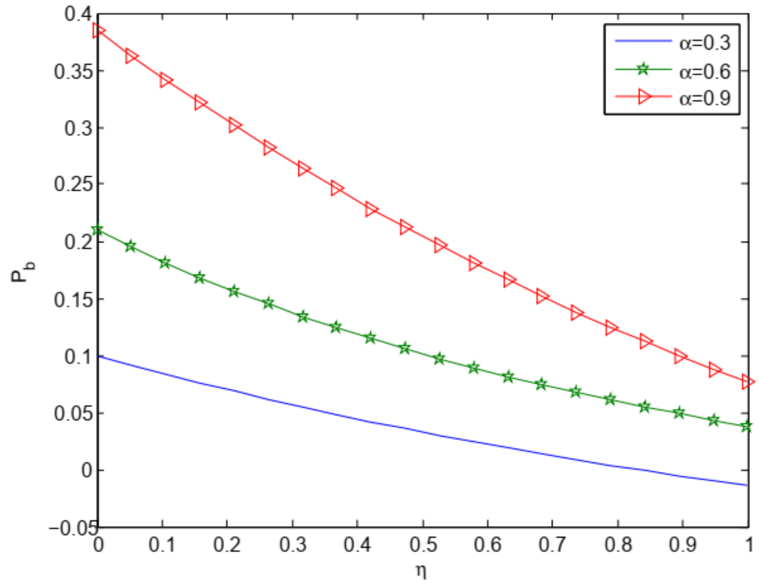

Figure 2: The probability that the server is busy with the change of eta.

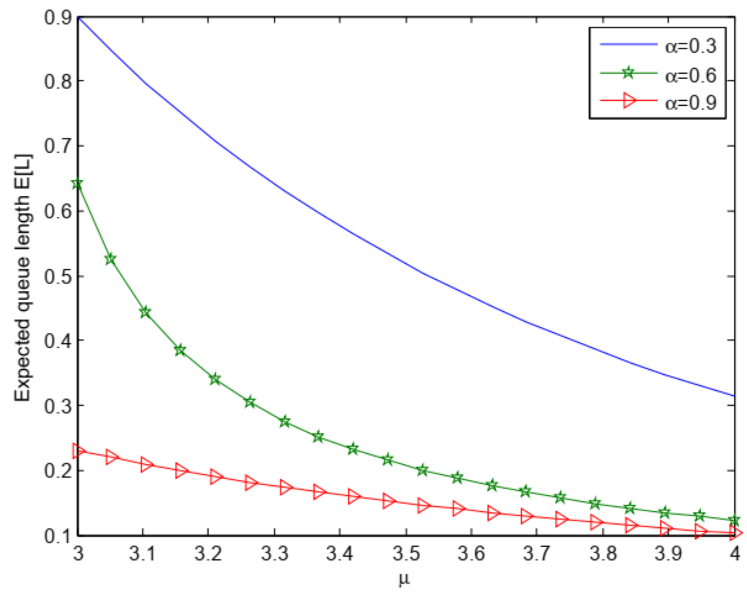

Figure 3: The expected queue length in the orbit with the change of mu.

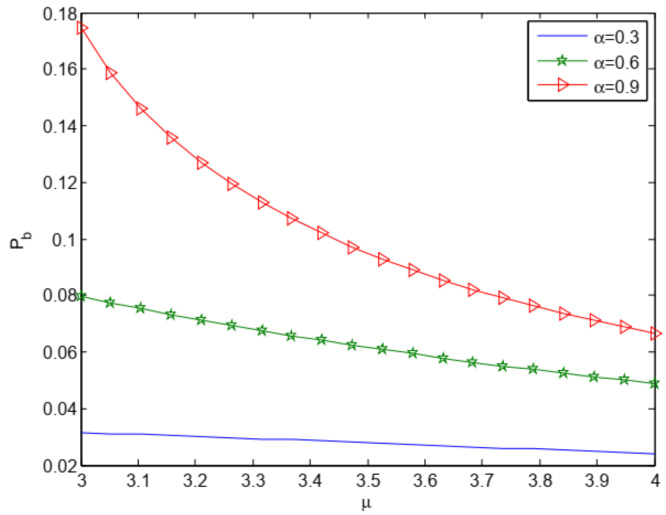

Figure 4: The probability that the server is busy with the change of mu.

5 Numerical Results

In this section, under the stationary condition, take $\lambda = 0.8$. $\theta = 0.4$, $\eta = 0.5$, $\mu = 1.4$, unless they are considered as variable or their values are mentioned in the respective figures, we present some numerical examples to illustrate the effect of some parameters on the expected queue length $E[L]$.

Fig.1 and Fig. 2 show the effect of working vacation service rate η on the expected queue length $E[L]$ and the probability that the server is busy P_b. We can see that $E[L]$ is increasing with the increasing value of η, but P_b is decreasing. As the value of η increases, the mean service speed in working vacation increases. So the quicker the service speed is, the smaller the probability that the server is busy is, which decreases P_b, and increases the value of $E[L]$. When $\alpha = 0.3$, we can see that the effect of α on $E[L]$ and P_b is not obvious, the reason is that the retrial time small. We can also find that $E[L]$ decreases as the values of α increase. While P_b has an opposite tendency.

In Fig.3 and Fig.4, with the change of normal service rate μ, the curves of the expected queue length $E[L]$ and the probability P_b are provided. We can find that the queue length $E[L]$ and the probability P_b both decrease quickly with an increasing value of μ. This is because that the server provides service at a high speed during the normal period. As a result, $E[L]$ and P_b both show a downward trend.

6 Conclusion

In this paper, we analyze an M/M/1 retrial queue with working vacation and collision. Using matrix-analytic method, we obtain the condition for the system to be stable, and the steady-state distributions and some performance measures are also derived. Finally, we present some numerical examples to study the effect of some parameters. For future research, using the supplementary variable method, one can consider the similar model but with general retrial times.

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