EULER-LAGRANGIAN EQUATIONS WITH KÄHLER-EINSTEIN METRIC AND EQUAL KÄHLER ANGLES ON FANO MANIFOLDS

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ABSTRACT
It is well-known that a classical field theory deals with the general idea of a quantity and it is a function of time and space. Classical field theory interested in classical mechanics. Classical mechanics explains the motion style of object with Euler-Lagrange equations. It is well-known that a classical field theory explain the study of how one or more physical .elds interact with matter which is used quantum and classical mechanics of physics branches. This manuscript set forth an attempt to introduce Lagrangian formalism for mechanical systems using with Kähler-Einstein metrics on Fano manifolds which represent an interesting multidisciplinary field of research. In this study, we will obtain the geodesic equations of moving objects on Fano manifolds. As a result of this study, partial differential equations will be obtained for movement of objects in space and solutions of these equations will be made using the Maple computation program. In addition to, the geometrical-physical results related to on Kähler-Einstein mechanical systems are also given.

Keywords: *Kähler-Einstein, Lagrangian, Mechanical System, Fano Varieties.*

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1. INTRODUCTION

A Kähler–Einstein metric on a complex manifold is a Riemannian metric that is both a Kähler metric and an Einstein metric. A manifold is said to be Kähler–Einstein if it admits a Kähler–Einstein metric. The most important problem for this area is the existence of Kähler–Einstein metrics for compact Kähler manifolds. In the case in which there is a Kähler metric, the Ricci curvature is proportional to the Kähler metric. Therefore, the first Chern class is either negative, or zero, or positive. Kähler–Einstein metric (or Einstein metric) is a Kähler metric on a complex manifold M^n whose Ricci curvature tensor is proportional to the metric tensor. This proportionality is an analog of the Einstein field equation in the general theory of relativity. A Kähler–Einstein manifold (or Einstein manifold) is a complex manifold equipped with a Kähler–Einstein metric. In this case the Ricci curvature tensor, considered as an operator on the tangent space, is just multiplication by a constant. De Leon presented many studies about Lagrangian dynamics, mechanics, formalisms, systems and equations [1]. *Tian* submitted an expository paper on Kähler metrics of positive scalar curvature [2]. *Vries* shown that the Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics [3]. Paracomplex analogue of the Euler-Lagrange equations was obtained in the framework of para-Kählerian manifold and the geometric results on a paracomplex mechanical systems were found by *Tekkoyun* [4]. Electronic origins, molecular dynamics simulations, computational nanomechanics, multiscale modelling of materials fields were contributed by *Liu* [5]. *Chen* et al. provided that any compact complex surface with $c_1 > 0$ admits an Einstein metric which is conformally related to a Kähler metric [6]. *Spotti* investigated how Fano manifolds equipped with a Kähler-Einstein metric can degenerate as metric spaces and some of the relations of this question with algebraic geometry [7]. *Le Brun* showed that $CP_2\#2\overline{CP_2}$ admits an Einstein metric [8]. *Heier* carry out Nadel's method of multiplier ideal sheaves to show that every complex del Pezzo surface of degree at most six whose automorphism group acts without fixed points has a Kähler–Einstein metric [9]. *Li* solved a folklore conjecture, it is often referred as the Yau-Tian-Donaldson conjecture, on Fano manifolds without nontrivial holomorphic vector fields [10]. *Coevering* gave many examples of Kähler-Einstein strictly pseudo convex manifolds on bundles and resolutions [11]. *Roček* had studied the relationship between the curvature of a Kähler-Einstein manifold with Kähler potential K and the curvature of the base manifold [12]. *Nadel* introduced a coherent sheaf of ideals and showed that it satisfies various global algebrogeometric conditions, including a cohomology vanishing theorem [13]. *Chen* and *Tian* proved that if M is a Kähler-Einstein surface with positive scalar curvature, if the initial metric has nonnegative sectional curvature [14]. *Koiso* and *Sakane* had considered $P^1(\mathbb{C})$-bundles over compact Kähler-Einstein manifolds to obtain non-homogeneous Kähler-Einstein manifolds with positive Ricci tensor [15]. *Alekseevskya* et al. examined that a para-Kähler manifold can be defined as a pseudo-Riemannian manifold (M, g) with a parallel skew-symmetric paracomplex structures K [16]. *Salavessa* and *Valli* considered $F: M \to N$ a minimal submanifold M of real dimension $2n$, immersed into a Kähler–Einstein manifold N of complex dimension $2n$, and scalar curvature R [17].

Kasap introduced Weyl-Euler-Lagrange equations of motion on flat manifold [18].

2 PRELIMINARIES

Definition 1. Let M be a differentiable manifold of dimension $2n$, and suppose J is a differentiable vector bundle isomorphism $J: TM \to TM$ such that $J_p: T_pM \to T_pM$ is a **complex structure** for T_pM, i.e. $J^2 = -I$ where I is the identity vector bundle isomorphism. Then J is called **an almost-complex structure** for the differentiable manifold M. A manifold with a fixed almost-complex structure is called an **almost-complex manifold.**

Theorem 1. If M is a smooth manifold of real dimension $2n$, then a smooth field $J = (J_x)$ of complex structures on TM is called an almost complex structure of M. An almost complex structure $J = J_x$ is called a complex structure if it comes from a complex structure on M as in $J_x X_j(x) = Y_j(x)$, $J_x Y_j(x) = -X_j(x)$. Any almost complex structure on a surface is a complex structure.

Theorem 2. A celebrated theorem of *Newlander* and *Nirenberg* says that an almost complex structure is a complex structure if and only if its *Nijenhuis* tensor, $N_J \in \wedge^2 T^*M \otimes TM$ of an almost complex structure $J \in T^*M \otimes TM$, or torsion N_J vanishes

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y], \qquad X,Y \in TM, \tag{1}$$

where the right hand side is calculated for arbitrary vector fields X,Y with the given values $X_p, Y_p \in T_apM$ at the point $p \in M$. In coordinates it has the formula: $N^i_{jk} = J^i_j \partial_l J^i_k - J^l_k \partial_l J^i_j - J^i_l \partial_j J^l_k + J^i_l \partial_k J^l_j$. Also, $N_J(X,Y) = (\nabla_X J)(JY) - (\nabla_Y J)(JX) + (\nabla_{JX} J)(Y) - (\nabla_{JY} J)(X)$, where ∇ (i.e. such a connection that $\nabla J = 0$) is any symmetric connection on M [19, 20].

Definition 2. Let M be a smooth manifold of dimension $n \geq 3$. Let ∇ be its Levi-Civita connection, a torsion free connection on the tangent bundle TM of M and let

$$g = <.,.>, \tag{2}$$

be a **pseudo-Riemann metric** on M of signature (p,q). (M,g) be called the **pseudo-Riemannian manifold** [21].

Definition 3. The **Ricci(curvature) tensor** r of a pseudo-Riemannian manifold (M,g) is the 2-tensor

$$r(X,Y) = tr(Z), \tag{3}$$

where tr donates the trace of the linear map $Z \to R(X,Z)Y$. Note that the Ricci tensor is symmetric.

Definition 4. A **Riemannian manifold** (M,g) consists of the following data: a compact C^∞ manifold M. A metric tensor field g which is a positive definite bilinear symmetric differential form on M. In other words, we associate with every point p of M a Euclidean structure g_p on the tangent space T_pM of M at p and require the association $p \to g_p$ to be C^∞. We say that g is a **Riemannian metric** on M. **A pseudo-Riemannian manifold** (also called a semi-Riemannian manifold) (M,g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g. Also, it is generalization of a Riemannian manifold in which the metric tensor need not be positive-definite. A pseudo-Riemannian manifold (M,g) is **Einstein manifold if there exists a real constant** λ such that

$$r(X,Y) = \lambda g(X,Y), \tag{4}$$

for $\forall p \in M$, $\forall X,Y \in T_pM$.

Theorem 3. Assume $n \geq 3$. Then an n-dimensional **pseudo-Riemannian manifold is Einstein** if and only if, for each p in M, there exists a constant λ_p such that

$$r_p = \lambda_p g_p. \tag{5}$$

Proof: The " only if" part is trivial. In the other direction, applying the divergence δ to both sides of (5), we get $\delta r = -\frac{1}{2} ds = -d\lambda$. So $\lambda - \frac{1}{2} s$ is a constant. Taking the trace of (5) with respect to g, we get $n\lambda = s$. So finally λ (and s) are constant [22].

A **pseudoholomorphic curve** (J-holomorphic curve) is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy–Riemann equation [23]. A closed two-form ω on a complex manifold M which is also the negative imaginary part of a Hermitian metric $h = g - i\omega$ is called a Kähler form. In this case, M is called a Kähler manifold and g, the real part of the Hermitian metric, is called a **Kähler metric**. The **Kähler form** combines the metric and the complex structure, indeed

$$g(X,Y) = \omega(X,JY), \tag{6}$$

where J is the almost complex structure. Since the Kähler form comes from a Hermitian metric, it is preserved by J, i.e., since $h(X,Y) = h(JX,JY)$. The equation $d\omega = 0$ implies that the metric and the complex structure are related. It gives M a Kähler structure, and has many implications. A Kähler metric g on a complex manifold M is Einstein if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\rho = \lambda\omega, \tag{7}$$

where ω is the fundamental form associated to g and

$$\rho(X,Y) = Ric(X,JY), \tag{8}$$

for $X,Y \in \chi(M)$. The pair (M,g), where M is a complex manifold and g a **Kähler-Einstein metric** is said a **Kähler-Einstein manifold** [24]. A **Kähler–Einstein metric** on a complex manifold is a Riemannian metric that is both a Kähler metric and an Einstein metric. A manifold is said to be Kähler-Einstein if it admits a Kähler-Einstein metric. A Kähler metric on a complex manifold whose Ricci tensor $Ric(\omega)$ is proportional to the metric tensor:

$$Ric(\omega) = \lambda\omega. \tag{9}$$

Let M be a complex manifold with complex structure J and compatible **Riemannian metric** $g =<.,.>$ as in $< JX,JY >=< X,Y >$, where X and Y any two vector fields. The alternating 2-form

$$\omega(X,Y) := g(JX,Y), \tag{10}$$

is called the associated Kähler form. We can retrieve g from ω,

$$g(X,Y) = \omega(X,JY). \tag{11}$$

We say that g is a Kähler metric and that M is a **Kähler manifold** if ω is closed and (M,g) is displayed in the form. Let M be a complex manifold. A Riemannian metric on M is called Hermitian if it is compatible with the complex structure J of M, $< JX,JY >=< X,Y >$. Then the associated differential two-form ω defined by

$$\omega(X,Y) =< JX,Y >, \tag{12}$$

is called the **Kähler form**. It turns out that ω is closed if and only if J is parallel. Then M is called a Kähler manifold and the metric on M a Kähler metric. Kähler manifolds are modelled on complex Euclidean space.

Theorem 4. Let M be a compact connected complex manifold and $c_1(M)_{\mathbb{R}}$ its first Chern class; if $c_1(M)_{\mathbb{R}} > 0$, M is Fano manifold, then M carries a unique (**Ricci-positive**) **Kähler–Einstein metric** ω such that for $\lambda = 1$,

$$Ric(\omega) = \omega. \tag{13}$$

In algebraic geometry, a Kähler manifold M with $c_1(M)_{\mathbb{R}} > 0$ is called a **Fano manifold** [25, 26].

3. THE KÄHLER ANGLE

The principal or canonical angles (and the related principal vectors) between two subspaces provide the best available characterization of the relative subspace positions. In any (finite-dimensional) real (Euclidean) vector space $V_{\mathbb{R}}(\simeq \mathbb{R}_m, m \in \mathbb{N},\ m \geq 2)$ equipped with the scalar product $< X,Y >_{\mathbb{R}} = \sum_{k=1}^{m} X_k Y_k$ for any pair of vectors $X,Y \in V_{\mathbb{R}}$ one can define an (real) angle $\theta < X,Y >$, $0 \leq \theta \leq \pi$, between these two vectors by means of the standard formula

$$cos\theta < X,Y >= \frac{<X,Y>_{\mathbb{R}}}{\|X\|\|Y\|}. \tag{14}$$

The Kähler Angle: In order to proceed further let us introduce the almost complex structure J, $J^2 = -I$, which acts as an operator in the real vector space $V_{\mathbb{R}}$ isometric to $V_{\mathbb{C}}$. In our coordinates the almost complex structure J performs the following transformations: $X_{2k-1} \to X_{2k}$, $X_{2k} - X_{2k-1}$, $k = 1,\ldots,n$. This is equivalent to the transformation $x \to ix$ in $V_{\mathbb{C}}$. A subspace P of $V_{\mathbb{R}}$ is called holomorphic, if it holds $P = JP$. It is called antiholomorphic (totally real, with a real Hermitian product), if it holds $P \perp JP$. Following the convention applied in a large fraction of the literature we introduce the notation $\tilde{X} = JX$, $X \in V_{\mathbb{R}}$. By writing

$$cos\theta_K < x,y >. sin\theta < x,y >= cos\theta_K < X,Y >. sin\theta < X,Y >= \frac{<\tilde{X},Y>_{\mathbb{R}}}{\|X\|\|Y\|}, \tag{15}$$

one can now introduce a further angle $\theta_K < x,y >= \theta_K < X,Y >$, $0 \leq \theta_K \leq \pi$, which is called the **Kähler Angle** between the vectors $x,y \in V_{\mathbb{C}}$ or the vectors $X,Y \in V_{\mathbb{R}}$, respectively [27].

Definition 5. Let N be a Kähler manifold with the complex structure J and the standard Kähler metric $<.,.>$, let M

be a Riemann surface; and let $\Psi: M \to N$ be an isometric minimal immersion of M into N. Then the Kähler angle θ of Ψ which is an invariant of the immersion Ψ related to J, is defined by

$$cos(\theta) = < Je_1, e_2 >. \tag{16}$$

where $\{e_1, e_2\}$ is an orthonormal basis of M [28].

4. HOLOMORPHIC PROPERTIES OF $F^*\omega$

Let (N, J, g) be a Kähler manifold of complex dimension $2n$ and g is a Kähler metric. Also $F: M \to N$ an immersed submanifold of real dimension $2n$ and minimal submanifold M. We denote by ω the Kähler form and $x, y \in \chi(M)$:

$$\omega(x, y) = g(Jx, y). \tag{17}$$

We take the induced metric on M

$$g_\mu = F^* g. \tag{18}$$

N is Kähler-Einstein manifold if its Ricci tensor is a multiple of the metric, $Ricci^N = Rg$. At each point $p \in M$, we identify $F^*\omega$ with a skew-symmetric operator of T_pM by using the musical isomorphism with respect to g_μ namely

$$g_\mu(F^*\omega(x), y) = F^*\omega(x, y). \tag{19}$$

We take its polar decomposition

$$F^*\omega = \tilde{g} J_\omega, \tag{20}$$

where $J_\omega: T_pM \to T_pM$ is a partial isometry with the same kernel κ_ω as of $F^*\omega$, and where $\tilde{g}$ is the positive semi-definite operator

$$\tilde{g} = |F^*\omega| = \sqrt{-(F^*\omega)^2}. \tag{21}$$

The holomorphic structures obtaining: Let's take a Kähler-Einstein metric g. If X and Y are orthonormal basis on M then $cos(\theta) = < JX, Y >$ according to (14) and (16). Also, $\omega(X, Y) = g(JX, Y) = < JX, Y >$ at (12) and (10). $\rho = Ric(\omega) = \omega$ for first Chern class ($\lambda = 1$) (7):

$$\rho = Ric(\omega) = \lambda\omega(X, Y) = g(JX, Y) = < JX, Y > = cos(\theta). \tag{22}$$

We take equation (22) into consideration (20) then $F^*\omega$ is as follows:

$$F^*\omega = cos(\theta) J_\omega. \tag{23}$$

Let $\{x_\alpha, y_\alpha\}_{1 \le \alpha \le n}$ be a g_μ-orthonormal basis of T_pM, that diagonalizes $F^*\omega$ at p, that is

$$F^*\omega \begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix} = \underset{0 \le \alpha \le n}{\bigoplus} \begin{bmatrix} 0 & -\cos\theta_\alpha \\ \cos\theta_\alpha & 0 \end{bmatrix} \begin{bmatrix} x_\alpha \\ y_\alpha \end{bmatrix}, \tag{24}$$

where $cos\theta_1 \ge cos\theta_2 \ge \ldots \ge cos\theta_n \ge 0$. The angles $\{\theta_\alpha\}_{1 \le \alpha \le n}$ are the Kähler angles of F at p. Thus, using (23) for $\forall\alpha$,

$$F^*\omega(x_\alpha) = cos\theta_\alpha y_\alpha, \qquad F^*\omega(y_\alpha) = -cos\theta_\alpha x_\alpha, \tag{25}$$

and if $k \ge 1$, where $2k$ is the rank of $F^*\omega$ at p, $J\omega(x_\alpha) = y_\alpha$,$\forall\alpha \le k$. M is a complex submanifold iff $cos\theta_\alpha = 1$, $\forall\alpha$, and is a Lagrangian submanifold iff $cos\theta_\alpha = 0$, $\forall\alpha$. We say that F has equal Kähler angles if $\theta_\alpha = \theta$, $\forall\alpha$. Complex and Lagrangian submanifolds are examples of such case. If F is a complex submanifold, then J is the complex structure induced by J of N. The Kähler angles are some functions that at each point p of M measure the deviation of the tangent plane T_pM of M from a complex or a Lagrangian subspace of $T_{F(p)}M$. This concept was introduced by Chern and Wolfson for oriented surfaces, namely $F^*\omega = cos\theta \mathrm{Vol}_{\ M}$ [29].

Theorem 5. If M is a real compact surface and N is a complex Kähler-Einstein surface with $R < 0$, and if F is minimal with no complex points, then F is Lagrangian [30].

Let we denote by $\nabla_x dF(y) = \nabla dF(x, y)$ the second fundamental form of F. If F is an immersion with no complex directions at p and $\{x_\alpha, y_\alpha\}$ diagonalizes $F^*\omega$ at p, then $\left\{dF(z_\alpha), dF(\bar{z}_\alpha), \left(JdF(z_\alpha)\right)^\perp, \left(JdF(\bar{z}_\alpha)\right)^\perp\right\}$ constitutes a complex basis of $T_{F(p)}^{\mathbb{C}}N$, where

$$z_\alpha = \frac{1}{2}(x_\alpha - \mathbf{i} y_\alpha), \qquad \bar{z}_\alpha = \frac{1}{2}(x_\alpha + \mathbf{i} y_\alpha), \tag{26}$$

are complex vectors of the complexfield tangent space of M at p. If F has equal Kähler angles, then

$$F^*\omega = \cos\theta J_\omega, \qquad \tilde{g} = \sin^2\theta g_\mu. \tag{27}$$

If we parallel transport a diagonalizing orthonormal basis $\{x_\alpha, y_\alpha\}$ of $F^*\omega$ at p_0 along geodesics, on a neighborhood of p_0. Similarly we that $\tilde{g}$ is parallel. If we extend $F^*\omega$ to the complexified tangent space $T_{p_0}^{\mathbb{C}}M$ then the holomorphic base structures, considering (25),(27) and (26), are as follows [29]:

$$F^*\omega(z_\alpha) = \mathbf{i}\cos\theta_\alpha z_\alpha, \qquad F^*\omega(\bar{z}_\alpha) = -\mathbf{i}\cos\theta_\alpha \bar{z}_\alpha. \tag{28}$$

A **Hermitian matrix** (or self-adjoint matrix) is a square matrix with complex entries that is equal to its own conjugate transpose-that is, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j: $a_{ij} = \overline{a_{ji}}$ or $A = A^T$, in matrix form. The above structure (24) are Hermitian matrix. Namely

$$(\bar{F}^*\omega)^T = \begin{bmatrix} 0 & \mathbf{i}\cos\theta_\alpha \\ -\mathbf{i}\cos\theta_\alpha & 0 \end{bmatrix}_{0\le\alpha\le n}, \tag{29}$$

from here $(\bar{F}^*\omega)^T = F^*\omega.$ We will be examine for holomorphic property of (28).

$$\begin{aligned} &1. F^{*2}\omega(z_\alpha) = F^*\omega o F^*\omega(z_\alpha) = F^*\omega(\tfrac{1}{2}(x_\alpha - \mathbf{i}y_\alpha)) = \tfrac{1}{2}F^*\omega(x_\alpha) - \tfrac{1}{2}\mathbf{i}F^*\omega(y_\alpha) = \mathbf{i}\cos\theta_\alpha z_\alpha, \\ &2. F^*\omega(\bar{z}_\alpha) = F^*\omega o F^*\omega(\bar{z}_\alpha) = F^*\omega(\tfrac{1}{2}(x_\alpha + \mathbf{i}y_\alpha)) = \tfrac{1}{2}F^*\omega(x_\alpha) + \tfrac{1}{2}\mathbf{i}F^*\omega(y_\alpha) = -\mathbf{i}\cos\theta_\alpha \bar{z}_\alpha. \end{aligned} \tag{30}$$

As we have seen above, these structures have the ability to (para)complex for $\theta_\alpha = 2k\pi,\ k \in \mathbb{Z}$.

5. EULER-LAGRANGE DYNAMICS EQUATIONS

Lemma 1. The function that reduces the closed 2-form on a vector field to 1 −form on the phase space defined of a mechanical system is equal to the differential of the energy function 1-form of the Lagrangian or the Hamiltonian mechanical systems [30].

Definition 6. Let M be an n-dimensional manifold and TM its tangent bundle with canonical projection $\tau_M: TM \to M$. TM is called the phase space of velocities of the base manifold M. Let $L: TM \to R$ be a differentiable function on TM called the **Lagrangian function**. Here, $L = T - V$ such that T is the kinetic energy and V is the potential energy of a mechanical system. In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$, so we have $L = m\dot{x}^2/2 - kx^2/2$. We consider the closed 2-form and base space (J) on TM given by $\Phi_L = -d\mathbf{d}_J L = -d(J(\mathbf{d}))$. Consider the equation

$$i_\xi \Phi_L = dE_L. \tag{31}$$

Where i_ξ is reduction function and $i_\xi \Phi_L = \Phi_L(\xi)$ is defined in the form. Then ξ is a vector field, we shall see that (31) under a certain condition on ξ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation (31) is named as **Lagrange dynamical equation** [31]**.** We shall see that for motion in a potential, $E_L = VL - L$ is an energy function and $V = J\xi$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, ξ) is known as **Lagrangian system** on the tangent bundle TM. If it is continued the operations on (31) for any coordinate system then infinite dimension **Lagrange's equation is** obtained the form below. The equations of motion in Lagrangian mechanics are the Lagrange equations of the second kind, also known as the Euler–Lagrange equations;

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \tag{32}$$

We have $\partial L/\partial\dot{x} = mx$ and $\partial L/\partial x = -kx$, so eq. (32) gives $m\ddot{x} = -kx$ which is exactly the result obtained by using $F = ma$ at Newton's second law for the mechanical problem. The Euler-Lagrange equation, eq. (32), gives $m\ddot{x} = -\frac{dV}{dx}$. In a three-dimensional setup written in terms of Cartesian coordinates, the potential takes the form $V(x, y, z)$, so the Lagrangian is $L = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2 - V(x, y, z)$. So, the three Euler-Lagrange equations may be combined into the vector statement $m\ddot{x} = -\nabla V$.

6. LAGRANGIAN MECHANICAL SYSTEMS

We, using (28), Lemma 1 and (31), get Euler-Lagrange equations for quantum and classical mechanics on Kähler-Einstein manifolds $(M, g, F^*\omega)$. Firstly, take $F^*\omega$ as the local basis element on Kähler-Einstein manifolds and $(z_\alpha, \bar{z}_\alpha)$ be its coordinate functions.

Proposition 2. Let ξ be the vector field decided and $Z^\alpha = \dot{z}_\alpha$, $\bar{Z}^\alpha = \dot{\bar{z}}_\alpha$ on Kähler-Einstein manifolds for the local coordinates (28) by

$$\xi = Z^\alpha \frac{\partial}{\partial z_\alpha} + \bar{Z}^\alpha \frac{\partial}{\partial \bar{z}_\alpha}. \tag{33}$$

The vector field described by

$$V = F^*\omega(\xi) = Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial}{\partial z_\alpha} - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial}{\partial \bar{z}_\alpha}, \tag{34}$$

$$VL = Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha}, \tag{35}$$

is said to be *Liouville vector field* with Kähler-Einstein metric and equal Kähler angles on Fano manifolds. The closed 2-form is given by $\Phi_L = -dd_{F^*\omega}L$ such that

$$\begin{aligned} d_{F^*\omega} &= F^*\omega(Z^\alpha dz_\alpha + \bar{Z}^\alpha d\bar{z}_\alpha) = \mathbf{i}\cos\theta_\alpha Z^\alpha dz_\alpha - \mathbf{i}\cos\theta_\alpha \bar{Z}^\alpha d\bar{z}_\alpha, \\ d_{F^*\omega}(L) &= \mathbf{i}\cos\theta_\alpha Z^\alpha(L) dz_\alpha - \mathbf{i}\cos\theta_\alpha \bar{Z}^\alpha(L) d\bar{z}_\alpha. \end{aligned} \tag{36}$$

Then we using have

$$\begin{aligned} \Phi_L = -d(d_{F^*\omega}L) = {} & \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} dz_\alpha \wedge dz_\alpha - \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} d\bar{z}_\alpha \wedge dz_\alpha \\ & +\mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} dz_\alpha \wedge d\bar{z}_\alpha - \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge d\bar{z}_\alpha, \end{aligned} \tag{37}$$

and then we calculate, using $dz^i\left(\frac{\partial}{\partial z^j}\right) = \frac{\partial z^i}{\partial z^j} = \delta_j^i = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$, $(f \wedge g)(v) = f(v)g - g(v)f$,

$$\Phi_L(\xi) = \begin{bmatrix} \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} dz_\alpha \wedge dz_\alpha - \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} d\bar{z}_\alpha \wedge dz_\alpha \\ +\mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} dz_\alpha \wedge d\bar{z}_\alpha - \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge d\bar{z}_\alpha \end{bmatrix} \left(Z^\alpha \frac{\partial}{\partial z_\alpha} + \bar{Z}^\alpha \frac{\partial}{\partial \bar{z}_\alpha}\right), \tag{38}$$

and then

$$\begin{aligned} \Phi_L(\xi) = \Phi_L(\xi) = {} & Z^\alpha \mathbf{i}\cos\theta_\alpha \left[\frac{\partial L}{\partial z_\alpha}\left(dz_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) dz_\alpha - dz_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) dz_\alpha\right) - \frac{\partial L}{\partial z_\alpha}\left(d\bar{z}_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) dz_\alpha - dz_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) d\bar{z}_\alpha\right)\right] \\ & +Z^\alpha \mathbf{i}\cos\theta_\alpha \left[\frac{\partial L}{\partial \bar{z}_\alpha}\left(dz_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) d\bar{z}_\alpha - d\bar{z}_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) dz_\alpha\right) - \frac{\partial \bar{z}_\alpha(L)}{\partial \bar{z}_\alpha}\left(d\bar{z}_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) d\bar{z}_\alpha - d\bar{z}_\alpha\left(\frac{\partial}{\partial z_\alpha}\right) d\bar{z}_\alpha\right)\right] \\ & +\bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \left[\frac{\partial L}{\partial z_\alpha}\left(dz_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) dz_\alpha - dz_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) dz_\alpha\right) - \frac{\partial L}{\partial z_\alpha}\left(d\bar{z}_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) dz_\alpha - dz_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) d\bar{z}_\alpha\right)\right] \\ & +\bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \left[\frac{\partial L}{\partial \bar{z}_\alpha}\left(dz_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) d\bar{z}_\alpha - d\bar{z}_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) dz_\alpha\right) - \frac{\partial L}{\partial \bar{z}_\alpha}\left(d\bar{z}_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) d\bar{z}_\alpha - d\bar{z}_\alpha\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) d\bar{z}_\alpha\right)\right]. \end{aligned} \tag{39}$$

or

$$\begin{aligned} i_\xi \Phi_L = {} & Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} dz_\alpha - Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} dz_\alpha + Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} d\bar{z}_\alpha + Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \\ & -\bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial Z_\alpha} dZ_\alpha - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} dZ_\alpha - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} d\bar{z}_\alpha + \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} d\bar{z}_\alpha. \end{aligned} \tag{40}$$

Energy function and its differential are like the following:

$$E_L = VL - L = Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial \bar{z}_\alpha} - L, \tag{41}$$

and

$$\begin{aligned} dE_L = {} & Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z^\alpha} dz_\alpha - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial L}{\partial z_\alpha} dz_\alpha - \frac{\partial L}{\partial z_\alpha} dz_\alpha \\ & +Z^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial z_\alpha(L)}{\partial \bar{z}_\alpha} d\bar{z}_\alpha - \bar{Z}^\alpha \mathbf{i}\cos\theta_\alpha \frac{\partial \bar{z}_\alpha(L)}{\partial \bar{z}_\alpha} d\bar{z}_\alpha - \frac{\partial L}{\partial \bar{z}_\alpha} d\bar{z}_\alpha. \end{aligned} \tag{42}$$

If we use (31) we obtain the equations given by

$$-\mathbf{i}\cos\theta_\alpha \left[Z^\alpha \frac{\partial}{\partial z_\alpha} + \bar{Z}^\alpha \frac{\partial}{\partial \bar{z}_\alpha}\right] \frac{\partial L}{\partial z_\alpha} = -\frac{\partial L}{\partial z_\alpha}, \qquad \mathbf{i}\cos\theta_\alpha \left[Z^\alpha \frac{\partial}{\partial z_\alpha} + \bar{Z}^\alpha \frac{\partial}{\partial \bar{z}_\alpha}\right] \frac{\partial L}{\partial \bar{z}_\alpha} = -\frac{\partial L}{\partial \bar{z}_\alpha}. \tag{43}$$

and then, using (33),

$$-\mathbf{i}\cos\theta_\alpha \xi\left(\frac{\partial L}{\partial z_\alpha}\right) = -\frac{\partial L}{\partial z_\alpha}, \qquad \mathbf{i}\cos\theta_\alpha \xi\left(\frac{\partial L}{\partial \bar{z}_\alpha}\right) = -\frac{\partial L}{\partial \bar{z}_\alpha}. \tag{44}$$

Definition 7. Suppose that ξ is a vector field a vector-valued function with Cartesian coordinates $(\xi_1, \ldots, \xi_n)$; and $\alpha(t)$ a parametric curve with Cartesian coordinates $(\alpha_1(t), \ldots, \alpha_n(t))$. Then $\alpha(t)$ is an integral curve of ξ if it is a solution of the following autonomous system of ordinary differential equations:

$$\frac{d\alpha_1}{dt} = \xi_1(\alpha_1(t), \ldots, \alpha_n(t)), \ldots, \frac{d\alpha_n}{dt} = \xi_1(\alpha_1(t), \ldots, \alpha_n(t)). \tag{45}$$

Such a system may be written as a single vector equation:

$$\xi(\alpha(t)) = \alpha'(t) = \frac{\partial}{\partial t}(\alpha(t)). \tag{46}$$

Considering the curve α, an integral curve of ξ, (46), we can find the equations at (44) as follows:

$$\begin{aligned} &1.\,\text{Equation:} \quad -\,\mathbf{i}\cos\theta_\alpha \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z_\alpha}\right) + \frac{\partial L}{\partial z_\alpha} = 0, \\ &2.\,\text{Equation:} \quad \mathbf{i}\cos\theta_\alpha \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \bar{z}_\alpha}\right) + \frac{\partial L}{\partial \bar{z}_\alpha} = 0, \end{aligned} \tag{47}$$

such that these equations are called Euler-Lagrangian equations with Kähler-Einstein metric and Equal Kähler angles constructed on Fano manifolds with equal Kähler angles and thus the triple (M, Φ_L, ξ) is named as a *mechanical system* on Fano manifolds $(N, g, F^*\omega)$.

7. SOLUTION OF EULER-LAGRANGE EQUATIONS

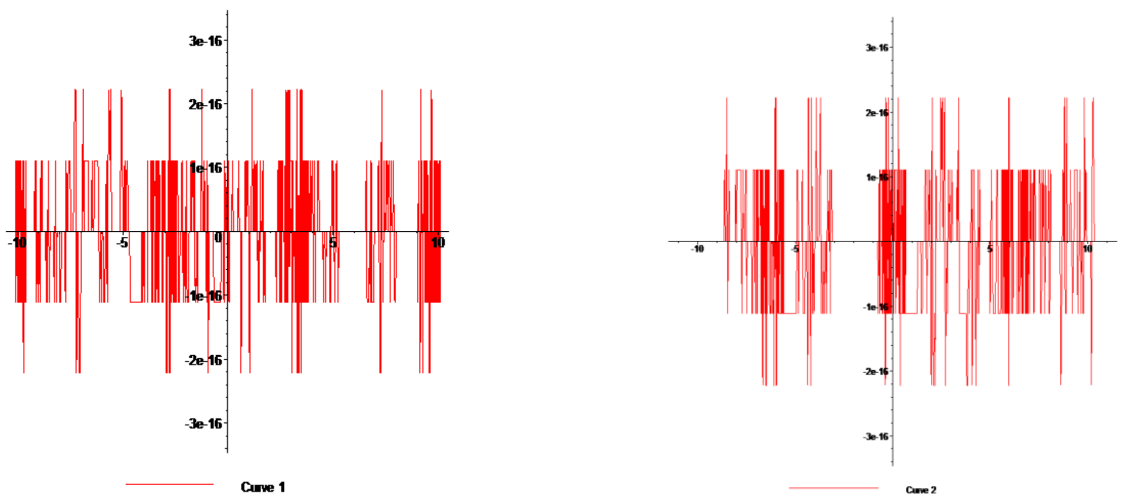
Those found (47) are partial differential equation based on three variables. The implicit solution of equations system (47) was found using the Maple program.

$$L(z, \bar{z}, t) = F_1(t) + exp\left(-\mathbf{i} * \frac{t}{cos(\theta)}\right) * F_2(z) + exp\left(\frac{t}{cos(\theta)} * \mathbf{i}\right) * F_3(\bar{z}), \quad \mathbf{i}^2 = -1. \tag{48}$$

Example 1. In the above expression; θ angle and closed functions to be specific selected and the following graphs obtained.

$L(z, \bar{z}, t) = \mathrm{t} + exp(-\mathbf{i} * \mathrm{t}) * \mathbf{i} + \mathrm{t} - \mathbf{i} * exp(\mathrm{t} * \mathbf{i}) * \mathbf{i}$ $for \;\; \theta = 0$

$L(z, \bar{z}, t) = exp(\mathbf{i} * \mathrm{t}) * \mathbf{i} + \mathbf{t} - \mathbf{i} * exp(\mathrm{t} * -\mathbf{i}) * \mathbf{i}$ $for \;\; \theta = \pi$



(Graph)

(49)

8. CONCLUSION

A classical field theory is just a mechanical system with a continuous set of degrees of freedom. Also, a classical field theory explain the study of how one or more physical fields interact with matter which is used quantum and classical mechanics of physics branches. An electromagnetic field is a physical field produced by electrically charged objects. How the movement of objects in electrical, magnetically and gravitational fields force is very important. For example, on a weather map, the surface wind velocity is defined by assigning a vector to each point on a map. So, said that each vector represents the speed and direction of the movement of air at this point. In this study, the Euler-Lagrange mechanical equations (47) derived on a generalized Euler-Lagrangian equations with Kähler-Einstein metric and Equal Kähler angles on Fano manifolds may be suggested to deal with problems in electrical, magnetically and gravitational fields for the path of movement (49) of defined space moving objects [32].

REFERENCES

[1]. M. De Leon and P.R. Rodrigues, Second-order differential equations and non-conservative Lagrangian Mechanics, *J. Phys. A: Math. Gen.*, **20** (1987), 5393-5396.

[2]. G.Tian, Kähler-Einstein metrics on Fano manifolds, *Japanese Journal of Mathematics*, Volume 10, Issue 1, (2015), 1-41.

[3]. H. de Vries, *Understanding Relativistic Quantum Field Theory, The Hamiltonian and Lagrangian Densities*, (2009).

[4]. M. Tekkoyun, On para-Euler Lagrange and para-Hamiltonian equations, *Physics Letters A*, Vol. **340**, (2005), 7-11.

[5]. W.K. Liu and S. Jun, Computational nanomechanics of materials, *Comput. Methods Appl. Mech. Eng*., 193, (2004) ,1529–1578.

[6]. X. Chen, C. LeBrun and B. Weber, On conformally Kähler-Einstein manifolds, *J. Amer. Math. Soc*,. 21, (2008), 1137-1168.

[7]. C. Spotti, Degenerations of Kähler-Einstein Fano manifolds, http://arxiv.org/abs/1211.5334v1, (2012), 1-131.

[8]. C. LeBrun, Einstein manifolds and extremal Kähler metrics, *Journal Für Die Reine und Angewandte Mathematik*, Issue 678, (2013), 69–94.

[9]. G. Heier, Existence of Kähler-Einstein metrics and multiplier ideal sheaves on Del Pezzo surfaces, *Math. Z.*, **264,** (2010), 727-743.

[10]. C. Li, K-Stability and Kähler-Einstein metrics, http://arxiv.org/pdf/1211.4669v2.pdf, (2013).

[11]. C. van Coevering, Kähler-Einstein metrics on strictly pseudoconvex domains, *Annals of Global Analysis and Geometr*, Vol. **42**, Issue 3, (2012), 287-315.

[12]. M. Roček, Kähler-Einstein Supermanifolds, http://www.eecs.berkeley.edu/~joschu/docs/superman.

[13]. A.M. Nadel, Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, *Alan Michael Nadel Annals of Mathematics Second Series*, Vol. **132**, No. 3, (1990), 549-596

[14]. X.X. Chen and G. Tian , Ricci flow on Kähler-Einstein surfaces, *Inventiones Mathematicae*, Vol. **147**, Issue3, (2002), 487-544.

[15]. N. Koiso and Y. Sakane, Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds II, *Osaka J. Math.*, **25,** (1988), 933-959.

[16]. D.V. Alekseevsky, C. Medori and A. Tomassini, Homogeneous para-Kähler Einstein manifolds, *Russian Mathematical Surveys*, Volume 64, Number 1, (2009), 1-44.

[17]. I.M.C. Salavessa and G. Valli, Minimal Submanifolds of Kähler-Einstein manifolds with equal Kähler angles, *Pacific Journal of Mathematics*, Vol. **205**, No.1, (2002).

[18]. Z. Kasap, Weyl-Euler-Lagrange equations of motion on flat manifold, *Advances in Mathematical Physics*, (2015), 1-16.

[19]. A. Newlander and L. Nirenberg, Complex analytic coordinates in almost complex manifolds, *Ann. of Math*., **65,** (1957), 391-404.

[20]. A.Nijenhuis and W.Wolf, Some integration problems in almost-complex and complex canifolds, *Ann. Math*., 77, (1963), 424-489.

[21]. P. Gilkey, S. Nikčevic′, Kähler and para-Kähler curvature Weyl manifolds, http://arxiv.org/pdf/1011.4844.pdf, (2010), 1-11.

[22]. A.L. Besse, *Einstein Manifolds*, Springer, (1987).

[23]. D. McDu and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, (1995).

[24]. M. Zedda, *Kähler Immersions of Kähler–Einstein Manifolds Into Infinite Dimensional Complex Space Forms*, Università degli Studi di Cagliari Dipartimento di Matematicae Informatica, (2009).

[25]. J. Fine, *Canonical Kähler Metrics and Quantisation*, Summer school in Cologne, 2012.

[26]. X.J. Wang and X. Zhu, Kähler-Ricci solitons on toric manifolds with positive first chern class, *Advances in Mathematics*, 188, (2004), 87–103.

[27]. K. Scharnhorst, Angles in complex vector spaces, *Acta Applicandae Mathematica*, Vol. **69**, Issue. 1, (2001), 95-103.

[28]. X. Mo, Minimal surfaces with constant Kähler angle in complex projective spaces, *Proceedings Of The.American Mathematical Society*., Vol. **121**, Num. 2, (1994), 569-571.

[29]. J.G. Wolfson, Minimal surfaces in Kähler surfaces and Ricci curvature, *J. Differential Geom.*, **29**, (1989), 281-294.

[30]. J. Klein, *Escapes Variationnels et Mécanique*, Ann. Inst. Fourier, Grenoble, **12,** (1962).

[31]. M. de Leon and P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North-Hol. Math. St.,152, Elsevier Sc. Pub. Com. Inc., Amsterdam, (1989).

[32]. B. Thidé, *Electromagnetic Field Theory*, (2012).