ENHANCING THE EFFICIENCY OF THE RIDGE REGRESSION MODEL USING MONTE CARLO SIMULATIONS

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ABSTRACT

Ridge regression (RR)estimator has been introduced as an alternative to the ordinary least squares estimator (OLS) in the presence of multicollinearity. In the ridge regression analysis, the estimation of ridge parameter k is an important problem. Several algorithms for the ridge parameter have been proposed in the literature. In this paper, a new method for estimating ridge parameter is suggested to solve multicollinearity problem. The investigation of the performance of the proposed estimator has been carried out using Monte Carlo simulations. The results indicate that under certain conditions, the proposed estimator performs better than other well-known estimators.

Keywords: *Ordinary Least Squares; Multicollinearity; Ridge Regression; Monte Carlo Simulation.*

1. INTRODUCTION

The concept of multicollinearity was first introduced by Frisch (1934) (Al-Somahi,A.A.,S.Mousa and L.I.Al-turk ,2015), which defines as the existence of nearly linear dependency among the independent variables in a multiple regression model (EL-Habil, A.M. ,2012).Multicollinearity can cause serious problem in estimation and prediction, increasing the variance of least squares of the regression coefficients and tending to produce least squares estimates that are too large in absolute value (EL-Habil, A.M. ,2012). In this case, the ordinary least squares (OLS) estimator is unsatisfactory due to their large variance

(Al-turk,L.I.,2015), to circumvent such problem, alternative methods are available in the literature; one of them is called the ridge regression (RR) that first proposed by Hoerl and Kennard(1970)(A.V.and D.N. Kashid). The main interest in this approach is to find the ridge parameter that reduces variance or mean square error. Several methods for estimating the ridge parameter have been proposed by many researchers. To mention a few (Hoerl and Kennard,1970a; Hoerl et.al., 1975; Lawless and Wang ,1976; Hocking et. al.,1976; Schaefer, R., L. Roi and R. Wolfe ,1984;Kibria , 2003; Khalaf and Shukur ,2005; Alkhamisi and Shukur, 2007; Dorugade and Kashid,2010; Khalaf, et al.,2012; Dorugade, 2014).

Because multicollinearity is a serious problem. So it is very important to find a better method to deal with this problem. Therefore, the main aim of this paper is to proposed new method for estimating ridge parameter to solve multicollinearity and yields to minimum MSE. A simulation study evaluates the performance of the proposed estimators based on the Mean Squared Error (MSE) criterion and indicates that, under certain conditions, the proposed estimators perform well compared to the OLS estimator and well-known estimator reviewed.

The rest of the paper is organized as follows: Section 2 discusses the methodology. Section 3 describes the Monte Carlo simulation. The results are discussed in Section 4 and finally the conclusions are summed up.

2. METHODOLOGY

The main aim of this paper is to suggest a new method for choosing the ridge parameter k, so this section starts by defining the ridge regression, then the different methods of estimating the ridge parameter k have been presented. Finally, the proposed estimators has been proposed.

2.1 The Ridge Parameter

The general linear regression model can be expressed as:

$$Y = X\beta + e \qquad (1)$$

Where Y is an n×1 vector of observed values of the dependent variable , X is an n× p matrix of the non stochastic values of the explanatory variables , β is a p×1 vector of the coefficient to be estimated, and e is an n×1 vector of random errors, assumed to be distributed $N(0, \sigma^2 I_n)$ (Lin ,K. and J.Kmenta ,1982) , $I_n$is a identity matrix of order n.

The most common estimator for β is the ordinary least squares estimator (OLS) by finding the parameter values which minimize the sum of squared residuals, given by(Lin ,K. and J.Kmenta ,1982; Dorugade, A. V. ,2014)

$\hat{\beta} = (X'X)^{-1}X'Y$ (2)

$\hat{\beta}$ is an unbiased estimator of β. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ denotes the eigenvalues of $X'X$. The mean squared error (MSE) of the components of $\hat{\beta}$ is given by (Yazid, M. H.,2010):

$MSE(\beta) = E(\hat{\beta} - \beta)'(\hat{\beta} - \beta) = \sigma^2 \sum_{i=1}^{p} \frac{1}{\lambda_i}$ (3)

Suppose, there exists an orthogonal matrix D such that $D'CD = \Lambda$,where $\Lambda =$ diagonal $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ are the eigenvalues of the matrix $C = X'X$. The orthogonal (canonical form) version of the multiple regression eq.(1) is (Muniz ,G., B. M. Kibria, K.Mansson and G.Shukur ,2012):

$Y = X^* \alpha + e$ (4)

Where $X^* = X\,D$ and $\alpha = D'\beta$. In case the matrix $X'X$ is ill-conditioned, however (in the sense of there is a near-linear dependence among the columns of the matrix) the OLS estimator of β has a large variance, and multicollinearity is said to be present. Ridge regression replaces $X'X$ with $X'X + kI$, (k > 0). Then the generalized ridge regression estimator of α are given as follows:

$\hat{\alpha}(k) = (X'X + kI)^{-1}X'Y$ (5)

Where k =diag $(k_1, k_2, \ldots, k_p)$, $k_i > 0$ and $\hat{\alpha}$ is the ordinary least squares estimates of α ,I is the identity matrix. When k = 0, the linear regression estimate is given by (2), and when k = 1, $\hat{\alpha}(k) = 0$, it follows from Hoerl and Kennard (1970a) that the value of k_i which minimizes the MSE($\hat{\alpha}(k)$), where:

$\text{MSE}(\hat{\alpha}(k)) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k_i)^2} + \hat{k}^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{(\lambda_i + k_i)^2}$ (6)

When:

$k_i = \frac{\sigma^2}{\alpha_i^2}$ (7)

Where σ^2 represents the error variance of model(1), α_i is the i^th element of α(Kibria, B. M. ,2003; Muniz ,G., B. M. Kibria, K.Mansson and G.Shukur,2012; Hoerl, A. E. and R. W. Kennard,1970a). Hocking,Speed and Lynn (1976) shows that for known optimal k_i, the generalized ridge regression estimator is superior to all other estimators within the class of biased estimators they considered. nevertheless, the optimal value of k_i fully depends on the unknown σ^2, α_i, which must be estimated from the observed data. Hoerl and Kennard (1970) suggested to replace σ^2 and α_i^2 by their corresponding unbiased estimators, That is:

$\hat{k}_i = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$ (8)

Where $\hat{\sigma}^2 = \frac{\sum e_i^2}{n-p}$ is the residual mean square estimate, which is an unbiased estimator of $\sigma^2$and $\hat{\alpha}_i$ is the i^th element of $\hat{\alpha}_i$, which is an unbiased estimator of α.

2.2 Methods for estimating ridge parameter

Several methods for estimating k have been proposed in the literature, some of the well- known methods are listed below.

1- Hoerl and Kennard (1970b) suggested k to be:(Muniz ,G., B. M. Kibria, K.Mansson and G.Shukur,2012; Hoerl,A. E. and R.W. Kennard,1970b)

$$\hat{k}_{HK} = \frac{\hat{\sigma}^2}{\hat{\alpha}_{max}^2} \quad (9)$$

Where $\hat{\alpha}_{max}$ is the maximum element of $\hat{\alpha}$ when σ^2 and $\underline{\alpha}$ are known then $\hat{k}_{HK}$ will give smaller MSE than the OLS.

2-Hoerl et al. (1975), proposed a different estimator of k by taking the harmonic mean of $\hat{k}_i$.That is:

$$\hat{k}_{HKB} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \alpha_i^2} = \frac{p\hat{\sigma}^2}{\hat{\alpha}'\hat{\alpha}} \quad (10)$$

3- Lawless & Wang method (1976) proposed the following estimator:

$$\hat{k}_{LW} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \lambda_i \hat{\alpha}_i^2} \quad (11)$$

4- Hocking, Speed and Lynn (1976) suggested the following estimator for k:

$$\hat{k}_{HSL} = \hat{\sigma}^2 \frac{\sum_{i=1}^{p} (\lambda_i \hat{\alpha}_i)^2}{\left(\sum_{i=1}^{p} \lambda_i \hat{\alpha}_i^2\right)^2} \quad (12)$$

5- Schaefer, R., Roi, L. and Wolfe, R. (1984) proposed the following estimator:

$$\hat{k}_{SRW} = \frac{1}{\hat{\alpha}_{max}^2} \tag{13}$$

6- Kibria(2003). proposed some estimator of *k* based on the generalized ridge regression.They are as follows:

- by using the arithmetic mean of $\hat{k}_i$, which produces the following estimator:

$$\hat{k}_{AM} = \frac{1}{P}\sum_{I=1}^{P}\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \tag{14}$$

- by using the geometric mean of $\hat{k}_i$, which produces the following estimator:

$$\hat{k}_{GM} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{P}\hat{\alpha}_i^2)^{\frac{1}{P}}} \tag{15}$$

-by using the median of $\hat{k}_i$,which produces the following estimator for $p \geq 3$.

$$\hat{k}_{MED} = Median\left\{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right\}, i = 1,2,3,\dots,p \tag{16}$$

7- Khalaf and Shukur (2005) suggested a new approach for choosing the ridge parameter *k* as:

$$\hat{k}_{KS} = \frac{(\lambda_{max})\hat{\sigma}^2}{(n-p-1)\hat{\sigma}^2 + \lambda_{max}(\hat{\alpha}_{\max})^2} \tag{17}$$

Where λ_{max} is the maximum eigenvalue of the matrix $X'X$

8- Alkhamisi, M., and G.Shukur. (2007) suggested a new approach for choosing the ridge parameter *k* as:

$$\hat{k}_{AS} = max\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} + \frac{1}{\lambda_i}\right) \qquad i = 1,2,\dots,p \tag{18}$$

where $VIF_j = \frac{1}{1-R_j^2}$, *j=1,2,...,p* is variance inflation factor of j^{th} regressor.

9-Dorugade, A.V. (2014) suggested a new approach for choosing the ridge parameter k as:

$$\hat{k}_{\sigma} = \frac{\hat{\sigma}^2}{min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} + \frac{1}{\lambda_i}\right)} \qquad i = 1,2,\dots,p \tag{19}$$

2.3 The Proposed Estimator

In this section, a new estimator called as $\hat{k}_{KSLW}$ will be proposed, the method is considered a modification of two others ridge parameters proposed, the $\hat{k}_{KSLW}$ is a modification of the ridge parameter, which is suggested by Lawless & Wang (1976) eq.(11) and Khalaf and Shukur (2005) eq.(17)

The main goal is to give new estimators with smaller MSE value compared with other previously ridge estimators. The modified ridge parameters is given by the following formulas:

$$\hat{k}_{KSLW} = \left(\frac{(\lambda_{max})\hat{\sigma}^2}{(n-p-1)\hat{\sigma}^2 + \lambda_{max}(\hat{\alpha}_{\max})^2}\right)^p \Big/ \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\lambda_i\hat{\alpha}_i^2} \tag{20}$$

Where: p is the number of explanatory variables in the model.

3. Simulation Study

The main aim of this paper is to propose a new method for estimating the ridge parameter *k.* A Monte Carlo comparison will be made using the MSE criterion to compare the performances of the proposed estimator with the OLS estimator and the well known estimator reviewed. A simulation study has been conducted using R Program.

The simulation procedure suggested by McDonald and Galarneau (1975), Gibbons (1981) and Kibria (2003) was used to generate the explanatory variables(Kibria, B. M.,2003):

$$x_{ij} = (1-\gamma^2)^{1/2} z_{ij} + \gamma z_{ip}, i = 1,2,\dots,n, j = 1,2,\dots,p \tag{21}$$

where z_{ij} are independent standard normal pseudo-random numbers, γ is specified so that the correlation between any two explanatory variables is given by γ^2 , and p is the number of explanatory variables.

The: *n* observations for the dependent variable y are determined by:

$$y_i = \beta_0 + \beta_1 x_{i1} +, \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + e_i \quad i = 1,2,\dots,n \tag{22}$$

Where e_i are independent normal (0, σ^2) pseudo-numbers and β_0 is taken to be identically zero.

Four different sets of correlation are considered corresponding to γ = 0.7, 0.8,0.9 and 0.99, the other factors varied were sample size (*n*) and the number of explanatory variables (*p*). Models consisting of 30,50,100 and 200 observations with 2,3,4 and 5 explanatory variables, the experiment is repeated 1000 times and obtains the average MSE (AMSE) Using eq.(3) for OLS estimator and eq.(6) for ridge estimator.

4. The Results

In this section, the results from our Monte Carlo study are presented in Tables 1–4. Those Tables show the effects of changing p, n, and γ on the performance of (OLS) ($\hat{k}_{KSLW}$), and other ridge estimators.

According to the simulation study, many conclusion can be drawn on the performance of the ($\hat{k}_{KSLW}$)and different ridge estimators. These conclusions can be summarized as follows:

1-the performance of the proposed ridge parameter($\hat{k}_{KSLW}$)is much better than other ridge parameters for all combinations of numbers of predictors (p), correlation between predictors(γ) and sample size (n)used in this simulation study.

2- When the sample size increases, the AMSE decreases for the ($\hat{k}_{KSLW}$)for all combinations of (p) and (γ) which indicate that increasing the sample size (n) enhance the performance of the estimator ($\hat{k}_{KSLW}$)

3- The high degree of correlation γ =0.99 causes the average AMSEs to be increased for all the estimators.

CONCLUSION

Several procedures for constructing ridge estimators have been proposed in the literature. In this paper , new method to obtain an estimator of the ridge parameter *k* is suggested, The performance of the proposed ridge parameter was evaluated and compared with OLS estimator and the well-known ridge estimators through the Monte Carlo simulation, for different combinations of correlation among the explanatory variables, the number of explanatory variables and sample size . For each combination, 1000 replications have been done. The results from the simulation study show that The performance of the new proposed ridge estimators based on AMSE criteria is shown to be better than the previously suggested ridge estimators for all different situations.

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APPENDICES

Table 1: Simulated AMSE of for different γ and n with p=2

n	γ	$\hat{k}_{OLS}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{SRW}$	$\hat{k}_{LW}$	$\hat{k}_{HSL}$	$\hat{k}_{GM}$	$\hat{k}_{MED}$	$\hat{k}_{KS}$	$\hat{k}_{AS}$	$\hat{k}_{\sigma}$	$\hat{k}_{KSLW}$
30	0.7	0.0266	0.0006	0.0017	0.0006	0.2560	0.0123	0.0078	0.0014	0.0006	0.0004	0.2687	0.0080	0.0003
	0.8	.0298	0.0005	0.0016	0.0006	0.2216	0.0107	0.0021	0.0017	0.0005	0.0004	0.3089	0.0079	0.0002
	0.9	0.0427	0.0008	0.0024	0.0016	0.1871	0.0113	0.0011	0.0032	0.0008	0.0006	0.3921	0.0079	0.0003
	0.99	0.2994	0.0129	0.0231	0.3942	0.1731	0.0446	0.0204	0.0282	0.0207	0.0101	0.7222	0.0079	0.0023
50	0.7	0.0144	0.0002	0.0001	0.0002	0.1480	0.0002	0.0043	0.0004	0.0002	0.0001	0.2644	0.0079	0.00009
	0.8	0.0152	0.0001	0.0001	0.0001	0.1354	0.0001	0.0008	0.0004	0.0001	0.0001	0.3023	0.0079	0.000087
	0.9	0.0206	0.00016	0.0001	0.0002	0.1207	0.0001	0.0002	0.0007	0.0002	0.0001	0.3966	0.0079	0.000107
	0.99	0.1337	0.0029	0.0027	0.1145	0.1180	0.0031	0.0031	0.0099	0.0031	0.0022	0.6995	0.0079	0.00160
100	0.7	0.0065	0.00004	0.0001	0.0003	0.1490	0.00004	0.0025	0.00008	0.00003	0.00002	0.2652	0.0079	0.000019
	0.8	0.0067	0.00002	0.00008	0.00003	0.1344	0.00002	0.0003	0.00008	0.00002	0.00002	0.3029	0.00793	0.000017
	0.9	0.0087	0.00002	0.00009	0.00004	0.1184	0.00002	0.00004	0.00012	0.00002	0.00002	0.3946	0.00793	0.000018
	0.99	0.0476	0.00033	0.00130	0.00082	0.1089	0.00038	0.00038	0.0021	0.00034	0.00027	0.6680	0.00799	0.00022
200	0.7	0.0031	0.000007	0.00002	0.000008	0.1487	0.000009	0.0018	0.00002	0.000007	0.000006	0.2649	0.0079	0.000004
	0.8	0.0031	0.000005	0.000018	0.000007	0.1341	0.000006	0.00021	0.00001	0.000005	0.000004	0.30375	0.00792	0.000003
	0.9	0.0038	0.000045	0.000017	0.000008	0.1189	0.000004	0.00001	0.00002	0.000004	0.000004	0.39627	0.00792	0.000003
	0.99	0.0183	0.00003	0.0002	0.0001	0.1070	0.00004	0.00004	0.0004	0.00004	0.00003	0.66625	0.0079	0.000029

Table 2: Simulated AMSE of for different γ and n with p=3

n	γ	$\hat{k}_{OLS}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{SRW}$	$\hat{k}_{LW}$	$\hat{k}_{HSL}$	$\hat{k}_{GM}$	$\hat{k}_{MED}$	$\hat{k}_{KS}$	$\hat{k}_{AS}$	$\hat{k}_{\sigma}$	$\hat{k}_{KSLW}$
30	0.7	0.03940	0.00062	0.00149	0.0084	0.21629	0.00085	0.00533	0.00389	0.3199	0.00049	0.2422	0.00795	0.000359
	0.8	0.04802	0.00067	0.00162	0.5548	0.1712	0.00082	0.00143	0.00697	0.3845	0.00054	0.26567	0.00794	0.000413
	0.9	0.07670	0.00106	0.00235	0.03966	0.13237	0.00117	0.001162	0.01427	0.50083	0.00087	0.35209	0.00794	0.00065
	0.99	0.631485	0.02420	0.03914	1.56695	0.1719	0.03567	0.03533	0.06548	0.63375	0.02115	0.75020	0.00794	0.005020
50	0.7	0.01955	0.00015	0.00039	0.00021	0.10938	0.00020	0.002756	0.00088	0.22456	0.00012	0.22715	0.00793	0.0000891
	0.8	0.2192	0.00012	0.00032	0.00028	0.08981	0.00015	0.000382	0.00134	0.30110	0.00011	0.24787	0.00793	0.0000854
	0.9	0.03577	0.00021	0.00054	0.03528	0.07434	0.00023	0.000244	0.00368	0.4257	0.00018	0.32812	0.00793	0.000140
	0.99	0.26917	0.00441	0.00930	0.9612	0.08619	0.00572	0.00568	0.02625	0.5947	0.00372	0.68165	0.00793	0.000988
100	0.7	0.00815	0.00002	0.0000681	0.00003	0.10796	0.00003	0.001445	0.00015	0.11349	0.000021	0.21468	0.00793	0.0000152
	0.8	0.00887	0.00001	0.0000537	0.00003	0.08918	0.000023	0.00011	0.00020	0.17297	0.0000182	0.235	0.00793	0.0000137
	0.9	0.01277	0.00002	0.0000594	0.00007	0.07199	0.000024	0.00002	0.00047	0.2938	0.000021	0.31471	0.00792	0.0000168
	0.99	0.0930	0.00035	0.00122	0.42208	0.06593	0.00036	0.00035	0.00855	0.50210	0.000314	0.6416	0.00792	0.000117
200	0.7	0.00367	0.0000050	0.0000137	0.00000	0.10726	0.000006	0.00093	0.00003	0.04684	0.0000044	0.2073	0.00792	0.0000030
	0.8	0.00390	0.0000037	0.00001	0.00000	0.088260	0.0000044	0.000058	0.00003	0.08327	0.0000034	0.22750	0.00792	0.0000025
	0.9	0.00527	0.0000036	0.0000096	0.00001	0.07178	0.000003	0.000005	0.00008	0.16179	0.0000035	0.30887	0.00792	0.0000028
	0.99	0.03384	0.0000337	0.000147	0.0043	0.06060	0.0000342	0.000033	0.00239	0.3514	0.0000315	0.63082	0.00792	0.0000152

Table 3: Simulated AMSE of for different γ and n with p=4

n	γ	$\hat{k}_{OLS}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{SRW}$	$\hat{k}_{LW}$	$\hat{k}_{HSL}$	$\hat{k}_{GM}$	$\hat{k}_{MED}$	$\hat{k}_{KS}$	$\hat{k}_{AS}$	$\hat{k}_{\sigma}$	$\hat{k}_{KSLW}$
30	0.7	0.05380	0.00073	0.00218	0.5354114	0.08629	0.00102	0.00366	0.008607	0.5293	0.00060	0.22114	0.00795	0.000483
	0.8	0.06623	0.00073	0.00210	0.23879	0.06877	0.00089	0.00108	0.01345	0.5455	0.00064	0.2376	0.00796	0.000554
	0.9	0.111610	0.00134	0.00343	0.67085	0.06296	0.00150	0.00146	0.026277	0.6222	0.00116	0.33161	0.00795	0.000925
	0.99	0.99624	0.03409	0.06242	4.574979	0.18560	0.04983	0.049134	0.110879	0.8596	0.03119	0.84316	0.00794	0.008317
50	0.7	0.02467	0.00014	0.00050	0.000538	0.08339	0.00021	0.001519	0.001946	0.5163	0.00013	0.19573	0.00793	0.000103
	0.8	0.02950	0.00014	0.00044	0.00972	0.06610	0.00016	0.000222	0.003618	0.5434	0.00013	0.2135	0.00793	0.000113
	0.9	0.05461	0.00023	0.00064	0.14951	0.05317	0.00024	0.000233	0.010000	0.6404	0.00022	0.33414	0.00793	0.000197
	0.99	0.44125	0.00798	0.01957	1.95744	0.09395	0.01143	0.01130	0.04838	0.7714	0.00697	0.7085	0.00793	0.001705
100	0.7	0.00730	0.00001	0.0000417	0.00002	0.05539	0.00001	0.000553	0.000148	0.4965	0.00001	0.182692	0.00792	0.000008
	0.8	0.00724	0.000007	0.0000231	0.000017	0.04585	0.000008	0.000028	0.000177	0.5401	0.000007	0.19786	0.00792	0.000005
	0.9	0.00850	0.000004	0.000013	0.000019	0.03797	0.000005	0.000005	0.000297	0.6109	0.0000049	0.27949	0.00792	0.000004
	0.99	0.02368	0.000004	0.000020	0.000100	0.03254	0.00000	0.000004	0.00148	0.7276	0.0000045	0.64575	0.00792	0.000003
200	0.7	0.41565	0.00611	0.01545	3.5088	0.08526	0.00828	0.008193	0.043191	0.7723	0.005400	0.70710	0.00793	0.001505
	0.8	0.413101	0.00639	0.01537	2.46115	0.08573	0.00689	0.006800	0.04326	0.7832	0.005619	0.71580	0.00793	0.001571
	0.9	0.42284	0.00720	0.01729	4.85026	0.09119	0.00924	0.00913	0.045310	0.7809	0.006312	0.783115	0.00793	0.001588
	0.99	0.4249721	0.00661	0.016489	6.34749	0.08665	0.00912	0.00901	0.042605	0.7571	0.005810	0.69684	0.00793	0.001559

Table 4: Simulated AMSE of for different γ and n with p=5

n	γ	$\hat{k}_{OLS}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{AM}$	$\hat{k}_{SRW}$	$\hat{k}_{LW}$	$\hat{k}_{HSL}$	$\hat{k}_{GM}$	$\hat{k}_{MED}$	$\hat{k}_{KS}$	$\hat{k}_{AS}$	$\hat{k}_{\sigma}$	$\hat{k}_{KSLW}$
30	0.7	0.069659	0.00081	0.00211	0.06804	0.06915	0.00113	0.00280	0.01513	0.502524	0.00071	0.20293	0.00794	0.0006202
	0.8	0.087324	0.00088	0.00209	0.49573	0.05644	0.00104	0.00110	0.02230	0.539373	0.00081	0.22565	0.00794	0.0007468
	0.9	0.161502	0.00189	0.00415	1.38241	0.05338	0.00209	0.00199	0.03708	0.629678	0.00167	0.32895	0.00794	0.0014424
	0.99	1.537037	0.05510	0.09503	4.40915	0.26970	0.08170	0.08063	0.18142	1.005335	0.05149	0.96081	0.00794	0.013677
50	0.7	0.030197	0.00016	0.00048	0.00116	0.06674	0.00022	0.00083	0.00036	0.496824	0.00014	0.17430	0.00793	0.0001232
	0.8	0.037179	0.00016	0.00043	0.01838	0.05201	0.00019	0.00021	0.00657	0.532993	0.00015	0.19270	0.00793	0.000142
	0.9	0.064308	0.00029	0.00071	0.11882	0.04218	0.00030	0.00029	0.01518	0.613194	0.00027	0.28101	0.00793	0.0002505
	0.99	0.588107	0.00984	0.02164	3.8896	0.09866	0.01380	0.01360	0.06115	0.814827	0.00887	0.73119	0.00793	0.002263
100	0.7	0.011097	0.00002	0.00006	0.0000443	0.06523	0.00002	0.00023	0.00050	0.497586	0.000019	0.15395	0.00793	0.0000162
	0.8	0.005203	0.000003	0.00000	0.0000082	0.04820	0.000003	0.000006	0.00015	0.526178	0.000002	0.15332	0.00792	0.0000026
	0.9	0.02183	0.00003	0.00065	0.079173	0.03677	0.00003	0.00030	0.00309	0.60574	0.000302	0.24983	0.00792	0.0000284
	0.99	0.187931	0.00062	0.00240	0.4710499	0.04289	0.00063	0.00062	0.01934	0.749923	0.000570	0.65488	0.00792	0.000246
200	0.7	0.004473	0.000003	0.00001	0.000006	0.06509	0.000004	0.00012	0.00007	0.496240	0.000003	0.14263	0.00792	0.0000025
	0.8	0.005203	0.000003	0.0000078	0.0000082	0.04820	0.000003	0.000006	0.00015	0.526178	0.000002	0.15332	0.00792	0.0000026
	0.9	0.007807	0.00003	0.000007	0.036644	0.03565	0.000003	0.0000039	0.00048	0.603479	0.000003	0.22977	0.00792	0.0000035
	0.99	0.064584	0.00003	0.000162	0.0868530	0.03043	0.000039	0.000039	0.00655	0.743056	0.00003	0.63897	0.00792	0.0000286