HOMOTOPY ANALYSIS METHOD FOR SOLVING INTEGRAL AND INTEGRO_DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, several integral equations are solved by Homotopy analysis method (HAM). A comparison of the exact solution and HAM have shown that the method is very effective and convenient for solving integral and integro – differential equations.

Keywords :*Homotopy analysis method, Integral equation, Approximate solution.*

1. INTRODUCTION

Various kinds of analytical methods and numerical methods were used to solve integral equations[1]. In this paper, we apply homotopy analysis method to solve integral equations.
In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method [7]. This method has been successfully applied to solve many types of nonlinear problems [2–7]. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer power and time. The HAM is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time.

2. BASIC IDEA OF HAM

We consider the following differential equation

$$N [u (\tau)] = 0 \qquad (1)$$

where N is a nonlinear operator, τ denotes independent variables, u (τ) is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [4] constructs the so called zero – order deformation equation.

$$(1-p)L[\phi(\tau;p)-u_o(\tau)] = phH(\tau)N[\phi(\tau;p)] \qquad (2)$$

where $p\in[0,1]$ is the embedding parameter, h $\neq$ 0 is a nonzero parameter, H(τ) $\neq$ 0 is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $u(\tau;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(\tau;0)=u_o(\tau) \quad , \quad \phi(\tau;1)=u(\tau) \qquad (3)$$

Respectively. Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guesses $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p)=u_o(\tau)+\sum_{m=1}^{\infty}u_m(\tau)p^m, \qquad (4)$$

where

$$u_m(\tau)=\frac{1}{m!}\frac{\partial^m\phi(\tau;p)}{\partial p^m}\bigg|_{p=0} \qquad (5)$$

If the auxiliary linear operator, the initial guess, the auxiliary h, and the auxiliary function are so properly chosen, the series (4) converges at $p=1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{6}$$

Define the vector

$$\overrightarrow{u_n} = \{u_0(\tau), u_1(\tau), ..., u_n(\tau)\}$$

Differentiating equation (2) m Times with respect to the embedding parameter p and then setting p = 0 and finally dividing them by m!, we obtain the *m*th – order deformation equation

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau)R_m(\overrightarrow{u_{m-1}}) \tag{7}$$

Where

$$R_m(\overrightarrow{u_{m-1}}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}} \bigg|_{p=0} \tag{8}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{9}$$

Applying L^{-1} on both side of equation (7), we get

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + hL^{-1}[H(\tau)R_m(\overrightarrow{u_{m-1}})] \tag{10}$$

In this way, it is easily to obtain u_m for m ≥ 1 ,at mth- order, we have

$$u(\tau) = \sum_{m=0}^{M} u_m(\tau), \tag{11}$$

when $M \to +\infty$, we get an accurate approximation of the original equation (1). For the convergence of the above method we refer the reader Liao's work [7]. If equation (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3. APPLYING HAM

In this section, we apply this method for solving integral equations.

3.1 Volterra integral equations of second kind

First, we consider the Volterra integral equations of the second kind, which read

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt \ , \tag{12}$$

where k (x, t) is the kernel of the integral equation.

Example 1. Consider the following Volterra integral equation of the second kind

$$u''(x) = 1 + xe^x - \int_a^x e^{x-t} u(t)dt \, , \tag{13}$$

with initial conditions $u(0) = 0$, $u'(0) = 1$.

To solve the equation (13) by means of homotopy analysis method, according to the initial conditions denoted in equation (13), it is natural to choose.

$$u_0(x) = e^x - 1 \tag{14}$$

We choose the linear operator

$$L[\phi(x.p)] = \frac{\partial^2 \phi(x;p)}{\partial x^2}, \qquad L^{-1} = \int_0^x \int_0^x dx dx \tag{15}$$

With the property $L[c_1 t + c_2] = 0$, where c_1 and c_2 are constants. We now define a nonlinear operator as

$$N[\phi(x;p)] = \frac{\partial^2 \phi}{\partial x^2} - 1 - xe^x + \int_0^x e^{x-t}\phi(t)dt \tag{16}$$

Using above definition, with assumption $H(\tau)=1$ we construct the zero order deformation equation

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = phH(\tau)N[\phi(\tau;p)] \tag{17}$$

Obviously, when $p=0$ and $p=1$,

$$\phi(\tau,0) = u_0(\tau) \quad , \quad \phi(\tau,1) = u(\tau) \tag{18}$$

Thus, we obtain the mth – order deformation equations

$$L[u_m - \chi_m u_{m-1}] = hR_m(\overrightarrow{u_{m-1}}) \tag{19}$$

Where

$$R_m(\overrightarrow{u_{m-1}}) = \frac{\partial^2 u_{m-1}}{\partial x^2} - (1+xe^x)(1-\chi_m) + \int_0^x e^{x-t}u_{m-1}(t)dt \tag{20}$$

Now, the solution of the mth-order deformation equation (19)

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + hL^{-1}[R_m(\overrightarrow{u_{m-1}})] \tag{21}$$

Finally, we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau),$$

We obtain

$$u_0(x) = e^x - 1$$
$$u_1(x) = 0$$
$$u_2(x) = 0$$
$$\vdots$$

Hence,

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + \ldots \\ &= e^x - 1 + 0 + \ldots = e^x - 1 \end{aligned}$$

Which is the exact solution of equation (13).

Example 2. Consider the following Fredholm integral equations [8]

$$u(x) = x + \int_0^x (t-x)u(t)dt \tag{22}$$

To solve the equation (21), it is natural to choose

$$u_0(x) = x \tag{23}$$

We choose The linear operator

$$L[\phi(x;p)] = \phi(x,p) \tag{24}$$

We now define a nonlinear operator as

$$N[\phi(x;p)] = \phi(x,p) - x - \int_0^x (t-x)\phi(t)dt \tag{25}$$

Thus, we obtain the mth-order deformation equation

$$L[u_m - \chi_m u_{m-1}] = hR_m(\overrightarrow{u_{m-1}}) \tag{26}$$

Where

$$R_m(\overrightarrow{u_{m-1}}) = u_{m-1}(x) - (1-\chi_m)x - \int_0^x (t-x)u_{m-1}(t)dt \tag{27}$$

Now, the solution of the mth-order deformation equation (26)

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + hL^{-1}[R_m(\overrightarrow{u_{m-1}})] \tag{28}$$

Finally, we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau),$$

We obtain

$$u_0(x) = x$$

$$u_1(x) = h\frac{x^3}{6}$$

$$u_2(x) = h\frac{x^3}{6} + h\left(-\frac{x^3}{6} - \frac{x^5}{5!}\right)$$

$$u_3(x) = h\frac{x^3}{6} + h\left(-\frac{x^3}{6} - \frac{x^5}{5!}\right) + h\left(\frac{x^5}{5!} + \frac{x^7}{7!}\right)$$

$$\vdots$$

Hence

$$u(x) = u_0(x) + u_1(x) + \ldots$$

$$= x + h\frac{x^3}{3!} + h\frac{x^3}{3!} + h\left(-\frac{x^3}{3!} - \frac{x^5}{5!}\right) + \ldots$$

When h = -1 we have

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Which is the exact solution of equation (22).

3.2 Fredholm integral equations of the second kind

Next we consider the Fredholm integral equations of the second kind, which read

$$u(x) = f(x) + \int_a^b h(x,t)u(t)dt \tag{29}$$

Where k(x,t) is the kernel of the integral equation.

Example 3. Consider the following Fredholm integral equations

$$u(x) = \cos x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x u(t)dt \tag{30}$$

Beginning with

$$u_0(x) = \cos x \tag{31}$$

We choose the linear operator

$$L[\phi(x;p)] = \phi(x,p) \tag{32}$$

We now define a nonlinear operator as

$$N[\phi(x;p)] = \phi(x,p) - \cos x - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\phi(t)dt \tag{33}$$

Thus, we obtain the mth-order deformation equation

$$L[u_m - \chi_m u_{m-1}] = hR_m(\overrightarrow{u_{m-1}}) \tag{34}$$

Where

$$R_m(\overrightarrow{u_{m-1}}) = u_{m-1}(x) - (1-\chi_m)\cos x - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\, u_{m-1}(t)dt \tag{35}$$

We obtain

$$u_0(x) = \cos x$$

$$u_1(x) = -\frac{1}{2}h\sin x$$

$$u_2(x) = -\frac{1}{2}h\sin x + \frac{1}{4}h\sin x$$

$$\vdots$$

Hence , $u(x) = u_0(x) + u_1(x) + ...$

$$= \cos x + -\frac{1}{2}h\sin x + -\frac{1}{2}h\sin x + \frac{1}{4}h\sin x + ..$$

Where h = -1 we have

$$u(x) = \cos x + \frac{1}{2}\sin x + \frac{1}{4}\sin x + \frac{1}{8}\sin x + ... = \cos x + \lim_{n\to\infty}\left(\frac{2^n - 1}{2^n}\right)\sin x \to \cos x + \sin x$$

which is the exact solution of equation (30).

4. CONCLUSIONS

In this paper, the Homotopy analysis method has been successfully applied to find the solution of integral and integro- differential equations. It is apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wide classes of and integral equations. It is worth pointing out that this method presents a rapid convergence for the solutions. They also do not require large computer memory and discretization of the variables x. The results show that HAM is powerful mathematical tool for solving linear equations

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