APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR SOLVING NON-DARCY FLOW PROBLEM IN POROUS MEDIA

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ABSTRACT
Many industrial problems are highly non-linear and are thus difficult to solve analytically. A typical problem of this nature in the petroleum industry is the flow of fluids in porous media. In this work, a dimensionless non-linear ordinary differential equation for transient flow behavior of oil production in porous media was derived and solved using the Homotopy Analysis Method (HAM).

The results obtained by the HAM solution were found to be in good agreement with the exact solution as demonstrated by the residual plots which confirms that the pressure distribution in the porous medium can be accurately predicted with the HAM solution with average relative error of ±0.5%. The solutions of the linear and non-linear dimensionless pressure differs as follows: for \( \lambda = 1 \), DV (differential value) and RDV (relative differential value) of Dimensionless Pressure are 1.27 and 25.3%, respectively; and the HAM solution to the three field examples (case 1, 2 and 3) gave 99.96% accuracy; an indication that the HAM solution can be used in predicting reservoir response with ±0.04% error.

The analyses used to compare the solutions of the linear and non-linear models lends better understanding of the effectiveness of HAM and shows that its computation is not as rigorous compared to other methods.

Keywords: Non-Darcy Flow, homotopy, diffusivity, non-linear flow, partial differential equation, ordinary differential equation

1. INTRODUCTION
Nonlinear phenomena play an important role in applied mathematics, engineering and physics. In recent times, significant theoretical advances have been made regarding nonlinear problems. A common nonlinear problem encountered in real life which occurs in the petroleum industry is the flow of fluids in reservoir rocks. This fluid flow is governed by the diffusivity equation, an equation describing the nonlinearities resulting from non-Darcy flow and the dependence of fluid and rock properties such as porosity, permeability, viscosity and fluid density on pressure.

Non-Darcy flow occurs in petroleum reservoirs with high conductivity to flow. Initially it was assumed that this phenomenon was only important in gas wells, but field observations and analyses have shown its relevance to oil wells as well. This was proven by Fetkovitch during a comprehensive field study of 40 oil wells [1]. Non-Darcy flow has been treated as a rate dependent skin factor by including the term “Dq” as an additional source of pressure loss in the vicinity of the wellbore. However, due to non-linearity of the problem, numerical approaches are common while analytical approaches seem to be intractable in most cases.

The basic idea of Homotopy in topology is premised on proposing a general analytic method for non-linear problems namely homotopy analysis method (HAM). The method was proposed by Liao [2]. This method is now widely used to solve many types of nonlinear problems. For example, Khani et al. [3] used HAM for the solutions and efficiency of the non-linear fin problem in which he showed the efficiency of HAM. Abbasbandy [4] applied HAM to solve a generalized Hirota-Satsuma coupled KdV equation. Liao [5] applied HAM for the solutions of the Blasius viscous flow problems. Other researchers have since employed HAM for solutions of seemingly difficult scientific and engineering problems [6-9]. In this paper, we introduce the basic idea of HAM and apply it to find an approximate analytical solution to non-Darcy flow problem in oil reservoir.
MATHEMATICAL FORMULATION

Here we consider a well in a bounded circular drainage area producing at constant rate \( q \). We modelled the non-linear transient flow behavior of well production in an underground formation by combining the continuity equation (material balance equations), non-darcy flow equation and equation of state, then transform the non-linear diffusivity equation (PDE) into dimensionless non-linear ordinary differential equation (ODE) and then apply the Homotopy Analysis Method procedure to solve the non-linear ordinary differential equation (ODE).

**Governing Equation**

\[
\frac{\partial^2 p}{\partial t^2} + G \left[ \frac{\partial p}{\partial r} \right]^2 + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi \mu c \omega k}{\omega k} \frac{\partial p}{\partial t}
\]

with

**Initial condition:**

\[ P = P_i \text{ at } t = 0 \text{ for all } r \]

where \( P_i \) is initial formation pressure in psi

**Well production (inner boundary) condition:**

Based on effective radius,

\[ \frac{kh}{\mu} \left( r \frac{\partial p}{\partial r} \right)_{r_{wa}} = \frac{qB}{2\pi} + C_s \frac{\partial P_w}{\partial t} \text{ for } t > 0 \]

where \( B \) is oil volume factor (dimensionless); \( C_s \) is wellbore storage coefficient, in bbl/Psi; \( P_w \) is wellbore pressure in Psi; \( q \) is well rate at wellhead in bbl/day; \( r_{wa} \) is effective wellbore radius in ft.

\[ r_{wa} = r_w e^{-S} \]

where \( r_w \) is real wellbore radius in ft; \( S \) is skin factor (dimensionless).

**Outer boundary conditions**

\[ \lim_{r_e \to \infty} P = P_i \text{ (infinite acting)} \]

\[ \frac{\partial p}{\partial r} \bigg|_{r=r_e} = 0 \text{ (Finite acting)} \]

The partial differential equation (PDE) equation 1 model the non-linear diffusivity equation of flow in porous media and in developing this model, the only assumptions made is the simplification of the Non-Darcy flow equation to be

\[ \nu = \frac{\omega \frac{\partial p}{\mu}}{\frac{\partial p}{\partial t}} \]

And

\[ \omega = \frac{1}{1 + \frac{\partial p}{\partial t}} \]

The ‘\( \omega \) (the non-Darcyflow factor)’ account for the increased pressure drop due to inertial effect \( \alpha \), as the main cause of non-linearity in the non-Darcy flow equation.

**Transformation of the Mathematical Model**

Introducing the dimensionless variables into eqn 1 and its initial and boundary conditions
\[ P_D = \frac{2\pi kh}{q\mu B} (P_i - P), \quad r_D = \frac{r}{r_w e^{-s}}, \quad r_{eD} = \frac{r}{r_e e^{-s}}, \quad C_D = \frac{C_s}{\phi C_t h r_w^2}, \quad t_D = \frac{kt}{\phi C_t h r_w^2}, \quad T_D = \frac{t_D}{C_D}, \]

\[ \beta = \frac{q\mu B G}{kh} \]

\[ P_D - \text{Dimensionless Pressure}, \quad r_D - \text{Dimensionless radius based on wellbore radius}, \quad r_{eD} - \text{Dimensionless radius based on external boundary}, \quad C_D - \text{Dimensionless wellbore storage coefficient}, \quad t_D - \text{Dimensionless time}, \quad \text{and} \beta - \text{Dimensionless coefficient of squared pressure gradient} \]

**Mathematical Model:**

\[ \frac{\partial^2 P_D}{\partial r_D^2} + \beta \left( \frac{\partial P_D}{\partial r_D} \right)^2 + \frac{1}{r_D} \frac{\partial P_D}{\partial r_D} = \frac{1}{\omega C_D e^{2s}} \left( \frac{\partial P_D}{\partial t_D} \right) \]

**Initial condition**

\[ P_D(r_D, 0) = 0 \text{ at } T_D = 0 \text{ for all } r_D \]

**Well production (inner boundary) condition:**

\[ \frac{\partial P_{wD}}{\partial T_D} - \left( \frac{\partial P_D}{\partial r_D} \right)_{r_D=1} = 1 \]

Where \( P_{wD} \) is dimensionless wellbore pressure.

**Outer boundary conditions**

\[ \lim_{r_{eD} \to \infty} P_D = 0 \text{ (infinite)} \]

\[ \frac{\partial P}{\partial r_{eD} = r_{eD}} = 0 \text{ (closed)} \]

2.1. The Homotopy Analysis Method (HAM) Solution

Equation (9) can be solved using Homotopy analysis method (HAM) but first we have to transform the partial differential equation (PDE) to ordinary differential equation (ODE) using the Boltzmann transformation

**Using Boltzmann transformation**

\[ \lambda = \frac{r_D^2}{4T_D} \]

equation (9) become:

\[ \frac{\partial^2 P_D}{\partial \lambda^2} + \beta \left( \frac{\partial P_D}{\partial \lambda} \right)^2 + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2s}} \right) \left( \frac{\partial P_D}{\partial \lambda} \right) = 0 \]

**Initial condition:**

\[ P_D(\lambda) = 0 \text{ at } \lambda = 0 \]

**Well production (inner boundary) condition:**

\[ \frac{\partial P_{wD}}{\partial \lambda} = -1 \]

Where \( P_{wD} \) is dimensionless wellbore pressure

**Outer boundary conditions**
\[ \frac{\partial P_D}{\partial \lambda} = 0 \] (closed)

In general, the analytic solution of Equation (15) is difficult. As a matter of fact, only if \( \beta = 0 \) and \( \omega = 1 \) will this equation pose an analytical solution. In engineering applications, an explicit approximation with an acceptable accuracy is preferred rather than an exact result in an implicit form with a rigorous proof. As a result, this paper intends to look for a solution algorithm which can generate an explicit approximate solution in terms of ordinary functions.

### 2.1.1 A Review of Ham

The basic idea of HAM which was introduced by Liao [1] is needed to deal with the goal of this work. The following definitions are needed for the start.

**Definition 1.** Let \( \phi \) be a function of the Homotopy-parameter q. Then

\[
D_m(\phi) = \frac{1}{m!} \frac{\partial^m \phi(\lambda, q)}{\partial q^m} |_{q=0} \quad q = 0
\]

is called the \( m \)-th order Homotopy-derivative of \( \phi \), where \( m \geq 0 \) is an integer.

**Definition 2:** For non-linear differential equation as in the case of eqn 16

\[
N[P(\lambda)] = 0
\]

where \( N \) is a non-linear operator, and \( P(\lambda) \) is an unknown function and is temporal independent variable. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way.

**Definition 3: Construction of the homotopy**

\[
\mathcal{H}[\phi(\lambda; q), P_0(\lambda), h, q] = (1-q)L[\phi(\lambda; q) - P_0(\lambda)] - qhN[\phi(\lambda; q)]
\]

where \( q \in [0,1] \) is the homotopy embedding parameter, \( h \) is a non-zero convergence parameter, \( H(\lambda) \) is an auxiliary function (\( H(\lambda) \neq 0 \)), \( L \) is an auxiliary linear operator, \( P_0(\lambda) \) is an initial guess of \( P(\lambda) \) (which satisfies the initial conditions) and \( \phi(\lambda; q) \) is a function of homotopy parameter \( q \)

**The zeroth order deformation equation:**

Setting the homotopy eqn 21 equal to zero,

\[
(1 - q)L[\phi(\lambda; q) - P_0(\lambda)] = qhN[\phi(\lambda; q)]
\]

Thus, as \( q \) varies from 0 to 1, the solution \( \phi(\lambda; q) \) varies from the initial guess \( P_0(\lambda) \) to the solution \( P(\lambda) \).

Expanding \( \phi(\lambda; q) \) using the Taylor series with respect to \( q \) gives the homotopy series;

\[
\phi(\lambda; q) = P_0(\lambda) + \sum_{m=1}^{\infty} P_m(\lambda) q^m
\]

where

\[
P_m(\lambda) = D_m[\phi(\lambda; q)]
\]

If the auxiliary linear operator, the initial guess, auxiliary parameter and auxiliary function are properly chosen, the series eqn 23 converges at \( q=1 \) and we get homotopy series solution;

\[
P(\lambda) = P_0(\lambda) + \sum_{m=1}^{\infty} P_m(\lambda)
\]

Which must be one of the solution of the original non linear equation \( N[P(\lambda)] = 0 \) and equation governing \( P_m \) is called the \( m \)-th order deformation equation.

The \( m \)-th order deformation equation can be deduced from the zero-order deformation eqn 22 as follow:
The mth order deformation equation:

Now, taking the homotopy derivative of both side of the zero-order deformation eqn 29:

\[ D_m[(1-q)\mathcal{L}[\phi(\lambda; q) - P_0(\lambda)]] = D_m[q\hbar H(\lambda)N[\phi(\lambda; q)]] \]

\[ \mathcal{L}[P_m(\lambda) - P_{m-1}(\lambda) + P_0D_m(q)] = \hbar H(\lambda)D_{m-1}(N[\phi(\lambda; q)]) \]

The LHS equals \( \mathcal{L}[P_m(\lambda)] \) when \( m = 1 \) and \( \mathcal{L}[P_m(\lambda) - P_{m-1}(\lambda)] \) when \( m > 1 \) respectively.

Thus, introducing Lemma 1

\[ X_m = \begin{cases} 
0 & \text{if } m \leq 1 \\
1 & \text{if } m > 1 
\end{cases} \]

Therefore the mth order deformation equation is;

\[ \mathcal{L}[P_m(\lambda) - X_mP_{m-1}(\lambda)] = \hbar H(\lambda)D_{m-1}(N[\phi(\lambda; q)]) \]

Let

\[ R_m(P_{m-1}, \lambda) = D_{m-1}(N[\phi(\lambda; q)]) \]

\[ \mathcal{L}[P_m(\lambda) - X_mP_{m-1}(\lambda)] = \hbar H(\lambda)R_m(P_{m-1}, \lambda) = \hbar H(\lambda)D_{m-1}(N(\sum_{n=0}^{\infty} P_n(\lambda) q^n)) \]

It is important to reiterate that \( P_m(\lambda) \) for \( m \geq 1 \) is governed by the linear eqn with the linear boundary condition which comes from the original problem and can be easily solved using symbolic computation software like Mathematica, Maple or Matlab.

2.2 Solution for the Pressure Distribution

In this section, we obtain analytical approximate solutions of Equation (15), and to apply HAM, it is necessary to introduce the Molabahrami and Khani’s Theorems [9].

Molabahrami and Khani [9] proved the following theorems.

**Theorem 3.4:** For homotopy series \( \phi = \sum_{n=0}^{\infty} P_n(\lambda) q^n \) it holds that

\[ D_m(\phi^k) = \sum_{r_1=0}^{m} P_{m-r_1} \sum_{r_2=0}^{r_1} P_{r_2-r_1} \sum_{r_3=0}^{r_2} P_{r_3-r_2} \cdots \sum_{r_k=0}^{r_{k-3}} P_{r_k-r_{k-3}} \sum_{r_{k-2}=0}^{r_k-r_{k-3}} P_{r_{k-2}-r_{k-3}} \sum_{r_{k-1}=0}^{r_k-r_{k-2}} P_{r_{k-1}-r_{k-2}} \]

**Corollary 3.4:** From Theorem 3.4, we have

\[ D_m(\phi^{k-1}\phi') = \sum_{r_1=0}^{m} P_{m-r_1} \sum_{r_2=0}^{r_1} P_{r_2-r_1} \sum_{r_3=0}^{r_2} P_{r_3-r_2} \cdots \sum_{r_k=0}^{r_{k-3}} P_{r_k-r_{k-3}} \sum_{r_{k-2}=0}^{r_k-r_{k-3}} P_{r_{k-2}-r_{k-3}} \sum_{r_{k-1}=0}^{r_k-r_{k-2}} P_{r_{k-1}-r_{k-2}} \]

And

\[ D_m(\phi^{k-1}\phi'') = \sum_{r_1=0}^{m} P_{m-r_1} \sum_{r_2=0}^{r_1} P_{r_2-r_1} \sum_{r_3=0}^{r_2} P_{r_3-r_2} \cdots \sum_{r_k=0}^{r_{k-3}} P_{r_k-r_{k-3}} \sum_{r_{k-2}=0}^{r_k-r_{k-3}} P_{r_{k-2}-r_{k-3}} \sum_{r_{k-1}=0}^{r_k-r_{k-2}} P_{r_{k-1}-r_{k-2}} \]

From the initial condition eqn 23, it is reasonable to express the solution by a set of base functions.
\{e_n(\lambda) | n \geq 0 \}

in the form of:

\[ P(\lambda) = \sum_{n=0}^{\infty} a_n e_n(\lambda) \]

Where \( a_n \) is a coefficient, \( e_n(\lambda) \) is determined to provide the so called rule of solution of expression \( P(\lambda) \)

Under the rule of solution denoted by eqn 44 and from eqn 15, it is straightforward to choose the auxiliary linear operator as

\[ \mathcal{L}[\phi(\lambda; q)] = \frac{\partial^2 \phi(\lambda;q)}{\partial \lambda^2} + \beta \left( \frac{\partial \phi(\lambda;q)}{\partial \lambda} \right)^2 + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) \frac{\partial \phi(\lambda;q)}{\partial \lambda} \]

The solution given by HAM denoted by eqn 29 can be represented by many different base functions. In this work we set \( e_n(\lambda) = \lambda^{2n} \)

According to eqn 15, we define the non-linear operator

\[ N[\phi(\lambda; q)] = \frac{\partial^2 \phi(\lambda;q)}{\partial \lambda^2} + \beta \left( \frac{\partial \phi(\lambda;q)}{\partial \lambda} \right)^2 + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) \frac{\partial \phi(\lambda;q)}{\partial \lambda} \]

From Theorem 3.4 and Corollary 3.4 eqn 28 results in

\[ R_m(\bar{P}_{D_m-1}(\lambda)) = D_{m-1}(\phi''') + \beta D_{m-1}(\phi')^2 + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) D_{m-1}(\phi') \]

For \( m=1 \),

\[ R_1(\bar{P}_{D_0}(\lambda)) = P''_{D_0} + \beta \left( P'_{D_0} \right)^2 + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) P'_{D_0} \]

For \( m=2 \),

\[ R_2(\bar{P}_{D_1}(\lambda)) = P''_{D_1} + \beta \left( 2P'_{D_0}P'_{D_1} \right) + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) P'_{D_1} \]

e.t.c where the primes denotes differentiation with respect to \( \lambda \).

Thus, for eqn 33

\[ R_m(\bar{P}_{D_{m-1}}(\lambda)) = P''_{D_{m-1}} + \beta \left( \sum_{r=0}^{m-1} P'_{D_r} P'_{D_{m-r-1}}(\lambda) \right) + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) P'_{D_{m-1}}(\lambda) \]

Then substitute eqn 34 into eqn 28

\[ \mathcal{L}\left\{ P_{D_m}(\lambda) - X_{m} P_{D_{m-1}}(\lambda) \right\} = \hbar H(\lambda) R_m\left( \bar{P}_{D_{m-1}}(\lambda) \right) + \hbar H(\lambda) \left( P''_{D_{m-1}} + \beta \left( \sum_{r=0}^{m-1} P'_{D_r} P'_{D_{m-r-1}}(\lambda) \right) + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) P'_{D_{m-1}}(\lambda) \right) \]

The inverse linear operator of eqn 35:

\[ P_{D_m}(\lambda) = X_{m} P_{D_{m-1}}(\lambda) + \mathcal{L}^{-1}\left\{ \hbar H(\lambda) \left( P''_{D_{m-1}} + \beta \left( \sum_{r=0}^{m-1} P'_{D_r} P'_{D_{m-r-1}}(\lambda) \right) + \left( \frac{1}{\lambda} + \frac{1}{\omega C_D e^{2S}} \right) P'_{D_{m-1}}(\lambda) \right) \right\} \]

where \( \mathcal{L}^{-1} \) is the inverse of the linear operator (i.e. inverse of differentiation which is integration).
Now, the solution of the $m$th-order deformation eqn 26 under eqn 32 with initial condition $P_m(0) = 0$ and $P_m'(1) = 0$, for $m \geq 1$ is

$$P_m(\lambda) = X_m P_{d_{m-1}}(\lambda) + \int_0^\lambda H(\lambda) \left( P_{''D_{m-1}} + \beta \left( \sum_{r=0}^{m-1} P_{d_r}(\lambda) P_{d_{m-r-1}}(\lambda) \right) + \left( \frac{1}{\lambda} + \frac{1}{\omega C_{D_{m}}^{2\lambda}} \right) P_{d_{m-1}}(\lambda) \right) d\lambda + d_1 \lambda + d_0$$

For simplicity, $\gamma = C_{D} e^{2\lambda}$

$$P_m(\lambda) = X_m P_{d_{m-1}}(\lambda) + \int_0^\lambda H(\lambda) \left( P_{''D_{m-1}} + \beta \left( \sum_{r=0}^{m-1} P_{d_r}(\lambda) P_{d_{m-r-1}}(\lambda) \right) + \left( \frac{1}{\lambda} + \frac{1}{\omega \gamma} \right) P_{d_{m-1}}(\lambda) \right) d\lambda d\lambda + d_1 \lambda + d_0$$

According to eqn 15 and 16 and the rule of solution expression eqn 44, it is straightforward to show that the initial approximation can be written in the form $P_0(\lambda) = \lambda$

According to the rule of solution expression, eqn 30 and eqn 37, the auxiliary function $H(\gamma)$ should be as follows:

$$H(\lambda) = 1$$

Now from eqn 39 or eqn 40, we can successively obtain:

$$P_1(\lambda) = \left( \frac{h(y - \beta \omega)}{2\omega} \right) \lambda^2 + \left( 1 + \frac{h(2 \ln(\lambda) \omega - 2 \omega)}{2\omega} \right) \lambda$$

$$P_2(\lambda) = \left( \frac{h(2 \gamma h - 4 \gamma h \beta \omega + 2 h \beta^2 \omega^2)}{12 \omega^2} \right) \lambda^3$$

$$+ \left( \frac{h(6 \omega \gamma - 6 \beta \omega^2 + 3 h \omega \gamma - 3 h \beta \omega^2 + 6 \gamma h \ln(\lambda) \omega - 6 h \ln(\lambda) \omega^2)}{12 \omega^2} \right) \lambda^2$$

$$+ \left( \frac{h(-12 \omega^2 - 12 \ln(\lambda) \omega + 6 h \ln(\lambda) \omega^2)}{12 \omega^2} \right) \lambda$$

In this way, we derive $P_m(\lambda)$ for $m = 1,2,3 \ldots$ successively

**Convergence of Solution**

The solution series obtained in this way will still have the auxiliary parameter $h$; this solution would be valid for a range of values of $h$. In order to get the optimum value of $h$, the so-called $h$ — curves of the solution are graphed by plotting $P_m(\lambda)$ and/or their first few derivatives determined at a specific value of $\lambda$ against the parameter $h$.

As long as eqn 15 has a unique solution under the given initial and boundary conditions, the partial sums and their derivatives will converge to the correct solution for all values of $h$ for which the solution converges. This means that the curves will be essentially horizontal over the range of $h$ for which the solution converges.
As long as we choose $\zeta$ in this horizontal region $-2 \leq \zeta \leq 2$, our solution must converge to the actual solution of eqn 15.

3. RESULTS AND DISCUSSION
The solution obtained from the Homotopy Analysis Method (HAM) under a set of initial and boundary conditions already defined is validated in figures 1b, 2 and 3 for $\beta = 0$, $\omega = 1.0$, and $\gamma = 10^6$ values of the parameters of the problem to show the behaviour of the solution which satisfy the boundary conditions.

*Fig 1a.  $h - curve for P''' and P''''*

*Fig 1b. Dimensionless Pressure Profile (HAM)*
$P_D$ values against $t_D$ were obtained from the HAM solution by substituting $\lambda = \frac{T_D^2}{4T}$, $T = \frac{t_D}{C_D}$, and $C_D = 10^6$. The $P_D - t_D$ values from HAM were compared with those used in van Everdigen and Hurst Laplace solution and the result are shown in figures 2a and 3a with the residual plot on figures 2b and 3b.

For infinite radial system, constant rate at inner boundary ($r_D = 1$) we obtain Table 1

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<tr>
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<td>3.9534</td>
<td>-0.095</td>
<td>0.095</td>
<td>0.024622</td>
<td>2.46216</td>
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</table>

AVERAGE 0.040587 4.058726

Figure 2a. Exact and HAM solution for an infinite radial system
For finite radial system with closed exterior boundary, constant rate at inner boundary \( r_D = 200 \), we have table 2

**Table 2 Exact and HAM solution for a finite radial system with the Residuals**

<table>
<thead>
<tr>
<th>( t_D )</th>
<th>( P_D ) (FINITE)</th>
<th>( P_D ) (HAM)</th>
<th>ABS ERROR</th>
<th>REL ERROR</th>
<th>% REL ERROR</th>
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</thead>
<tbody>
<tr>
<td>1000</td>
<td>3.858</td>
<td>3.893</td>
<td>-0.035</td>
<td>0.035</td>
<td>0.009072</td>
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<tr>
<td>1200</td>
<td>3.949</td>
<td>3.969</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.005065</td>
</tr>
<tr>
<td>1400</td>
<td>4.026</td>
<td>4.042</td>
<td>-0.016</td>
<td>0.016</td>
<td>0.003974</td>
</tr>
<tr>
<td>1600</td>
<td>4.092</td>
<td>4.101</td>
<td>-0.009</td>
<td>0.009</td>
<td>0.002199</td>
</tr>
<tr>
<td>1800</td>
<td>4.15</td>
<td>4.146</td>
<td>0.004</td>
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<tr>
<td>2000</td>
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<td>-0.067</td>
<td>0.067</td>
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<tr>
<td>2500</td>
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<td>4.352</td>
<td>-0.049</td>
<td>0.049</td>
<td>0.011387</td>
</tr>
<tr>
<td>3000</td>
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<td>4.401</td>
<td>-0.022</td>
<td>0.022</td>
<td>0.005024</td>
</tr>
<tr>
<td>3500</td>
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<td>4.428</td>
<td>0.006</td>
<td>0.006</td>
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<tr>
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<tr>
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<td>0.000437</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>7500</td>
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<td>0.004</td>
<td>0.004</td>
<td>0.000872</td>
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<tr>
<td>8000</td>
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<td>4.613</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.004354</td>
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<tr>
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<td>4.678</td>
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<td>0.017399</td>
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<tr>
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<td>4.836</td>
<td>-0.232</td>
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<tr>
<td>15000</td>
<td>4.605</td>
<td>4.878</td>
<td>-0.272</td>
<td>0.272</td>
<td>0.059283</td>
</tr>
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</table>

**Average**

\[ 0.009827 \text{ 0.982656} \]
Fig 3a. *Exact and HAM solution for a finite radial system*

Fig 3b. *The Absolute error of HAM for Finite case*

Figures 4 to 8 show the effect of the non-linear parameters $\beta, \gamma$ and $\omega$ on the dimensionless pressure profile.
Fig 4. The effect of \( \omega \) on HAM solution
Fig 5. The effect of $\beta$ on HAM solution
Fig 6. The effect of $\beta$ and $\omega$ on HAM solution
Fig 7. The effect of $\beta$ and $\omega$ on HAM solution
Figure 1 demonstrates the pressure distribution in a porous medium (reservoir rock) with coefficient of square of pressure gradient ($\beta$), non-darcy flow factor ($\omega$) and wellbore storage coefficient carrying skin factor ($\gamma$) at $\beta = 0$, $\omega = 1$ and $\gamma = 10^1, 10^2, 10^3, 10^4, 10^5, 10^6$. From this Figure, the efficiency and robustness of the HAM in solving approximately the non-linear flow problem in the reservoir is evident.

Table 3: Theoretical offset of curves caused by the nonlinear term ($\beta = 1$)

| $\Lambda$ | $P_{D\beta=0}$ | $P_{D\beta=1}$ | $DV = |P_{D\beta=0} - P_{D\beta=1}|$ | $RDV = \frac{|P_{D\beta=0} - P_{D\beta=1}|}{P_{D\beta=0}}$ |
|-----------|----------------|----------------|----------------------------------|----------------------------------|
| 0.01      | 0.662959801    | 0.663955766    | 0.000995965                      | 0.150230029                     |
| 0.1       | 2.575599086    | 2.63215675     | 0.056536589                      | 2.195084992                     |
| 1         | 5.016998589    | 6.286850241    | 1.269851652                      | 25.31098284                     |

$Fig. 8$. The effect of $\beta$ and $\omega$ on HAM solution
Figures 2a and 3a shows the comparison between of HAM and the exact solution and from Figure 2b and 3b, it is observed that the results obtained by the HAM and exact solution are in good agreement as shown on the residual plots which confirms that the pressure distribution in the porous medium with coefficient of square of pressure gradient, non-Darcy flow factor and wellbore storage coefficient carrying skin factor can be accurately predicted with the HAM solution with average relative error of ±0.5%.

Figures 4 to 8 show how the Dimensionless pressure curves of nonlinear flow models deviate gradually from those of linear flow models with time. “DV”and “RDV” show the quantitative differences between the two curves where DV is the differential value between linear and nonlinear models and RDV is the relative differential value between linear and nonlinear models. The pressure distribution in the reservoir with coefficient of square of pressure gradient, non-darcy flow factor and wellbore storage coefficient carrying skin factor for different values of β, ω and γ are also shown and from these Figures, an excellent agreement is found between the results of the HAM and the exact solution. It is also observed that the pressure PD(λ) increases as the β, ω and γ increases.

Table 3 show the quantitative differences of nonlinear influence on curves for β = 0.01 β = 0.1 and β = 1 respectively. Dimensionless pressure values in Tables 4.5 was calculated by setting γ = 10^6 and ω = 1, the corresponding curves are shown in Fig. 7. The table show dimensionless pressure differs between linear and nonlinear models. For β = 0.1, λ = 0.01, DV (differential value) and RDV (relative differential value) of pressure are 0.0000995965 and 0.150230029%, respectively; when λ = 0.1, DV and RDV of pressure are 0.056536589 and 2.195084992%, respectively; when λ = 1, DV and RDV of pressure are 1.269851652 and 25.31098284%, respectively. Finally, the validation and application of the HAM solution with 99.96% accuracy is an indication that the Homotopy analysis (HAM) solution can be used in predicting reservoir response with ±0.04% error.

4. CONCLUSION
This study has provided an easy and accurate way of solving non-linear fluid flow problems in reservoir analysis with coefficient of square of pressure gradient β, non-darcy flow factor ω and wellbore storage coefficient carrying skin factor using the homotopy analysis method. It was found that

i. The HAM method accurately predicted the pressure distribution in the reservoir

ii. The HAM method is found to be better than other existing methods in that being able to choose the auxiliary parameter h, helps to adjust and control the convergence of the solution series.

iii. The effect of non-Darcy flow parameter ω on the pressure distribution in the reservoir as investigated in this work showed that the homotopy analysis method is an efficient and capable technique in handling a wide variety of engineering problems.

5. NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
<td>P</td>
<td>Reservoir Pressure</td>
</tr>
<tr>
<td>h</td>
<td>Reservoir Thickness</td>
</tr>
<tr>
<td>r_w</td>
<td>Wellbore radius</td>
</tr>
<tr>
<td>r_e</td>
<td>Drainage radius</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
</tr>
<tr>
<td>K</td>
<td>Permeability</td>
</tr>
<tr>
<td>ϕ</td>
<td>Porosity</td>
</tr>
<tr>
<td>µ</td>
<td>Fluid viscosity</td>
</tr>
<tr>
<td>β</td>
<td>Non- linear Parameter</td>
</tr>
<tr>
<td>ω</td>
<td>Non-Darcy flow factor</td>
</tr>
<tr>
<td>s</td>
<td>Skin factor</td>
</tr>
<tr>
<td>γ</td>
<td>Dimensionless wellbore coefficient carrying skin</td>
</tr>
<tr>
<td>PD</td>
<td>Dimensionless Pressure</td>
</tr>
<tr>
<td>r_D</td>
<td>Dimensionless radius</td>
</tr>
<tr>
<td>t_D</td>
<td>Dimensionless time</td>
</tr>
<tr>
<td>C_D</td>
<td>Dimensionless wellbore coefficient</td>
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h
λ
H(λ)
q
χ_m
D_m
$L$
ℕ

Auxiliary Parameter
Similarity variable
Auxiliary function
Homotopy embedding Parameter
Two-valued function
mth order Homotopy derivative
Linear operator
Non-linear operator

6. ACKNOWLEDGEMENTS
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7. REFERENCE
[1]. M.J. Fetkovich, Decline Curve Analysis Using Type Curves JPT June (1980) 1065-1077