INDIRECT BOUNDARY ELEMENT METHOD FOR NONLINEAR PROBLEMS

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ABSTRACT

A new boundary element method is presented to solve general nonlinear problems. The analog equation method (AEM) is a new problem that transforms the original problem into a basic solution known, but the virtual source is unknown. In this paper, we use this transformation, combined with the virtual source approximation of boundary element discrete and radial basis functions (RBFs) to obtain the boundary equation system of non-coupling domain. The method in this paper uses a simple, basic solution to the known homogeneous isotropic problem to establish the integral equation. An additional field function was introduced, determined by a complementary domain boundary integral equation. The nonlinear problem is transformed into a boundary integral based on the global approximation of the radial basis function, and then its numerical solution is estimated from the known basic solution. The advantages of this paper are as follows: 1) The presented method preserves the pure boundary properties of BEM, and element dispersion and integration are limited to boundaries. 2) Convert the governing equations of complex problems into simple governing differential equations of the same order, such as the Poisson equation. This greatly reduces the complexity of the solving problem.

Key words: *boundary element, nonlinear problem, radial basis function, Poisson equation*.

1. INTRODUCTION

Nonlinear science has become a common scientific problem in many basic research and engineering application research [1-3]. Quantitative research on nonlinear problems relies on quantitative solutions to nonlinear differential equations. Unlike linear problems, the solution of nonlinear differential equations is generally very difficult, and only a few simple problems can find exact solutions. In the past century, although people have made great efforts in the two main solutions, analytical methods and numerical methods of nonlinear differential equations, however, there is still a lack of a universal method that can directly obtain various types of weak nonlinear problems and high-precision approximate solutions of strong nonlinear problems [4-5].

After years of development, the boundary element method has become a powerful numerical method for solving scientific and engineering problems. It is widely used in linear flow problems such as potential flow, elastic statics, dynamics, and acoustic propagation, and various non-uniform materials [6-10]. Compared with the domain method, the boundary element method has the following advantages: First, the boundary element method only needs discrete boundaries, so the high-dimensional problem can be reduced to a low-dimensional problem; secondly, the boundary element method can strictly satisfy the boundary at infinity. The condition is therefore very suitable for solving the infinite domain problem; in addition, the boundary element method can accurately determine the field solution gradient at any point in the domain. Because the calculation formula of the field solution gradient can be analytically derived from the basic boundary integral equation of the computational field solution, the field solution gradient and the field solution itself generally have a certain level of precision. Finally, the boundary element method can accurately and efficiently calculate the ultra-thin thin body. Problems with coating structure [11]. However, unlike the domain method, the boundary element method needs to estimate the singular kernel integral, and the validity of its calculation is the key to the successful implementation of the boundary element method. To this day, research in this area is still in progress, which is determined by the characteristics of the method itself [12-14].

This paper proposes a new boundary element method for solving nonlinear problems [15-16]. The method is based the AEM [17-19], under the premise that the basic solution is known and the boundary conditions are consistent, an equivalent non-homogeneous linear equation is used instead of the nonlinear governing equation. The solution of the replacement equation consists of a two-part solution of the homogeneous equation and the non-homogeneous equation. Among them, the non-homogeneous term is an unknown imaginary domain source distribution, which is approximated by a truncated radial basis function sequence. Then, using the boundary element method, the field functions involved in the governing equation and their derivatives are represented by unknown sequence coefficients and are established by the configuration equations at discrete points within the domain. Therefore, the method only requires boundary discretization, and additional configuration points within the domain do not destroy the pure boundary features of the method. Numerical examples of three nonlinear problems are given below to verify the validity and accuracy of the presented method.

2. INDIRECT BOUNDARY ELEMENT METHOD

2.1 Govern equation

For the sake of simplicity, without limiting its generality, the following second-order differential equations are used to describe nonlinear problems.

$$L\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial x\partial y}\right)=g(x,y)\qquad (x,y)\in\Omega \tag{1}$$

$$\beta_1 u+\beta_2 q=\beta_3\qquad (x,y)\in\Gamma \tag{2}$$

Where, the domain Ω is a two-dimensional plane, $u=u(x,y)$ is a field function; $q=u,_n$ is a normal vector on the boundary, and $\beta_i=\beta_i(x,y)$ $i=1,2$ is a known function about the boundary.

In general, solving problem (1-2) cannot be solved by the traditional method because it requires the establishment of a basic solution of the control operator, which usually does not work. The average source boundary node method mentioned in this paper is based on the simulation equation, which converts the original equation into a Poisson equation of the same boundary condition. The specific process is given in the next section.

2.2 Indirect boundary element method based on AEM

Let $u=u(\mathbf{x})$ be the solution of equation (1-2) and the second order continuous differentiable in the domain. Therefore, applying the Laplace will get

$$\nabla^2 u=b(\mathbf{x}) \tag{3}$$

Where, $b(\mathbf{x})=b(x,y)$ is an unknown function, and we assume $b=\sum_{j=1}^{M}\alpha_j f_j$, $f_j=\sqrt{r^2+c^2}$, $r=\left\|x-x_j\right\|$ $(j=1,2,...ME)$ here is a series of radial basis functions.

Under the same boundary conditions, the solution of equation (1) can be approximated by Poisson equation (3). The solution of the Poisson equation (4) into the sum of the general solution of the homogeneous equation and the non-homogeneous special solution,

$$u=u_h+u_p \tag{4}$$

Here, the homogeneous solution can be solved by the method of boundary element

$$u_h=\int_\Gamma \varphi u^* ds \tag{5}$$

$$q_h=\frac{\partial u_h}{\partial \mathbf{n}}=\int_\Gamma \varphi\frac{\partial u^*}{\partial \mathbf{n}}ds \tag{6}$$

Also $\nabla^2 u_p=\sum_{j=1}^{ME}a_j f_j$, the special solution can be expressed as

$$u_p=\sum_{j=1}^{ME}a_j\overline{u_j} \tag{7}$$

$$q_p=\frac{\partial u_p}{\partial \boldsymbol{n}}=\sum_{j=1}^{ME}\alpha_j\frac{\partial \overline{u_j}}{\partial \boldsymbol{n}} \tag{8}$$

Where, $\overline{u_j}$ can be obtained by the integral of the equation $\nabla^2\overline{u_j}=f_j\,(j=1,2,...ME)$

$$\overline{u_j}=-\frac{c^3}{3}\ln(\mathrm{c}\sqrt{r^2+c^2}+\mathrm{c}^2)+\frac{1}{9}(r^2+4c^2)\sqrt{r^2+c^2} \tag{9}$$

Therefore, the boundary condition (2) can be rewritten as

$$\beta_1 u_h+\beta_2 q_h=\beta_3-(\beta_1 u_p+\beta_2 q_p) \tag{10}$$

Substituting equations (5) and (7) into equation (4), equation (4) is rewritten as

$$u = u_h + u_p = \int_\Gamma \varphi u^* ds + \sum_{j=1}^{M} a_j \overline{u_j} \tag{11}$$

Solve the first and second derivatives for equation (9).

$$u_x = \int_\Gamma \varphi u_x^* ds + \sum_{j=1}^{ME} a_j (\overline{u_j})_x \tag{12}$$

$$u_y = \int_\Gamma \varphi u_y^* ds + \sum_{j=1}^{ME} a_j (\overline{u_j})_y \tag{13}$$

$$u_{xx} = \int_\Gamma \varphi u_{xx}^* ds + \sum_{j=1}^{ME} a_j (\overline{u_j})_{xx} \tag{14}$$

$$u_{yy} = \int_\Gamma \varphi u_{yy}^* ds + \sum_{j=1}^{ME} a_j (\overline{u_j})_{yy} \tag{15}$$

$$u_{xy} = \int_\Gamma \varphi u_{xy}^* ds + \sum_{j=1}^{ME} a_j (\overline{u_j})_{xy} \tag{16}$$

Using the boundary condition (10), φ can be expressed as α_j. Applying equation (1) at the collocation point in the domain will result in an equation.

$$L_i(u^i, u^i{}_x, u^i{}_y, u^i{}_{,xx}, u^i{}_{yy}, u^i{}_{xy}) = g^i \quad i = 1,2,...,ME \tag{17}$$

Apply equation (11) and equations (12)-(16) to approximate the value of u^i and its derivatives at the collocation point, so equation (17) can be replaced with

$$L_i(\alpha_j) = \mathrm{g}^i \qquad i, j = 1,2,...,ME \tag{18}$$

Therefore, the coefficient of $b(\mathbf{x}) = b(x, y)$ can be obtained. Since equation (19) represents a linear system of equations, the Gaussian elimination method can be used to solve the boundary value problem.

3. NUMERICAL EXAMPLES

In this paper, three numerical examples are chosen to verify the accuracy, stability and effectiveness of the presented method. In order to better show the calculation results, the average relative error (ARE) has the following definition

$$ARE = \sqrt{\sum_{k=1}^{M} (I_{num}^k - I_{exact}^k) / \sum_{k=1}^{M} \left| I_{exact}^k \right|}$$

Where, NE is used to indicate the number of boundary elements, ME is the number of points in the domain, M is the total number of points calculated, I_{num}^k and I_{exact}^k respectively represent the calculated numerical solution and exact solution at the first point.

Example 1: Consider the surface whose mean curvature is constant, and have the following governing equation

$$[1+(\frac{\partial u}{\partial x_2})^2]\frac{\partial^2 u}{\partial {x_1}^2} - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} + [1+(\frac{\partial u}{\partial x_1})^2]\frac{\partial^2 u}{\partial {x_1}^2}$$
$$-\kappa[1+(\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2]^{3/2} = 0 \qquad \mathbf{x} \in \Omega$$

Where, the domain Ω is a square domain with a side length of 5, $\kappa = -\sqrt{2}/5$, and the boundary condition is

$$u(x_1, 0) = (50 - {x_1}^2)^{1/2},\ u(x_1, 5) = (25 - {x_1}^2)^{1/2}$$
$$u(0, x_2) = (50 - {x_2}^2)^{1/2},\ u(5, x_2) = (25 - {x_2}^2)^{1/2}$$

The discriminant of the fore formula is

$$[1+(\frac{\partial u}{\partial x_1})^2][1+(\frac{\partial u}{\partial x_2})^2] - (\frac{\partial u}{\partial x_1})^2(\frac{\partial u}{\partial x_2})^2$$
$$= 1 + (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2$$
$$\geq 0$$

Therefore, the governing equation is elliptical.
The exact solution of the equation is

$$u(x_1, x_2) = [50 - (x_1^2 + x_2^2)]^{1/2}$$

In the calculation, the boundary is divided into 100 units, the number of points in the domain $M = 100$, and the shape parameters $c = 1$ are taken. The table2 gives the temperature values along the middle line $x_2 = 2.5$ and is compared with the exact solution, and the relative error is less than 10^{-2}. It can be concluded from Fig.2 that the presented method can effectively solve the two-dimensional nonlinear problem, and the presented method has good stability.

Table 1 Value of temperature along the middle line $x_2 = 0.5$ in a rectangular plate

	CX_1	CX_2	Exact value	Numerical results	Relative error
1	0.5	2.5	0.6595453E+01	0.6599356E+01	-0.5918494E-03
2	1.0	2.5	0.6538348E+01	0.6543136E+01	-0.7322570E-03
3	1.5	2.5	0.6442049E+01	0.6450185E+01	-0.1262927E-02
4	2.0	2.5	0.6304760E+01	0.6315163E+01	-0.1649953E-02
5	2.5	2.5	0.6123724E+01	0.6138945E+01	-0.2485533E-02
6	3.0	2.5	0.5894913E+01	0.5912750E+01	-0.3025812E-02
7	3.5	2.5	0.5612486E+01	0.5629521E+01	-0.3035215E-02
8	4.0	2.5	0.5267827E+01	0.5279598E+01	-0.2234583E-02
9	4.5	2.5	0.4847680E+01	0.4848934E+01	-0.2588007E-03

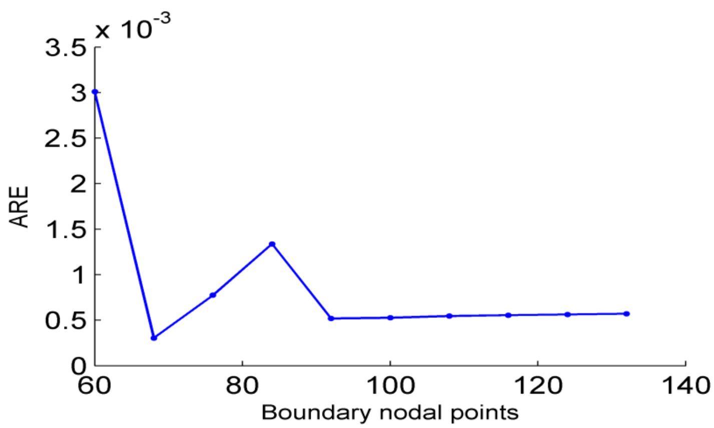


Fig.1 Average relative error varies with the number of boundary nodes

Example 2: On the rectangular plate $0 \le x_1, x_2 \le 1$, consider the following nonlinear heat transfer problem

$$(k\frac{\partial u}{\partial x_1})_{x_1} + (k\frac{\partial u}{\partial x_2})_{x_2} = 0$$

Boundary conditions are

$$u(0, x_2) = 300,\ u(1, x_2) = 400$$

$$u_n(x_1, 0) = 0,\ u_n(x_1, 1) = 0$$

In this example, we don't consider the factors of the heat source. And conductivity k is related to temperature

$$k = k_0[1 + \beta(u - u_0)/u_0]$$

Here, u_0 is the initial value, k_0 and β are constants. We substitute the fore formula into the original governing equation

$$k\nabla^2 u + \frac{k_0}{u_0}\beta(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}) = 0$$

In the calculation, we take the initial temperature value $u_0 = 300$ and the material constant $k_0 = 1$, $\beta = 3$ are taken separately, divide the boundary into 100 units, the number of points $M = 49$ in the domain, and take the shape parameters $c = 1$. Using the method in this paper, Table1 gives the temperature values along the middle line $x_2 = 0.5$ and is compared with the value of DRM and Kirchhoff in Ref [16]. It can be concluded from Table1 that the algorithm can effectively solve the two-dimensional nonlinear problem. It can be seen from Fig.1 that the solution value of the algorithm is in good agreement with the values of DRM and Kirchhoff .

Table 2 Value of temperature along the middle line $x_2 = 0.5$ *in a rectangular plate*

CX_1	CX_2	Kirchhoff	DRM	Present method
0.1000000E+00	0.5000000E+00	314.0	314.15	0.3140642E+03
0.3000000E+00	0.5000000E+00	337.82	338.34	0.3380341E+03
0.5000000E+00	0.5000000E+00	358.11	358.49	0.3583365E+03
0.7000000E+00	0.5000000E+00	376.08	376.27	0.3762143E+03
0.9000000E+00	0.5000000E+00	392.36	392.43	0.3923727E+03

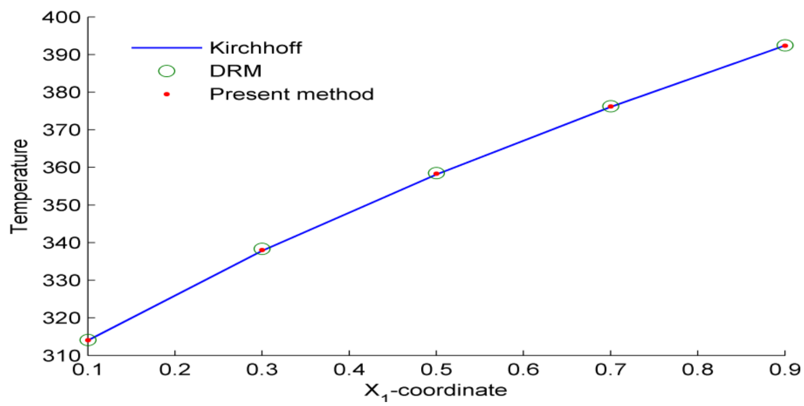


Figure 2 Temperature values along the middle line $x_2 = 0.5$

Example 3: In a multi-circle domain (Fig.3), solve the following two-dimensional anisotropic Helmholtz equation [20]

$$\frac{\partial^2 u}{\partial x_1^{\,2}} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^{\,2}} + \lambda^2 u = 0$$

Where, $\lambda = \frac{\sqrt{3}}{2}\mu$, boundary condition is $u = \sin(\frac{\sqrt{3}}{2}\mu x_1) + \cos[\mu(x_2 - \frac{x_1}{2})]$.

Its exact solution is

$$u = \sin(\frac{\sqrt{3}}{2}\mu x_1) + \cos[\mu(x_2 - \frac{x_1}{2})]$$

Calculate the anisotropy problem on this multi-circle domain, $NE = 48$, $ME = 96$, $C = 1$, $\mu = 1.9$. Table3 gives the numerical solution of the field function of the inner point and its relative error compared with the exact solution. Fig.4 (a) shows the average relative error of the field function as a function of the shape parameters. Fig. 4 (b) shows the average relative error of the field function under different internal configuration points. Fig.4 (a) (b)show the effectiveness of the presented method.

Table 3 Comparison of numerical solutions and exact solutions

	CX_1	CX_2	Exact value	Numerical results	Relative error
1	0.6	0.0	0.1676425E+01	0.1677254E+01	0.4944568E-03
2	0.8	0.2	0.1896470E+01	0.1896827E+01	0.1882381E-03
3	-0.7	0.2	-0.4116014E+00	-0.4134622E+00	0.4520805E-02
4	0.4	-0.8	0.2883879E+00	0.2861312E+00	0.7825312E-02

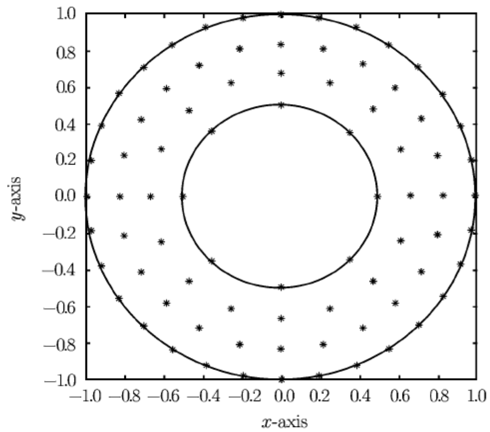
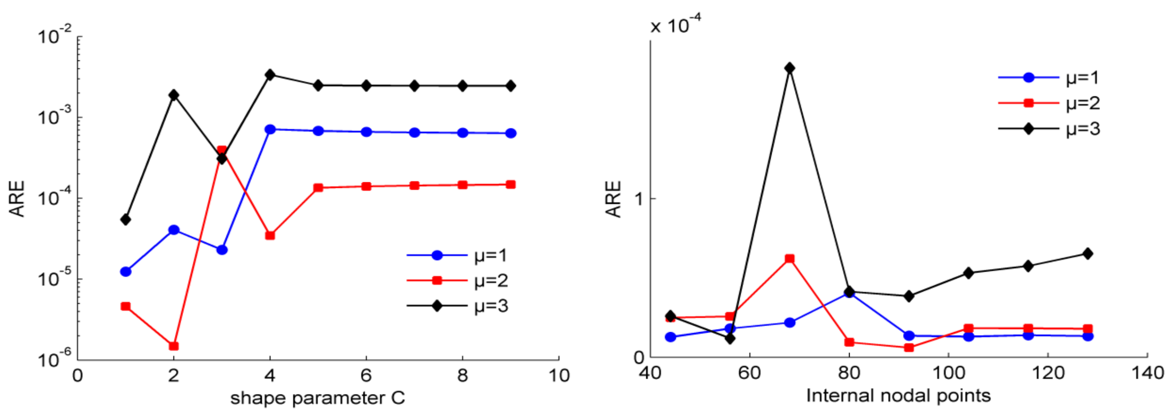


Fig.3 multi-circle domain



(a) *(b)*

Fig.4 Average relative error varies with (a) shape parameters c, (b) the number of collocation nodes.

4. Conclusion

For the general nonlinear problem, this paper proposes a new boundary element method. The proposed method preserves the pure boundary properties of the boundary elements, so that the discrete and integrated elements are only performed on the boundary. The simple basic solution applied in this paper depends only on the order of the differential equation, regardless of its specific differential operator. Using the method proposed in this paper, the governing equations of complex problems are transformed into simple governing differential equations of the same order, which reduces the complexity of the processing problem. In the numerical examples, the average relative error of the field function is analyzed when the number of points and shape parameters are configured internally.

The accuracy, stability and effectiveness of the proposed algorithm are shown. And the method in this paper is general, and the method is applicable to general nonlinear problems.

5. REFERENCES

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