AN EXTENSION OF THE TAYLOR SERIES EXPANSION FOR ANALYTIC MATRIX FUNCTION

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ABSTRACT
In this paper, we define and study Taylor series representations for analytic functions of two complex matrices. Also, we discuss Cauchy's inequality for the analytic matrix functions in complete Reinhardt domains, spherical regions and hyper elliptical regions. Moreover, convergence of Taylor series for functions of two complex matrices are investigated.

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1. INTRODUCTION
The subjects of series representation for functions of single complex variable and several complex variables have a great position as due to their relationship to all aspects of mathematics, physics and engineering. Taylor series from important series which can be used in solving many mathematical problems.

Taylors theorem is named after the mathematician Brook Taylor, who stated a version of it in 1712. Yet an explicit expression of the error was not provided until much later on by Joseph-Louis Lagrange. An earlier version of the result was already mentioned in 1671 by James Gregory. Taylors theorem is taught in introductory level calculus courses and it is one of the central elementary tools in mathematical analysis. Since that period, many results concerning the Taylor's series of functions in one or several variables were introduced (see e.g.[5, 6, 7, 9, 10]).

In this work, we introduce Taylor series representation for analytic functions of two complex matrices. Besides we discuss Cauchy's inequality for the analytic matrix functions in complete Reinhardt domains, spherical regions and hyper elliptical regions. Finally, a direct method for evaluating the radius of regularity of a matrix power series is stated.

2. SOME BASIC NOTATIONS AND RESULTS
Throughout this paper, we consider the complex space $\mathbb{C}^{N\times N}$ of complex matrices of common order N. The matrices I and 0 stand for the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. A matrix X is a positive stable matrix in $\mathbb{C}^{N\times N}$ if $Re(\lambda) > 0$ for all $\lambda \in \sigma(X)$, where $\sigma(X)$ is the set of all eigenvalues of X.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set $\Omega \subset \mathbb{C}$ and X is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(X) \subset \Omega$, then (see [3, 4])

$$f(X)g(X) = g(X)f(X).$$

Hence, if Y in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(Y) \subset \Omega$ and if $XY = YX$, then

$$f(X)g(Y) = g(Y)f(X).$$

Definition 2.1. Let $f(X)$ be a complex matrix function, then its derivative is defined by (see [8])

$$\frac{d}{dX} f(X) = \lim_{h\to 0} \frac{f(X + IX) - f(X)}{h}, \qquad (2.1)$$

where X is a square matrix over the complex field and h is a scalar complex variable.

The matrix function corresponding to the polynomial $p(z) = a_0 + a_1 z + \cdots + a_k z^k$ is

$$p(z) = a_0 I + \cdots + a_k A^k. \qquad (2.2)$$

Suppose now that the Taylor series of the scalar function f is convergent for expansion point υ:

$$f(z) = \sum_{s=0}^{\infty} \frac{f^{(s)}(\upsilon)}{s!} (z - \upsilon)^s. \qquad (2.3)$$

If we generalize polynomials using (2, 2), the matrix generalization of $(z - \upsilon)^s$ is $(A - \upsilon I)^s$. Therefore, in complete analogy with the polynomial matrix function (2, 2) we can define the matrix function via a Taylor series. The following definition is the matrix generalization of (2, 2) and (2, 3).

Definition 2.2. [4] The Taylor definition with expansion point $\upsilon \in \mathbb{C}$ of the matrix function associated with $f(z)$ is given by

$$f(A) = \sum_{s=0}^{\infty} \frac{f^{(s)}(\upsilon)}{s!} (A - \upsilon I)^s. \qquad (2.4)$$

This definition is valid for functions with convergent matrix-valued Taylor series, which turns out to be the case if the function is analytic in sufficiently large domain. This is illustrated in the following theorem. In particular, a consequence of the following result is that (2. 4) is well-defined if the scalar-valued function is analytic in the complex plane.

Theorem 2.1. (convergence of Taylor definition [4]). Suppose $f(z)$ is analytic in $\bar{D}(\upsilon, r)$ and suppose $r > \|A - I\upsilon\|$. Then, with definition (2, 4) of $f(A)$ and $\gamma \coloneqq \frac{\|A - I\upsilon\|}{r} < 1$, there exists a constant Kindependent of N such that

$$\left\| f(A) - \sum_{s=0}^{N} \frac{f^{(s)}(\upsilon)}{s!} (A - I\upsilon)^s \right\| < K\gamma^N \to 0 \; as \; N \to \infty$$

3. TAYLOR SERIES FOR FUNCTIONS OF TWO COMPLEX MATRICE

In this section, we establish Taylor series representation for the analytic functions of two complex matrices by the following theorem:

Theorem 3.1. Let $f(X_1, X_2)$ be analytic function of the two complex matrices $X_1 = [f_{ij}(z_1)]$ and $X_2 = [f_{ij}(z_2)]$ in the domain $D = D_1 \times D_2$ then

$$f(X_1, X_2) = \sum_{n_1, n_2 = 0}^{\infty} A_{n_1, n_2} (z_1 - a_1)^{n_1} (z_2 - a_2)^{n_2} \qquad (3.1)$$

Where A_{n_1, n_2} is matrix coefficients given by

$$A_{n_1, n_2} = \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{f(Y_1, Y_2)}{(w_1 - a_1)^{n_1+1} (w_2 - a_2)^{n_2+1}} dw_1 dw_2$$

$$= \frac{d^{n_1+n_2}}{dw_1^{n_1} dw_2^{n_2}} \frac{f(Y_1, Y_2)}{n_1! \, n_2!} \bigg|_{w_1 = a_1, w_2 = a_2} ; \; n_1, n_2 = 0, 1, \ldots,$$

Γ_1 and Γ_2 are a simple closed contour containing a_1, a_2, z_1 and z_2 lie in the domain D.

Proof. From the Cauchy's integral formula for functions of several complex matrices in [1, 2], we have

$$f(X_1, X_2) = \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{f(Y_1, Y_2)}{(w_1 - z_1)(w_2 - z_2)} dw_1 dw_2$$

$$= \frac{1}{2\pi i} \oint_{\Gamma_1} \left\{ \frac{1}{2\pi i} \oint_{\Gamma_2} \frac{f(Y_1, Y_2)}{(w_1 - z_1)(w_2 - z_2)} dw_1 \right\} dw_2$$

Where $X_1 = [f_{ij}(z_1)]$, $X_2 = [f_{ij}(z_2)]$, $Y_1 = [f_{ij}(w_1)]$ and $Y_2[f_{ij}(w_2)]$.

Now, we can written

$$\frac{1}{w_1 - z_1} = \frac{1}{(w_1 - a_1) - (z_1 - a_1)} = \frac{1}{(w_1 - a_1)\left[1 - \frac{z_1 - a_1}{w_1 - a_1}\right]},$$

$$\frac{1}{w_1 - z_1} = \frac{1}{w_1 - a_1}\left[1 + \frac{z_1 - a_1}{w_1 - a_1} + \cdots + \left(\frac{z_1 - a_1}{w_1 - a_1}\right)^{n_1 - 1} + \frac{\left(\frac{z_1 - a_1}{w_1 - a_1}\right)^{n_1}}{1 - \frac{z_1 - a_1}{w_1 - a_1}}\right]$$

$$= \sum_{r_1=0}^{n_1-1} \frac{(z_1 - a_1)^{r_1}}{(w_1 - a_1)^{r_1+1}} + \frac{\left(\frac{z_1 - a_1}{w_1 - a_1}\right)^{n_1}}{w_1 - z_1}.$$

Similary, we find that

$$\frac{1}{w_2 - z_2} = \frac{1}{(w_2 - a_2) - (z_2 - a_2)} = \frac{1}{(w_2 - a_2)\left[1 - \frac{z_2 - a_2}{w_2 - a_2}\right]},$$

$$\frac{1}{w_2 - z_2} = \frac{1}{w_2 - a_2}\left[1 + \frac{z_2 - a_2}{w_2 - a_2} + \cdots + \left(\frac{z_2 - a_2}{w_2 - a_2}\right)^{n_2 - 1} + \frac{\left(\frac{z_2 - a_2}{w_2 - a_2}\right)^{n_2}}{1 - \frac{z_2 - a_2}{w_2 - a_2}}\right]$$

$$= \sum_{r_2=0}^{n_2-1} \frac{(z_2 - a_2)^{r_2}}{(w_2 - a_2)^{r_2+1}} + \frac{\left(\frac{z_2 - a_2}{w_2 - a_2}\right)^{n_2}}{w_2 - z_2}.$$

Therefor, we have

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \left[\sum_{r_1=0}^{n_1-1} \frac{(z_1 - a_1)^{r_1}}{(w_1 - a_1)^{r_1+1}} + \frac{\left(\frac{z_1 - a_1}{w_1 - a_1}\right)^{n_1}}{w_1 - z_1}\right]\left[\sum_{r_2=0}^{n_2-1} \frac{(z_2 - a_2)^{r_2}}{(w_2 - a_2)^{r_2+1}} + \frac{\left(\frac{z_2 - a_2}{w_2 - a_2}\right)^{n_2}}{w_2 - z_2}\right]$$

$$= \sum_{r_1=0}^{n_1-1} \sum_{r_2=0}^{n_2-1} \frac{(z_1 - a_1)^{r_1}}{(w_1 - a_1)^{r_1+1}} \cdot \frac{(z_2 - a_2)^{r_2}}{(w_2 - a_2)^{r_2+1}}$$

$$+ \sum_{r_1=0}^{n_1-1} \frac{(z_1 - a_1)^{r_1}}{(w_1 - a_1)^{r_1+1}} \cdot \frac{\left(\frac{z_2 - a_2}{w_2 - a_2}\right)^{n_2}}{w_2 - z_2}$$

$$+ \sum_{r_2=0}^{n_2-1} \frac{(z_2 - a_2)^{r_2}}{(w_2 - a_2)^{r_2+1}} \cdot \frac{\left(\frac{z_1 - a_1}{w_1 - a_1}\right)^{n_1}}{w_1 - z_1}$$

$$+\frac{\left(\frac{z_1-a_1}{w_1-a_1}\right)^{n_1}}{w_1-z_1}\cdot\frac{\left(\frac{z_2-a_2}{w_2-a_2}\right)^{n_2}}{w_2-z_2}$$

Thus,

$$f(X_1,X_2)=\sum_{r_1=0}^{n_1-1}\sum_{r_2=0}^{n_2-1}\frac{f^{r_1,r_2}(A_1,A_2)}{r_1!\,r_2!}(z_1-a_1)^{r_1}(z_2-a_2)^{r_2}$$

$$+\sum_{r_1=0}^{n_1-1}\frac{(z_1-a_1)}{r_1!}\cdot\frac{1}{2\pi i}\oint_{\Gamma_1}\frac{\left(\frac{z_2-a_2}{w_2-a_2}\right)^{n_2}}{w_2-z_2}f^{r_1,0}(A_1,Y_2)dw_2$$

$$+\sum_{r_2=0}^{n_2-1}\frac{(z_2-a_2)}{r_2!}\cdot\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{\left(\frac{z_1-a_1}{w_1-a_1}\right)^{n_1}}{w_1-z_1}f^{0,r_2}(Y_1,A_1)dw_1$$

$$+\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{\left(\frac{z_1-a_1}{w_1-a_1}\right)^{n_1}}{w_1-z_1}\cdot\frac{\left(\frac{z_2-a_2}{w_2-a_2}\right)^{n_2}}{w_2-z_2}f(Y_1,Y_2)dw_1dw_2$$

$$f(X_1,X_2)=\sum_{r_1=0}^{n_1-1}\sum_{r_2=0}^{n_2-1}A_{r_1,r_2}(z_1-a_1)^{r_1}(z_2-a_2)^{r_2}+\alpha_{n_1,n_2}+\beta_{n_1,n_2}+\gamma_{n_1,n_2} \qquad (3.2)$$

$$A_{r_1,r_2}=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{f(X_1,X_2)}{(w_1-a_1)^{r_1+1}(w_2-a_2)^{r_2+1}}dw_1dw_2$$

$$=\frac{d^{r_1+r_2}}{dw_1^{r_1}dw_2^{r_2}}\frac{f(Y_1,Y_2)}{r_1!\,r_2!}\bigg|_{w_1=a_1,w_2=a_2};\ r_1,r_2=0,1,2,\ldots,$$

$$\alpha_{n_1,n_2}=\sum_{r_1=0}^{n_1-1}\frac{(z_1-a_1)}{r_1!}\cdot\frac{1}{2\pi i}\oint_{\Gamma_1}\frac{\left(\frac{z_2-a_2}{w_2-a_2}\right)^{n_2}}{w_2-z_2}f^{r_1,0}(A_1,Y_2)dw_2$$

$$\beta_{n_1,n_2}=\sum_{r_2=0}^{n_2-1}\frac{(z_2-a_2)}{r_2!}\cdot\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{\left(\frac{z_1-a_1}{w_1-a_1}\right)^{n_1}}{w_1-z_1}f^{0,r_2}(Y_1,A_1)dw_1$$

$$\gamma_{n_1,n_2}=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{\left(\frac{z_1-a_1}{w_1-a_1}\right)^{n_1}}{w_1-z_1}\cdot\frac{\left(\frac{z_2-a_2}{w_2-a_2}\right)^{n_2}}{w_2-z_2}f(Y_1,Y_2)dw_1dw_2.$$

As n_1 and n_2 tends infinity then $|\alpha_{n_1,n_2}|$, $|\beta_{n_1,n_2}|$ and $|\gamma_{n_1,n_2}|$ tends to zero

Taking limit as $n_1,n_2\to\infty$ in (3. 2) we get

$$f(X_1,X_2)=\sum_{n_1,n_2}^{\infty}A_{n_1,n_2}(z_1-a_1)^{n_1}(z_2-a_2)^{n_2} \qquad (3.3)$$

Where

$$A_{n_1,n_2} = \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{f(X_1,X_2)}{(w_1-a_1)^{n_1+1}(w_2-a_2)^{n_2+1}} dw_1 dw_2$$

$$= \frac{d^{n_1+n_2}}{dw_1^{n_1} dw_2^{n_2}} \frac{f(Y_1,Y_2)}{n_1!\, n_2!}\bigg|_{w_1=a_1, w_2=a_2} ;\ n_1,n_2 = 0,1,2,\ldots,$$

Remark 3.1. This is called the Taylor series for the analytic functions of two complex matrices and at $a_1 = 0, a_2 = 0$ it's also called the Maclaurin's series.

4. CAUCHY'S INEQUALITY

In this section, we discuss Cauchy's inequality for the analytic matric function $f(X_1, X_2)$ in complet Reinhardt domains, spherical regions and hyperelliptical regions. Suppose that

$$f(X_1,X_2) = \sum_{n_1,n_2}^{\infty} A_{n_1,n_2}(z_1-a_1)^{n_1}(z_2-a_2)^{n_2},$$

Where

$$A_{n_1,n_2} = \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{f(X_1,X_2)}{(w_1-a_1)^{n_1+1}(w_2-a_2)^{n_2+1}} dw_1 dw_2$$

$$= \frac{1}{n_1!\, n_2!}\left\{\frac{\partial^{n_1+n_2}}{\partial w_1^{n_1+1}\partial w_2^{n_2+1}} f(Y_1,Y_2)\right\}_{(w_1,w_2)=(a_1,a_2)} ;\ n_1,n_2 = 0,1,2,\ldots,$$

Where $\Gamma_1: |w_1 - a_1| = R_1$, $\Gamma_2: |w_2 - a_2| = R_2$. At $a_1 = a_2 = 0$, we have

$$f(X_1,X_2) = \sum_{n_1,n_2=0}^{\infty} A_{n_1,n_2} z_1^{n_1} z_2^{n_2}, \tag{4.1}$$

$$A_{n_1,n_2} = \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{f(Y_1,Y_2)}{w_1^{n_1+1} w_2^{n_2+1}} dw_1 dw_2$$

$$= \frac{1}{n_1!\, n_2!}\left\{\frac{\partial^{n_1+n_2}}{\partial w_1^{n_1}\partial w_2^{n_2}} f(Y_1,Y_2)\right\}_{(w_1,w_2)=(a_1,a_2)} ;\ n_1,n_2 = 0,1,2,\ldots,$$

$$\left[A_{n_1,n_2,ij}\right] = \left|A_{n_1,n_2}\right| = \left(\frac{1}{2\pi i}\right)^2 \left[\oint_{\Gamma_1} \oint_{\Gamma_2} \frac{f(Y_1,Y_2)}{w_1^{n_1+1} w_2^{n_2+1}} dw_1 dw_2\right]$$

$$\leq \left[\left(\frac{1}{2\pi i}\right)^2 \frac{M_{ij}(R_1,R_2)}{R_1^{n_1+1} R_2^{n_2+1}} . 2\pi R_1 . 2\pi R_2\right],$$

$$\left[A_{n_1,n_2,ij}\right] = \left|A_{n_1,n_2}\right| \leq \left\|\frac{M(R_1,R_2)}{R_1^{n_1} R_2^{n_2}}\right\| \tag{4.2}$$

Where

$$M(R_1, R_2) = \max_{0 \le i,j \le N} M_{ij}(R_1, R_2); \quad i, j = 1, 2, \dots$$

This inequality is called Cauchy's inequality for the matrix function $f(X_1, X_2)$ in the closed complete Reinhardt domain $\bar{\Gamma}_{R_1,R_2}$ of radii $R_1 > 0$ and $R_2 > 0$;

$$\bar{\Gamma}_{R_1,R_2} = \{(z_1, z_2) \in C^2 : |z_1| \le R_1, |z_2| \le R_2\},$$

The open complete Reinhardt domain is given by

$$\Gamma_{R_1,R_2} = \{(z_1, z_2) \in C^2 : |z_1| < R_1, |z_2| < R_2\},$$

Thus, the relation (4. 2) become in the form

$$[A_{n_1,n_2,ij}] \le \left\| \frac{M(f, \bar{\Gamma}_{R_1,R_2})}{R_1^{n_1} R_2^{n_2}} \right\|; \quad n_1, \quad n_2 = 0, 1, 2, \dots \qquad (4.3)$$

Now, if $R_1 = Rt_1, R_2 = Rt_2;\ t_1^2 + t_2^2 = 1, 0 \le t_1, t_2 \le 1$, we get

$$[A_{n_1,n_2,ij}] \le \left\| \max_{|t|=1} \frac{M(f, \bar{\Gamma}_{R_1,R_2})}{(Rt_1)^{n_1}(Rt_2)^{n_2}} \right\| = \left\| \frac{M(f, \bar{S}_R)}{R^{n_1+n_2}} \sigma_{n_1,n_2} \right\| \qquad (4.4)$$

This inequality is called Cauchy's inequality for the matrix function $f(X_1, X_2)$ in the closed spherical region $\bar{S}_R$ of radius $R > 0$;

$$\bar{S}_R = \left\{ (z_1, z_2) \in C^2 : \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} \le R \right\},$$

Where

$$M(f, \bar{S}_R) = \max_{i,j} \cdot \max_{|t|=1} \cdot \max_{|z_1|=Rt_1, |z_2|=Rt_2} \left| \{f(X_1, X_2)\}_{ij} \right|$$

And

$$\sigma_{n_1,n_2} = \max_{|t|=1} \frac{1}{t_1^{n_1} t_2^{n_2}} = \max_{t_1^2+t_2^2} \frac{1}{t_1^{n_1} t_2^{n_2}} = \begin{cases} 1; & n_1 \ or\ n_2 \\ \dfrac{(n_1+n_2)^{\frac{n_1+n_2}{2}}}{(n_1)^{\frac{n_1}{2}} (n_2)^{\frac{n_2}{2}}}; & n_1 \ne 0, n_2 \ne 0 \end{cases} \qquad (4.5)$$

At $R_1 = \rho_1 t_1, R_2 = \rho_2 t_2$, we obtain Cauch's inequality for the matrix function $f(X_1, X_2)$ in the closed hyper elliptical region $\bar{E}_{\rho_1,\rho_2}$in the form

$$[A_{n_1,n_2,ij}] \le \left\| \max_{|t|=1} \frac{M\left(f, \bar{E}_{\rho_1 t_1, \rho_2 t_2}\right)}{\left(\rho_1 t_1\right)^{n_1} \left(\rho_2 t_2\right)^{n_2}} \right\| = \left\| \frac{M\left(f, \bar{E}_{\rho_1,\rho_2}\right)}{\rho_1^2 \rho_2^2} \sigma_{n_1,n_2} \right\|; \qquad (4.6)$$

Where

$$\bar{E}_{\rho_1,\rho_2} = \left\{ (z_1, z_2) \in C^2 : \left\{ \frac{|z_1|^2}{\rho_1^2} + \frac{|z_2|^2}{\rho_2^2} \right\}^{\frac{1}{2}} \le 1 \right\},$$

And

$$M\left(f,\overline{\mathrm{E}}_{\rho_1 t_1,\rho_2 t_2}\right)=\max_{i,j}\cdot\max_{|t|=1}\cdot\max_{|z_1|=\rho_1 t_1,|z_2|=\rho_2 t_2}\left|\{f(X_1,X_2)\}_{ij}\right|$$

5. RADUIS OF REQULARITY

Suppose that

$$f(X_1,X_2)=\sum_{n_1,n_2=0}^{\infty}A_{n_1,n_2}(z_1-a_1)^{n_1}(z_2-a_2)^{n_2}, \tag{5.1}$$

Where

$$A_{n_1,n_2}=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{f(Y_1,Y_2)}{(w_1-a_1)^{n_1+1}(w_2-a_2)^{n_2+1}}dw_1dw_2$$

$$=\frac{1}{n_1!\,n_2!}\left\{\frac{\partial^{n_1+n_2}}{\partial w^{n_1}\partial w^{n_2}}f(Y_1,Y_2)\right\}_{(w_1,w_2)=(a_1,a_2)};\ n_1,n_2=0,1,2,\ldots,$$

When $a_1=a_2=0$, we have

$$f(X_1,X_2)=\sum_{n_1,n_2=0}^{\infty}A_{n_1,n_2}z_1^{n_1}z_2^{n_2}, \tag{5.2}$$

$$A_{n_1,n_2}=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{f(Y_1,Y_2)}{w_1^{n_1+1}+w_2^{n_2+1}}dw_1dw_2=$$

$$=\frac{1}{n_1!\,n_2!}\left\{\frac{\partial^{n_1+n_2}}{\partial w^{n_1}\partial w^{n_2}}f(Y_1,Y_2)\right\}_{(w_1,w_2)=(a_1,a_2)};\ n_1,n_2=0,1,2,\ldots,$$

Wesay that the matrix function $f(X_1,X_2)$ is convergent if $\|f(X_1,X_2)\|$ converges. Therefor, we will investigate the convergence of $\|f(X_1,X_2)\|$.

Let R_1 be any positive number such that $R_1<R$, and let us consider the series

$$\|f(X_1,X_2)\|\le\sum_{n_1,n_2=0}^{\infty}\left\|U_{n_1,n_2}\right\|, \tag{5.3}$$

Where

$$\left\|U_{n_1,n_2}\right\|=\left\|A_{n_1,n_2}\right\|\left\{\max_{\bar{S}_{R_1}}\left|z_1^{n_1}z_2^{n_2}\right|\right\}.$$

We will use the relation due to Abdalla [1], in the form:

$$\max_{\bar{S}_{R_1}}\left|z_1^{n_1}z_2^{n_2}\right|=\frac{R_1^{n_1+n_2}}{\sigma_{n_1,n_2}};\ n_1,n_2\ge 0.$$

Thus, it follows that

$$\left\|U_{n_1,n_2}\right\|=\left\|A_{n_1,n_2}\right\|\frac{R_1^{n_1+n_2}}{\sigma_{n_1,n_2}}.$$

Hence,

$$\lim_{n_1+n_2\to\infty} sup \left\|U_{n_1,n_2}\right\|^{\frac{1}{n_1+n_2}} = \frac{R_1}{R} < 1 \qquad (5.4)$$

If R_2 is real number such that $R_1 < R_2 < R$, then, from (5.4), there exists an integer m_0 such that

$$\left\|U_{n_1,n_2}\right\| < \left(\frac{R_2}{R}\right)^{n_1+n_2} . n_1 + n_2 > m_0.$$

i.e.

$$\sum_{n_1,n_2=0}^{\infty} \left\|U_{n_1,n_2}\right\| < K + \sum\nolimits_{n_1>m_0} (n_1+1)\left(\frac{R_2}{R}\right)^{n_1} < K + (1-\frac{R_2}{R})^{-2} < \infty,$$

where K denotes a positive finite numbers independent of n_1 and n_2, and which does not retain the same values at different occurrences, here

$$K = \sum_{0\le n_1+n_2\le m_0}^{\infty} \left\|U_{n_1,n_2}\right\| < \infty.$$

Therefore the series (5.2) is absolutely and uniformly convergent in S_{R_1}. Since R_2 can be chosen arbitrary near to R, then the following theorem follows.

Theorem 5.1. The matrix power series

$$f(X_1,X_2) = \sum_{n_1,n_2=0}^{\infty} A_{n_1,n_2} z_1^{n_1} z_2^{n_2},$$

Is absolutely and uniformly convergent in the open hypersphere $S_R; R > 0$ if

$$\lim_{n_1+n_2} \sup \left\{\frac{\left\|A_{n_1,n_2}\right\|}{\sigma_{n_1,n_2}}\right\}^{\frac{1}{n_1+n_2}} = \frac{1}{R} < \infty. \qquad (5.5)$$

Remark 5.1. Clearly if either n_1 or n_2 is equal 0, then $f(X_1,X_2)$ in (5.2) becomes of one matrix $X_1\ or\ X_2$ respetively.

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