JOINT CHANCE-CONSTRAINED PROGRAMMING WITH GENERALIZED EXPONENTIAL RANDOM PARAMETERS

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ABSTRACT
In this paper Joint Chance-constrained Programming (JCCP) models are considered when random parameters are independent and follow the generalized exponential distributions $GE\ (\lambda_i, \mu_i, \alpha_i)$. The main objective of this paper is to suggest an iterative approach to transform a probabilistic programming model into a sequence of equivalent deterministic linear programming models, which allow introducing several alternative policies for decision maker, then the linear models are solved using the Simplex method that is through two phases. Also, the suggested approach allows to process sensitivity analysis for the unknown probability of each single constraint included in the set of joint constraints. Finally, a numerical example introduced to illustrate the two phases and the alternative polices as well.

Keywords: *Probabilistic Programming, Chance-Constrained Programming, Joint Constrained Programming, Simplex method, Sensitivity analysis, Generalized Exponential distribution, alternative Polices.*

1. Introduction

Probabilistic Programming approach is a Mathematical Programming (MP) approach, it is used when some or all of the model parameters are random variables that are either the RHS parameters or some/ all LHS input coefficients, or some/ all objective function coefficients are random, taking into consideration the probability distributions of the random parameters in the underlying problem (Prékopa, 1995).

Chance constrained programming (CCP) technique is one of the probabilistic programming techniques. It is developed by Charnes and Cooper (1959, 1962, 1963) where the individual probabilistic constraints are assumed to be satisfied with probability at least γ_i, where γ_i is the tolerance measure of the i^{th} individual constraint. The main idea of CCP is to transform the probabilistic model into an equivalent deterministic model, then solving the later model using a suitable mathematical programming method (El-Dash, 2015). For theoretical background see (Prékopa, 2003).

Miller and Wagner (1965) were the first to introduce the idea of joint constraints where n probabilistic constraints should hold simultaneously with single probability γ. They assumed that the right hand side parameters are random and independently distributed.

CCP technique requires the knowledge of the probability distribution functions of the random variables and their inverses (El-Dash, 1984). In real life problems, probability distributions with non-negative range are applicable for a wide class of economic, demographic and reliability models where some of the model parameters have to be nonnegative, such as demand, supply, prices, life time,…etc (Sengupta, 1972 and Hafez, 2018).

Several non-negative distributions were used by many researchers in the context of CCP, for example; the Chi-square distribution (Sengupta, 1972; El-Dash, 1984), the Gamma distribution (Lingaraj and Wolfe, 1974; Atalay and Apaydin, 2011), the Weibull distribution (Jeeva et al., 2004; Ismail et al., 2018), and the Exponential distribution (Miller and Wagner 1965, Sengupta, 1972; El-Dash, 1984; Biswal, 1998, Hafez et al., 2018 (a,b)).

In this paper; The Generalized exponential distribution (GE) is our concern as one of the nonnegative distributions. Gupta and Kundu (1999) introduced the GE distribution as a three-parameter distribution (with scale, location and shape parameters) with probability density function $f(x;\ \lambda, \mu, \alpha)$ and distribution function $F(x;\ \lambda, \mu, \alpha)$ as follows:

$$f(x;\ \lambda, \mu, \alpha) = \frac{\alpha}{\lambda}\left(1 - e^{-\left(\frac{x-\mu}{\lambda}\right)}\right)^{\alpha-1} e^{-\left(\frac{x-\mu}{\lambda}\right)}\ ; x > \mu\ ;\ \lambda, \alpha > 0 \tag{1.1}$$

$$F(x;\ \lambda, \mu, \alpha) = \left(1 - e^{-\left(\frac{x-\mu}{\lambda}\right)}\right)^{\alpha}\ ; x > \mu\ ;\ \lambda, \alpha > 0 \tag{1.2}$$

where λ is the scale parameter, μ is the location parameter and α is the shape parameter.

There is a similarity of the density and distribution functions of the GE distribution and the Gamma and Weibull distributions where the three distributions are generalizations of the exponential distribution but in different ways, where if the shape parameter α equals 1 then the three distributions are reduced to the two-parameter exponential distribution.

The GE distribution has nice properties that overcame the drawbacks of both the gamma distribution and the Weibull distribution. The distribution function of the GE distribution is in closed form, unlike the Gamma distribution whose distribution function cannot be obtained in a closed form if the shape parameter is not an integer and could be obtained based on mathematical tables or computer software (Gupta and Kundu, 1999). As for the Weibull distribution, its distribution function could be obtained conveniently, but the distribution of the sum of independent and identically distributed Weibull random variables is not simple to obtain and approximation methods are need for using the CCP technique (Ismail et al., 2018).

In this paper; we propose a suggested iterative approach to obtain an equivalent deterministic model of joint chance constrained (JCC) model where RHS parameters are random and independently distributed with GE distributions. The suggested approach consists of two phases, where in phase (I) a JCCP model is transformed into a sequence of different individual CCP models with different individual tolerance measures, which are generated to provide several policies for decision maker, while in phase (II) the corresponding programming models for each alternative policy in phase (I) are solved, then a table is constructed to summarize optimal solution of alternative polices.

The suggested approach has two advantages; firstly, it keeps on the original size of the problem, while other approaches allows to increase the number of constraints as will be illustrated in section 4. Secondly, for real life JCCP problems with large size, Phase (I) allows to provide several alternative policies for decision makers easily. Thirdly, in Phase (II), the JCCP model is transformed into a sequence of linear programming models which are solved using the simplex method. fourthly, it keeps on the feasible area of the original problem unchanged unlike other approaches which require approximating the problem, which in turn changes the feasible area.

Consider the following linear JCCP model

$$Max.(Min.)Z = \sum_{j=1}^{n} c_j x_j \tag{1.3}$$

S.T.

$$P_r\left(\sum_{j=1}^{n} a_{ij}x_j \geq \tilde{b}_i \ ; i = 1,2,\dots,m_1\right) \geq \gamma_1 \tag{1.4}$$

$$P_r\left(\sum_{j=1}^{n} a_{ij}x_j \leq \tilde{b}_i \quad ; i = m_1+1, m_1+2,\dots,m_2\right) \geq \gamma_2 \tag{1.5}$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad ; i = m_2+1, m_2+2,\dots,m_3 \tag{1.6}$$

$$x_j \geq 0 \quad ; j = 1,2,\dots,n \tag{1.7}$$

Where $a_{ij}, c_j, b_i, j = 1,\dots,n; i = 1,2,\dots,m_3$ are constants, $\tilde{b}_i, i = 1,2,\dots,m_2$are random variables such that $\tilde{b}_i \sim GE(\lambda_i, \mu_i, \alpha_i)$, and γ_1, γ_2 are two tolerance measures

This paper is organized as follows; Section 2 presents how the JCCP model (1.3)- (1.7) is transformed into equivalent deterministic model. Section 3 introduces the Suggested approach for solving the equivalent deterministic model of JCCP models with independent GE distributed RHS parameters. Section 4 illustrates a Numerical example, and finally Section 5 is for summary and Conclusions.

2. Transformation of joint constraints

In this section, the joint chance constraints (1.4) and (1.5) are transformed into equivalent deterministic constraints, which in turn are transformed to linear constraints.

Case 1: the joint constraints in (1.4) can be written as

$$\left[\prod_{i=1}^{m_1} F_{\tilde{b}_i}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\right] \geq \gamma_1 \tag{2.1}$$

Where $F_{\tilde{b}_i}$ is the cumulative distribution function of a random variable $\tilde{b}_i, i = 1,\dots,m$. Since $\tilde{b}_i \sim GE(\lambda_i, \mu_i, \alpha_i)$, then (2.1) is equivalent to (El-Dash, 2018) :

$$\prod_{i=1}^{m_1}\left(1 - e^{-\frac{\sum_{j=1}^{m} a_{ij}x_j - \mu_i}{\lambda_i}}\right)^{\alpha_i} \geq \gamma_1 \rightarrow$$

$$\prod_{i=1}^{m_1} y_i \geq \gamma_1 \quad ; y_i \geq \left(1 - e^{-\frac{\sum_{j=1}^{n} a_{ij}x_j - \mu_i}{\lambda_i}}\right)^{\alpha_i} ;0 \leq y_i \leq 1 \rightarrow \tag{2.2}$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq \mu_i - \lambda_i \ln\left[1 - {y^*_i}^{\frac{1}{\alpha_i}}\right] \ i = 1,2,\dots,m \tag{2.3}$$

Where y_i^* are given feasible numerical values which satisfy constraints (2.2) and are determined through phase I in our suggested approach, as will be illustrated in next section. Constraints (2.3) are linear constraints in $x_j; j = 1, \dots, n$

Case II: Similarly; constraints (1.5) can be written as

$$\prod_{i=m_1+1}^{m_2}\left[1 - F_{\tilde{b}_i}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\right] \geq \gamma_2 \qquad (2.4)$$

$$\prod_{i=m_1+1}^{m_2}\left[1 - \left(1 - e^{-\frac{\sum_{j=1}^{m} a_{ij}x_j - \mu_i}{\lambda_i}}\right)^{\alpha_i}\right] \geq \gamma_2 \rightarrow$$

$$\prod_{i=m_1+1}^{m_2} h_i \geq \gamma_2 \quad ; h_i \geq 1 - \left(1 - e^{-\frac{\sum_{j=1}^{n} a_{ij}x_j - \mu_i}{\lambda_i}}\right)^{\alpha_i} ; 0 \leq h_i \leq 1 \qquad (2.5)$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu_i - \lambda_i \ln\left[1 - (1 - h^*{}_i)^{\frac{1}{\alpha_i}}\right];\ i = m_1 + 1, , \dots, m_2 \qquad (2.6)$$

Where h_i^* are given feasible numerical values which are determined through phase I in our suggested approach, as will be illustrated in next section. Again, Constraints (2.6) are linear constraints in $x_j; j = 1, \dots, n$.

3. Suggested approach

In this section we present a suggested iterative approach to solve the equivalent deterministic model of the JCCP model (1.3)-(1.7) based on the equivalent deterministic constraints (2.3) and (2.6). The suggested approach consists of two phases; phase (I) aims to obtain a sequence of feasible numerical values of y_i^* *and* h_i^* which satisfy constraints (2.3) and (2.6) ,phase (II) uses the feasible values of y_i^* *and* h_i^* obtained in phase (I) to solve sequence of linear equivalent deterministic models of the JCCP model.

3.1 Phase (I)

In this section, K number of the feasible values $y_i^*, i = 1, \dots, m_1$ and h_i^*, $i = m_1 + 1, \dots, m_2$ are suggested which satisfy the conditions (2.3)-(2.6) respectively, through K iterations.

From previous section, y_i^* *and* h_i^* are feasible numerical values which satisfy constraints (2.3) and (2.6), respectively. In the following, a simple algorithm is illustrated to compute y_i^*, h_i^*

Step 1: 1) Compute

$$\gamma_{01} = \sqrt[m_1]{\gamma_1} \qquad ,\gamma_{02} = \sqrt[m_2]{\gamma_2} \qquad (3.1)$$

Where $0 < \gamma_{01} < 1 \quad , \quad 0 < \gamma_{02} < 1$

Step 2: 1) let

$$\gamma_{01} \leq y_i^{*(t)} < \gamma_{01} + \epsilon_1^{(t)}, \qquad \epsilon_1 \rightarrow (1 - \gamma_{01}) \qquad (3.2)$$

$$\gamma_{02} \leq h_i^{*(t)} < \gamma_{02} + \epsilon_2^{(t)}, \qquad \epsilon_2 \rightarrow (1 - \gamma_{02}) \qquad (3.3)$$

Where t denotes the number of iteration, $t = 1,2, \dots, k$

2) Compute

$$\delta_1 = \frac{1 - \gamma_{01}}{K} \quad ,\epsilon_1^{(t)} = (t-1)\delta_1 \qquad (3.4)$$

$$\delta_2 = \frac{1 - \gamma_{02}}{K} \quad ,\epsilon_2^{(t)} = (t-1)\delta_2 \qquad (3.5)$$

Step 3: 1) put $t = 1$, $y^{*(1)} = \gamma_{01}$, $h^{*(1)} = \gamma_{02}$

2) put $t = t + 1$, then compute:

$$y_i^{*(t)} = \gamma_{01} + \epsilon_1^{(t)} \quad ,h_i^{*(t)} = \gamma_{02} + \epsilon_2^{(t)} \qquad (3.6)$$

3) if $t < K,\ put\ t = t + 1$, go to step 2 (number 2)

4) 2) if $t = K$, for all $i = 1, .., m_2$ stop.

Step 4: go to phase 2

3.2 Phase II

In this phase, the K sequence of LP models are solved using Simplex method through the following algorithm:

Step (1): 1) put $t = 1$

2) Construct the linear programming model (t)

$$Model\ (t): \qquad Max.(Min.)Z^{(t)} = \sum_{j=1}^{n} c_j x_j \tag{3.7}$$

S.T.

$$\sum_{j=1}^{n} a_{ij} x_j \le \mu_i - \lambda_i \ln\left[1 - (y_i^{*(t)})^{\frac{1}{\alpha_i}}\right] \quad ; \ i = 1,2,\dots,m \tag{3.8}$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge \mu_i - \lambda_i \ln\left[1 - \left(1 - h_i^{*(t)}\right)^{\frac{1}{\alpha_i}}\right]; \quad i = m_1 + 1,,\dots,m_2 \tag{3.9}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad ; i = m_2 + 1, m_2 + 2, \dots, m_3 \tag{3.10}$$

$$x_j \ge 0 \quad ; j = 1,2,\dots,n \tag{3.11}$$

Step (2): substitute with $y_i^{*(t)}, h_i^{*(t)}$ from phase (I) in (3.8) and (3.9), respectively.

Step (3): solve the above linear programming model by simplex method and find the optimal solution $Z^{*(t)}$, $x_j^{*(t)}$ $; j = 1,\dots,n$

Step (4):1) If $t < k$, put $t = t + 1$ go to step (1), number (2)

2) If $t = k$, stop and construct the following table

Table (3.1): optimal solution of alternative policies

Policy (t)	$y_i^{*(t)}$	$h_i^{*(t)}$	$x_1^{*(t)} \dots x_n^{*(t)}$	$Z^{*(t)}$
(1)	γ_{01}	γ_{02}	$x_1^{*(1)} \dots x_n^{*(1)}$	$Z^{*(1)}$
(2)	$\gamma_{01} + \epsilon_1^{(2)}$	$\gamma_{02} + \epsilon_2^{(2)}$	$x_1^{*(2)} \dots x_n^{*(2)}$	$Z^{*(2)}$
⋮	⋮	⋮	⋮	⋮
(k)	$\gamma_{01} + \epsilon_1^{(k)}$	$\gamma_{02} + \epsilon_2^{(k)}$	$x_1^{*(k)} \dots x_n^{*(k)}$	$Z^{*(k)}$

4. Comparison with Another approach

In this section, we compare our suggested approach with another approach some other researchers use in order to solve JCCP models, such as (An and Eheart, 2007; Sun et al., 2013). This approach is based on decomposing the JCC's into individual constraints with individual tolerance measures γ_{ik} and deal with γ_{ik}'s as decision variables , such that the product of $\gamma_{ik}'s$ satisfy the overall tolerance measure γ_k. Accordingly; model (1.3)- (1.7) becomes as follows:

$$Max.(Min.)Z = \sum_{j=1}^{n} c_j x_j \tag{4.1}$$

$$\text{S.t. } P_r\left(\sum_{j=1}^{n} a_{ij} x_j \ge \tilde{b}_i\ \right) \ge \gamma_{i1}\ ; i = 1,2,\dots,m_1 \tag{4.2}$$

$$P_r\left(\sum_{j=1}^{n} a_{ij} x_j \le \tilde{b}_i\ \right) \ge \gamma_{i2} \quad ; i = m_1 + 1, m_1 + 2, \dots, m_2 \tag{4.3}$$

$$\prod_i \gamma_{ik} \ge \gamma_k \ \ ; k = 1,2 \tag{4.4}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad ; i = m_2 + 1, m_2 + 2, \dots, m_3 \tag{4.5}$$

$$\delta \le \gamma_{ik} \le 1 - \delta \ \ ; \ \delta = 0.01 \ \ ; \ \ i = 1,\dots,m_2\ ; k = 1,2 \tag{4.6}$$

$$x_j \ge 0 \quad ; j = 1,2,\dots,n \tag{4.7}$$

The CCP model (4.1) -(4.7) is a nonlinear programming model because of constraint (4.4), and the equivalent deterministic model is also nonlinear model which could be obtained similarly as in constraints (2.3) and (2.6); such that the $y_i^*\ and\ h_i^*$ are now decision variables and replaced by γ_{ik} $i = 1,\dots,m_2\ ; k = 1,2$.

$$Max.(Min.)Z = \sum_{j=1}^{n} c_j x_j \tag{4.8}$$

$$\text{S.t.} \quad \sum_{j=1}^{n} a_{ij}x_j + \lambda_i \ln\left[1 - \gamma_{i1}^{\frac{1}{\alpha_i}}\right] \le \mu_i \; ; i = 1,2,\dots,m_1 \tag{4.9}$$

$$\sum_{j=1}^{n} a_{ij}x_j + \lambda_i \ln\left[1 - (1-\gamma_{i2})^{\frac{1}{\alpha_i}}\right] \ge \mu_i \; ; i = m_1+1, m_1+2,\dots,m_2 \tag{4.10}$$

$$\prod_i \gamma_{ik} \ge \gamma_k \; ; k = 1,2 \tag{4.11}$$

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad ; i = m_2+1, m_2+2,\dots,m_3 \tag{4.12}$$

$$\delta \le \gamma_{ik} \le 1-\delta \;\; ; \; \delta = 0.01 \;\; ; \;\; i = 1,\dots,m_2 \; ; k = 1,2 \tag{4.13}$$

$$x_j \ge 0 \;\; ; j = 1,2,\dots,n \tag{4.14}$$

Since the equivalent deterministic model (4.8)-(4.14) is also a nonlinear model because of the nonlinear constraints (4.9)-(4.11), it could be solved using one of the nonlinear techniques (Mokhtar, 2006). However; for large size problems this nonlinear model is very difficult to be solved. Therefore, in this paper we suggest to approximate model (4.8)-(4.14) into a linear programming model using Taylor approximation at initial point γ_{ik}^* as in model (4.15)-(4.22), which could be solved using the simplex method, as illustrated in next section.

$$Max.(Min.)Z = \sum_{j=1}^{n} c_j x_j \tag{4.15}$$

$$\text{S.t.} \quad \sum_{j=1}^{n} a_{ij}x_j - \left(\frac{\lambda_i \gamma_{i1}^{*\frac{1-\alpha_i}{\alpha_i}}}{\alpha_i\left[1-\gamma_{i1}^{*\frac{1}{\alpha_i}}\right]}\right)\gamma_{i1} \le \mu_i - \lambda_i \ln\left[1-\gamma_{i1}^{*\frac{1}{\alpha_i}}\right] - \frac{\lambda_i \gamma_{i1}^{*\frac{1}{\alpha_i}}}{\alpha_i\left[1-\gamma_{i1}^{*\frac{1}{\alpha_i}}\right]} \quad ; i = 1,2,\dots,m_1 \tag{4.16}$$

$$\sum_{j=1}^{n} a_{ij}x_j + \left(\frac{\lambda_i (1-\gamma_{i2}^*)^{\frac{1-\alpha_i}{\alpha_i}}}{\alpha_i\left[1-(1-\gamma_{i2}^*)^{\frac{1}{\alpha_i}}\right]}\right)\gamma_{i2} \ge \mu_i - \lambda_i \ln\left[1-(1-\gamma_{i2}^*)^{\frac{1}{\alpha_i}}\right] + \frac{\lambda_i (1-\gamma_{i2}^*)^{\frac{1-\alpha_i}{\alpha_i}}\gamma_{i2}^*}{\alpha_i\left[1-(1-\gamma_{i2}^*)^{\frac{1}{\alpha_i}}\right]} \; ; i = m_1+1, m_1+2,\dots,m_2 \tag{4.17}$$

$$\sum_{i=1}^{m_1}\left(\gamma_{i1}\prod_{\substack{q=1\\q\ne i}}^{m_1}\gamma_{q1}^*\right) \ge \gamma_1 - (m_1-1)\prod_{i=1}^{m_1}\gamma_{i1}^* \tag{4.18}$$

$$\sum_{i=m_1+1}^{m_2}\left(\gamma_{i2}\prod_{\substack{q=m_1+1\\q\ne i}}^{m_2}\gamma_{q2}^*\right) \ge \gamma_2 - (m_2-m_1-1)\prod_{i=m_1+1}^{m_2}\gamma_{i2}^* \tag{4.19}$$

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad ; i = m_2+1, m_2+2,\dots,m_3 \tag{4.20}$$

$$\delta \le \gamma_{ik} \le 1-\delta \;\; ; \; \delta = 0.01 \;\; ; \;\; i = 1,\dots,m_2 \; ; k = 1,2 \tag{4.21}$$

$$x_j \ge 0 \;\; ; j = 1,2,\dots,n \tag{4.22}$$

However; we have to bear in mind that (i) the size of the problem increases by k constraints unlike our suggested approach, (ii) approximating the nonlinear equivalent deterministic model to a linear one leads to changes in the feasible area of the original problem.

5. Numerical example

In this section, we demonstrate how the suggested approach is applied through phase I and II. Consider the following JCCP model:

$$Max.\ Z = 5x_1 + 2x_2 \tag{5.1}$$

S.T.

$$P_r\begin{pmatrix} 2x_1 + 3x_2 \le \tilde{b}_1 \\ 3x_1 - x_2 \ge \tilde{b}_2 \\ x_1 + 2x_2 \le \tilde{b}_3 \end{pmatrix} \ge 0.90 \tag{5.2}$$

$$10\,x_1 + 7x_2 \le 70 \tag{5.3}$$

$$x_1 + 2x_2 \le 10 \tag{5.4}$$

$$x_1, x_2 \ge 0 \tag{5.5}$$

Where $\tilde{b}_1 \sim GE(\lambda_1 = 1, \mu_1 = 6, \alpha_1 = 1.5)$, $\tilde{b}_2 \sim GE(\lambda_2 = 1.5, \mu_2 = 5, \alpha_2 = 1)$ and $\tilde{b}_3 \sim GE(\lambda_3 = 2\,, \mu_3 = 3, \alpha_3 = 2)$

Phase I:

Assume that $K = 3\,, and\ t = 1,2,3$

Step 1: $\gamma_{01} = \sqrt[3]{0.90} \cong 0.97$, let

Step 2: $0.97 \le y_i^{(t)} \le 0.97 + \epsilon^{(t)}$; $\epsilon^t \to (0.03)$

$$\delta_1 = \frac{1 - 0.97}{3} = 0.01 \quad , \qquad \epsilon^{(t)} = 0.01\,(t-1)$$

Step 3: 1) put $t = 1 \to \epsilon^{(1)} = 0 \to y_i^{(1)} = 0.97 \quad , i = 1,2,3$

2) put $t = t + 1 = 2 \to \epsilon^{(2)} = (2-1)(0.01) = 0.01 \to y_i^{(2)} = 0.98 \quad , t < k$

3) Put $t = t + 1 = 3 \to \epsilon^{(3)} = (3-1)(0.01) = 0.02 \to \ y_i^{(3)} = 0.99$

4) $t = 3 = k,\ \ stop$

Phase II:

1) Given $y_i^{*(1)} = 0.97 \to$ the deterministic model

$$\begin{aligned} &Max.Z^{(1)} = 5x_1 + 2x_2 \\ &\text{S.t. } 2x_1 + 3x_2 \le 9.91 \\ &3x_1 - x_2 \ge 5.05 \\ &x_1 + 2x_2 \le 11.38 \\ &10x_1 + 7x_2 \le 70 \\ &x_1 + 2x_2 \le 10 \\ &x_1, x_2 \ge 0 \end{aligned} \tag{5.6}$$

By Simplex method the optimal solution of model (5.6) is:

$$Z^{*(1)} = 24.77 \quad , x_1^{*(1)} = 4.95\,, x_2^{*(1)} = 0 \tag{5.7}$$

Similarly; when $y_i^{*(2)} = 0.98$ and $y_i^{*(3)} = 0.99$ we obtain the optimal solution of the model as summarized in table (5.1)

Table (5.1): optimal solution of the three alterative policies:

Policy (t)	$y_i^{(t)}$	$x_1^{(t)}$	$x_2^{(t)}$	$Z^{(t)}$
(1)	0.97	4.95	0	24.77
(2)	0.98	5.16	0	25.79
→ (3)	0.99	5.50	0	27.52

If we use the other approach of section (4), then the equivalent deterministic nonlinear model is:

$$Max.\ Z = 5x_1 + 2x_2 \tag{5.8}$$

$$\text{S.t. } 2x_1 + 3x_2 + \ln\left[1 - \gamma_1^{\frac{1}{1.5}}\right] \le 6 \tag{5.9}$$

$$3x_1 - x_2 + 1.5\ \ln[1 - (1 - \gamma_2)] \ge 5 \tag{5.10}$$

$$x_1 + 2x_2 + 2\ln\left[1 - \gamma_1^{\frac{1}{2}}\right] \le 3 \tag{5.11}$$

$$\prod_i^3 \gamma_i \ge 0.9 \tag{5.12}$$

$$10x_1 + 7x_2 \le 70 \tag{5.13}$$

$$x_1 + 2x_2 \le 10 \tag{5.14}$$

$$\delta \le \gamma_i \le 1 - \delta \ \ ; \ \delta = 0.01 \ \ ; \ \ i = 1,2,3 \tag{5.15}$$

$$x_1, x_2 \ge 0 \tag{5.16}$$

If we approximate constraints (5.9)-(5.12) to linear constraints given the initial point $(\gamma_1 = 0.97, \gamma_2 = 0.97, \gamma_3 = 0.97)$, then the linear model is:

$$
\begin{aligned}
& Max. Z = 5x_1 + 2x_2 \\
\text{S.t.} \quad & 2x_1 + 3x_2 - 33.504\,\gamma_1 \leq -22.592 \\
& 3x_1 - x_2 + 1.546\,\gamma_2 \geq 6.546 \\
& x_1 + 2x_2 - 67.178\,\gamma_3 \leq -53.779 \\
& 0.941\gamma_1 + 0.941\gamma_2 + 0.941\gamma_3 \geq -0.925 \\
& \delta \leq \gamma_i \leq 1 - \delta \;\; ; \; \delta = 0.01 \;\; ; \; i = 1,2,3 \\
& 10x_1 + 7x_2 \leq 70 \\
& x_1 + 2x_2 \leq 10 \\
& x_1, x_2 \geq 0
\end{aligned}
\tag{5.17}
$$

And the optimal solution is as follows for initial values $\gamma_i^{(0)} = 0.97$; $i = 1,2,3$:
$Z^* = 26.44 \quad , x_1^* = 5.29 , x_2^* = 0$; $\gamma_i = 0.99$; $i = 1,2,3$
It could be noticed that for $\gamma_i = 0.99$; $i = 1,2,3$, the optimal objective value of model (5.17) is 26.44 which is less than the corresponding optimal objective value (policy 3) obtained by applying the suggested approach of section (3). That is, the suggested approach of section 4 is better than applying linear approximation of the model, because the resultant feasible area is changed due to applying the linear approximation.

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