MELNIKOV CHAOS IN A PERIODICALLY DRIVEN THE NONLINAR SCHRODINGER EQUATION

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ABSTRACT
A numerical behavior of a KdV combined mKdV equation is obtained using the Melnikov method. Melnikov method is proved to be elegant, and successful alternative to characterizing the complex dynamics of multi-stable oscillators. Based on the Melnikov theory we present the homoclinic and heteroclinic orbits in the unperturbed system. It is show us whether the system is chaotic or not. In this work, we study the relation among the parameters of the linnonlinear dispersion term. Furthermore, we discuss the control threshold of the chaos. Chaos appear in the system due to the absence of damping term.

Keywords: *KdV combined mKdV equation, Melnikov method, Fiber-optic signal transmission.*

1. INTRODUCTION
As is well known, the fiber-optic signal plays an important role in real life. It is this optical soliton propagation in optical fiber long time to maintain shape, amplitude and velocity constant light pulses. The use of optical solitons can achieve ultra-long-distance, large-capacity optical communications.

Optical solitary has been the subject of intense current research, which is motivated by their important applications in the areas of high-capacity fiber telecommunications and all-optical switches due to their capability of propagating over a long distance without attenuation and changing their shapes [1].

Compactons are new class of solitary solutions for families of the nonlinear dispersive Schrödinger equation as well as other nonlinear dispersive equations, such as nonlinear dispersive KdV. Unlike the solitons, which, although highly localized, still have infinite span, these solutions have compact support; they vanish identically outside a finite region. Hence, these solitary waves have been christened as compactons [2].

A great deal of research work has been invested during the past decades for the study dispersive KdV Equation

$$u_t + auu_x + u_{xxx} = 0, \qquad (1.1)$$

and the modified KdV equation (mKdV)

$$u_t + au^2u_x + u_{xxx} = 0, \qquad (1.2)$$

The KdV and MKdV equations are most popular soliton equations and have been extensively investigated. But the nonlinear terms of KdV and MKdV equations often simultaneously exist in practical problems such as fluid physics, physics and quantum field theory and optical fiber communications.

The KdV equation and the modified KdV equation are completely integrable, equations that have multiple-soliton solutions and possess infinite conservation quantities. The two equations with time-variable coefficients have been examined in the context of ocean waves.

However, if the quadratic and the cubic nonlinear terms of the KdV equation and the mKdV equation respectively are combined, then the resulting equation [7],

$$u_t + \mu uu_x - u^2u_x + u_{xxx} = 0, \qquad (1.3)$$

In this study, we implemented the Melnikov method for finding the behaviors of the combined KdV-MKdV equation, it is the steady of the meaningful kdv-Mkde equation, and the character of the combined KdV and MKdV equation under external periodic perturbation.

In this paper, considered herein is the chaotic behaviors of the equation (1.3) under the periodic perturbation term and the controller, respectively. There has been a great deal of models in the control of chaotic behaviors by using Melnikov analysis. For chaos control of the three-well duffing system with two external forcings [7,8], one can apply melnikov's function to calculate the distance between the homoclinic orbits and yield conditions of chaos occurrence in a nearly Hamiltonian system. Furthermore, the chaotic behaviors of duffing-type system can be presented by the Melnikov analysis. This method can be applied to the analysis of smooth dynamical system. With the extensively application of the melnikov theory, this method also can be used in the liénard system, Rayleigh-Duffing oscillator and the KdV equation .

The organization of the paper is as follows: In section 2, we analyze the homoclinic and heteroclinic orbits of the unperturbed system. In section 3, the chaotic analysis and chaotic control threshold is shown. Numerical results relating the system is given under different parameters in section 4. Finally, in the appendix we give remark to conclude this paper.

2. ANALYSIS OF SYSTEM

According to the Exp-function method, we introduce a complex variation ξ defined as $\xi = kx + k^3 t$. Thus becomes an ordinary different equation, which reads

$$k^3 u' + k\mu u u' - k u^2 u' + k^3 u''' = 0 \tag{2.1}$$

After the first integral of Eq. (2.1), we can get

$$k^3 u + \frac{k}{2}\mu u^2 - \frac{k}{3}u^3 + k^3 u'' = 0 \tag{2.2}$$

Eq. (2.2) is the fiber-optic signal transmission system in ideal environment. But it seems that the signal propagation cannot exists in pure environment. It is always influenced by external environmental perturbations.

Consequently, we will consider the following the fiber-optic signal transmission system with perturbation

$$k^3 u + \frac{k}{2}\mu u^2 - \frac{k}{3}u^3 + k^3 u'' = d_1\cos(\omega_1\xi) + d_2\cos(\omega_2\xi), \tag{2.3}$$

where the new variables d_i, ω_i, $(i=1,2)$ denote the amplitude and the frequency of the parametric excitation respectively. Here d_i and ω_i are real positive parameters.
Eq. (2.3) can be transformed into first-order non-autonomous equation

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 - \frac{\mu}{2k^2}x_1^2 + \frac{1}{3k^2}x_1^3 + d_1\cos(\omega_1\xi) + d_2\cos(\omega_2\xi). \end{aligned} \tag{2.4}$$

If $d_1 = 0$, $d_2 = 0$, Eq. (2.4) is considered as an unperturbed system and be written as

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \frac{\mu}{2k^2}x_1^2 + \frac{1}{3k^2}x_1^3. \end{aligned} \tag{2.5}$$

The system (2.5) is a Hamiltonian system with Hamiltonian function

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{\mu}{6k^2}x_1^3 - \frac{1}{12k^2}x_1^4 \ . \tag{2.6}$$

The associated potential energy is given as

$$V(x_1) = \frac{1}{2}x_1^2 + \frac{\mu}{6k^2}x_1^3 - \frac{1}{12k^2}x_1^4 \tag{2.7}$$

is called the potential function. By the analysis of the fixed points (x_1, x_2) and their stabilities for system (2.3), we can obtain without any difficulty the following results.

Lemma 1

(i) for $\frac{\mu^2}{4k^4}-\frac{4}{3}k<0$, *there is a saddle pointer* $(0,0)$.

(ii) for $\frac{\mu^2}{4k^4}-\frac{4}{3}k=0$, *there is a saddle pointer* $(0,0)$ and a center.

(ii) for $\frac{\mu^2}{4k^4}-\frac{4}{3}k>0$, *there are three fixed points:* $(0,0)$ being a saddle pointer, and

$(\frac{1}{4}(-3\mu-\sqrt{3}\sqrt{-16k^4+3\mu^2}),0)$, $(\frac{1}{4}(-3\mu+\sqrt{3}\sqrt{-16k^4+3\mu^2}),0)$ being center.

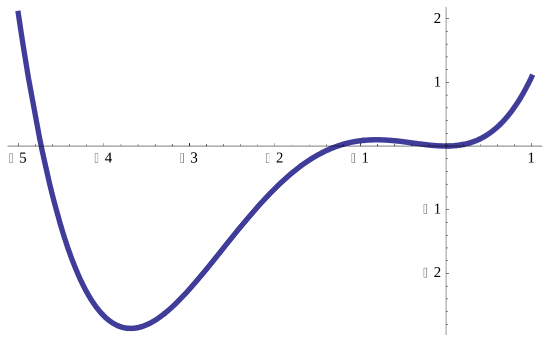


Fig.ure 1. The two-well potential function of the system.

In essence we use perturbation methods to study the system (2.5), we therefore study how the dynamics of unperturbed system (2.6) are changed under the periodic and quasi-periodic perturbations in the folloing parts.

3. MELNIKOV THEORETIC ANALYSIS

Melnikov theory has proved to be a simple, elegant, and successful alternative to characterizing the complex dynamics of multi-stable oscillators. This section, thus, develops a global analysis technique, known as Melnikov's method, to find the necessary conditions for homoclinic bifurcation to occur. For a detailed derivation of Melnikov's method, there are several texts of varying rigor and sophistication to which the reader is referred [14-17].

3.1. Melnikov criterion for chaos

By Lemma 1(ii) the unperturbed system for system (2.7) has two homoclinic orbits $\Gamma^{\pm}_{ho}$. When the perturbation is added, the closed homoclinic orbits break, and may have transverse homoclinic or heteroclinic orbits. By the Smale-Birkhoff Theorem [14,18], the existence of such orbits results in chaotic dynamics. We therefore apply the Melnikov method to system (2.6) for finding the criteria of the existence of homoclinic or heteroclinic bifurcation and chaos.

The Melnikov method derives a function to describe the first order distance between perturbed stable and unperturbed manifolds. Suppose that the unperturbed homoclinic or heteroclinic orbits are written as $(x_1,x_2)=(x_1(\xi),x_2(\xi))$. Satisfying the conditions for a double-well potential gives rise to a homoclinic orbit in the system's phase space for $m=0$ .The homoclinic trajectory can be found by setting $H(a,b)=0$. Solving for the resulting displacement and differentiating to determine velocity, the homoclinic trajectory is given as follows:

$$(x_1,x_2)=(\frac{\mu(b_0^2-4)}{b_0 2\cos(\xi)+b_0^2},\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}),$$

where $b_0=\sqrt{\frac{4\mu^2}{\mu^2-6k^2}}$.

We note that $b(\xi)$ is a function of time form $+\infty$ to $-\infty$. We therefore choose the initial conditions is $\xi = 0$, $x_1 = \dfrac{\mu(b_0^2-4)}{2b_0+b_0^2}$ and $x_2 = 0$, and $b(\xi)$ would be an odd function of time for the homoclinic orbit and an even function for heteroclinic orbit. Then the Melnikov Suppose that the unperturbed homoclinic or heteroclinic orbits are written as

$$M(\xi_0) = \int_{-\infty}^{+\infty} x_2^{\pm}(\xi)[d\cos(\omega\xi + \omega\xi_0)]d\xi\,, \qquad (3.1.1)$$

where ξ_0 is the cross section time of the Poincaré map and ξ_0 can be interpreted as the initial time of the forcing term. $(x_1, x_2) = (x_1(\xi), x_2(\xi))$, then the Melnikov function for system (2.5) can be given by

$$M(\xi_0) = \int_{-\infty}^{+\infty} x_2^{\pm}(\xi)[d\cos(\omega\xi + \omega\xi_0)]d\xi,$$

Because it is difficult to give analytical expression of x_2, we will compute x_2 numerically in section 5. For the homoclinic orbits $\Gamma_{ho}^{\pm}$, the Melnikov function can be simplified as

$$\begin{aligned}M(\xi_0) &= \int_{-\infty}^{+\infty} x_2^{\pm}(\xi)[d\cos(\omega\xi + \omega\xi_0)]d\xi\\ &= \int_{-\infty}^{+\infty}\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}[d\cos(\omega\xi+\omega\xi_0)]d\xi\\ &= -\int_{0}^{+\infty}\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}[d\sin(\omega\xi)\sin(\omega\xi_0)]d\xi\\ &= -Ad\sin(\omega\xi_0)\,, \qquad (3.1.2)\end{aligned}$$

where $A = 2\int_{0}^{+\infty}\dfrac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}\sin(\omega\xi)d\xi$ is functions of the frequency ω .

Using the previous results and Melnikov's theorem [23,24] the following is stated : If $M(\xi_0)=0$ and $M(\xi_0)\neq 0$ for some ξ_0 and some set of parameters, then horseshoes exist, and chaos occurs [14,19]. If $M(\xi_0)$ has a simple zero and the corresponding critical parameter value is

$$\left(\frac{m}{d}\right)_0 = \left|\frac{A}{B}\right|, \qquad (3.1.3)$$

where m denotes the controller' strength, $B = \int_0^{\infty} x_2(\xi_0)^2\, d\xi$ is a constant once $x_2(\xi_0)^2$ is given, then in the system with fractional order displacement (2.5) the deterministic chaos may appear for certain parameter values which satisfy the relation

3.2. Control of chaos

Because the fiber-optic transmission system in the chaotic state is very sensitive to its initial condition and chaos often causes irregular behavior, chaos is undesirable. It’s not hard to see that system (2.5) is similar to duffing system, except the absence of damping in the former. Therefore, we will select the controller that have the same function with the damping. To suppress chaos, we will add controller mu' to the system (2.4).

$$\omega u + \frac{k}{2}\mu u^2 + \frac{k}{3}u^3 + k^3 u'' + mu' = d\cos(\theta x)\ . \qquad (3.2.1)$$

Eq. (3.2.1) can be transformed into first-order non-autonomous equation

$$x_1' = x_2,$$
$$x_2' = -\omega x_1 - \frac{\mu}{2k^2}x_1^2 - \frac{1}{3k^2}x_1^3 - mx_2 + d\cos(\theta\xi). \qquad (3.2.2)$$

Now, the Melnikov function for system (3.2.1) can be given by

$$M_1(\xi_0) = \int_{-\infty}^{+\infty} x_2^{\pm}(\xi)[-mx_2^{\pm}(\xi) + d\cos(\theta\xi + \theta\xi_0)]d\xi\,, \qquad (3.2.3)$$

the Melnikov function can be simplified as

$$\begin{aligned} M_1(\xi_0) &= \int_{-\infty}^{+\infty} x_2^{\pm}(\xi)[-mx_2^{\pm}(\xi) + d\cos(\theta\xi + \theta\xi_0)]d\xi \\ &= -m\int_{-\infty}^{+\infty} x_2^{\pm}(\xi)^2 d\xi + d\int_{-\infty}^{+\infty} x_2^{\pm}(\xi)\cos(\theta\xi + \theta\xi_0)d\xi \\ &= -m\int_0^{+\infty}[\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}]^2 d\xi - 2d\int_0^{+\infty}\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}\sin(\theta\xi)\sin(\theta\xi_0)d\xi \\ &= -mB - Ad\sin(\theta\xi_0)\,, \qquad (3.2.4) \end{aligned}$$

where $B = \int_0^{+\infty}[\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}]^2 d\xi$, $A = 2\int_0^{+\infty}\frac{2b_0\sin(\xi)\mu(b_0^2-4)}{(b_0 2\cos(\xi)+b_0^2)^2}\sin(\theta\xi)d\xi$.

Thus, if

$$m < \frac{|dA|}{B} \equiv R_1(\theta) \qquad (3.2.5)$$

then there is a $\bar{\xi}_0$ such that $M_1(\bar{\xi}_0)$ and $\frac{\partial M_1}{\partial \xi_0}|_{\xi=\bar{\xi}_0} \neq 0$, $\bar{\xi}_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and the following theorem can be obtained.

If ξ_0 has a simple zero and the corresponding critical parameter value is

$$(\frac{k}{d})_0 = \left|\frac{A_i^{\pm}}{B_i^{\pm}}\right|, \qquad (i = 1,2)\,. \qquad (3.2.7)$$

4. NUM ERICAL SIMULATIONS

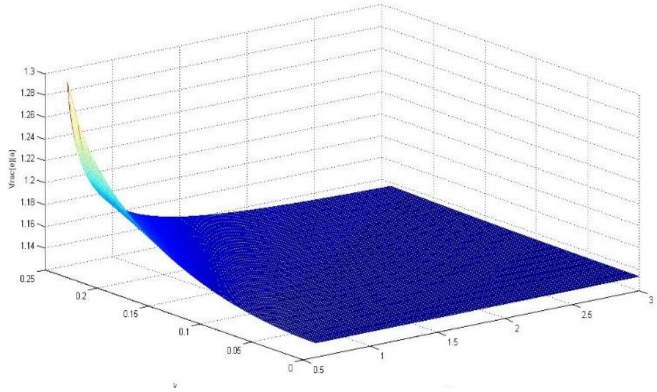
In this section, we give numerical simulations to support the theoretical results obtained in the previous sections and to find other new dynamics.

The interesting problem is to analyze the parameter regions for optical fiber signals stable propagation of the control led system. The controlled fiber-optic transmission system has several parameters, each of them plays different and virtual roles in the system. We will analyze the influence on optic-fiber signals propagation of controlled system (3.2.7) when the parameter of system changes with the fixed controller.

Let us study the intersections of the invariant manifolds of the saddle point. It is known, that these intersections are the necessary conditions for the existence of chaos. Since the Melnikov function theory measures the distance between the perturbed stable and unstable manifolds in the Poincare section, to preserve the homoclinic loops under a perturbation requires that at ξ_0, if $M(\xi_0)$ has a simple zero, then a homoclinic bifurcation occurs, signifying the possibility of chaotic behavior. This means that only necessary conditions for the appearance of strange attractors are obtained from Poincare-Melnikov-Arnold analysis, and therefore one always has the chance of finding sufficient conditions for the elimination of even transient chaos. Then the general necessary condition for which the invariant manifolds intersect is given by

$$(\frac{m}{d})_0 = \frac{A}{B}\,. \qquad (4.1)$$

We give numerical simulations to support the theoretical results as following:



Figur 2. According to chaotic threshold in $(k,\mu,\frac{m}{d})$ *space is given in Fig. 2, when the value of* $\frac{e}{d}$ *below the surface, the system may be chaotic state. And we can observe that within the change of* θ*, the system (2.5) signals that have large vibration, with* μ *and* k*, while increasing the chaotic state of the system has been reduced and variable smaller.*

5. CONTROL OF CHAOS NUMERICAL SIMULATIONS

In this section, we give numerical simulations to support the theoretical results of control of chaos.

First, numerical simulations are performed for the case of periodic perturbation of system (3.2.1), we will discuss the behaviors of the fiber-optic signal transmission under perturbation. The system (2.4) is integrated using the Runge-Kutta technique of order four to conduct numerical simulation.

Now we will study the chaotic control of system (2.5) with $\mu = 2$, $k = 0.25$, $\theta = 0.05$, $m = 0.5$ and setting d as the variable with the initial condition [1.0,0.0].

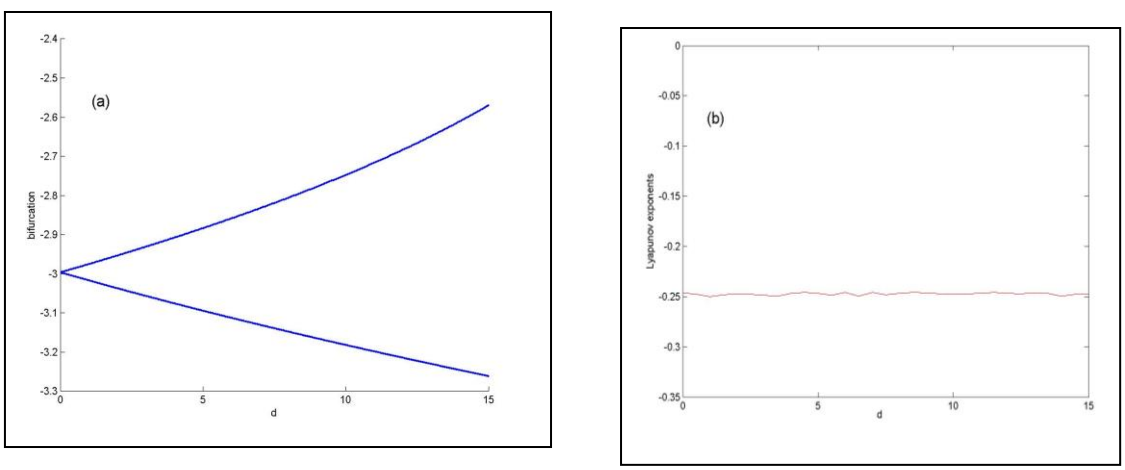


Figure.3. Bifurcation diagram and Lyapunove exponents spectrum of ξ *(*$d \in (0,15)$ *),*
and (b) The Maximum Lyapunov exponents corresponding to (a).

From the bifurcation diagrams of system (3.2.1) in (d,ξ) plane and Lyapunov exponents are given in Fig. 3. We can get that chaos can be suppressed to the stable state comparing with Fig. 4.

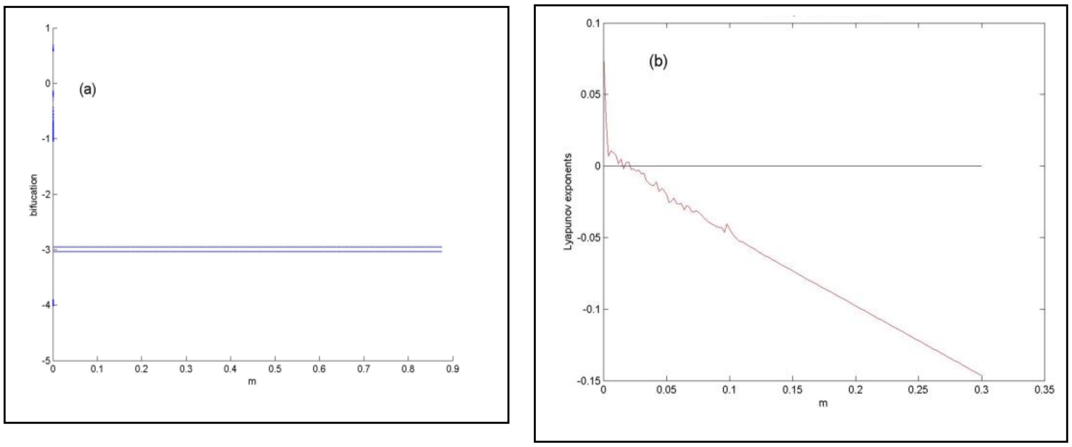


Figure.4. Bifurcation diagram and Lyapunov exponents. (a) bifurcation diagram of χ ($m \in (0,0.9)$)
(b) The Maximum Lyapunov exponents corresponding to(a).

Now we will study the chaotic control of system (2.5) with $\mu = 2, k = 0.25, \theta = 0.05, d = 2.0$ and setting m as the variable with the initial condition [1.0,0.0]. According to the bifurcation diagram and Lyapunov exponents in Fig. 4, we can obtain that the behavior of (2.3.1) is still chaotic within $m \in (0,0.004)$ for the controller being too weak to inhibit the chaos. Chaos of system (2.4) can be suppressed to the stable station with the larger m.

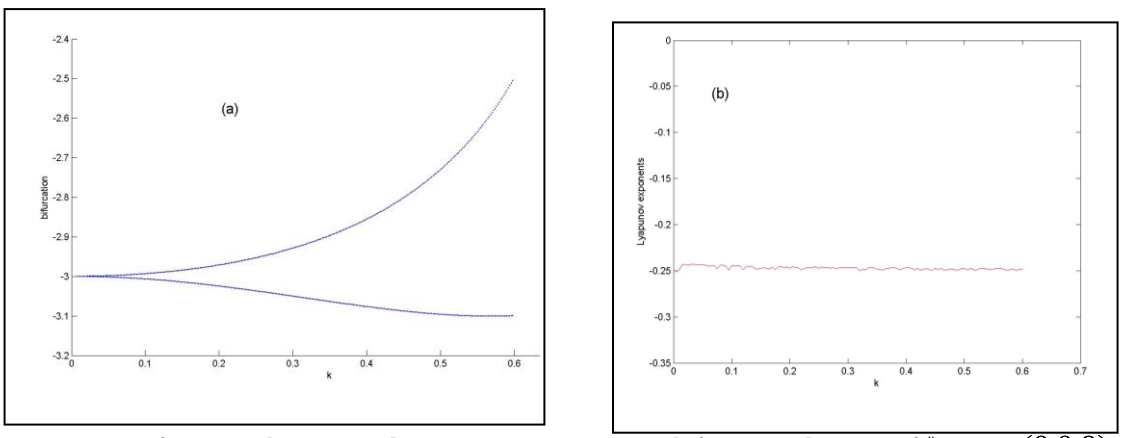


Figure.5. Bifurcation diagram and Lyapunov exponents. (a) bifurcation diagram of χ ($m \in (0,0.9)$)
(b) The Maximum Lyapunov exponents corresponding to(a).

6. CONCLUSION

Using Melnikov's method, chaos has been predicted to occurs easily in the NLSE under the effect of a cosine function perturbation. This phenomenon will cause the distortion in the process of information transmission. One can add a controller to suppress the chaos. A same function with the damping was considered from duff system and the Melnikov theorem was applied with an active control strategy to suppress chaos in the system. The result of this analysis was an inequality describing the set of parameters where chaos occurs, which is a useful design tool for tailoring the controller's parameters so homoclinic chaos either occurs or does not occur as desired. For a representative parameter set, numerical simulations showed that chaos does occur in regions of parameter space satisfying the criterion from Melnikov's method and helped to better understand the complicated dynamics of the oscillator. By varying controller coefficient ℓ of the system, many interesting attractors are found, including periodic orbits and chaos. The phase portraits, the maximum Lyapunov exponent and the bifurcation diagrams show that the stability of the NLSE will lose once the damping term exceeds a critical value. All these results strongly demonstrate that the controller term can produce a considerable effect on the NLSE. Some of the results obtained for other NLSE can be corresponding extended.

7. ACKNOWLEDGEMENTS

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