MECHANICS EQUATIONS OF FRENET-SERRET FRAME ON MINKOWSKI SPACE

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ABSTRACT
It is well known that Minkowski space is the mathematical space setting and Einstein's theory of special relativity is most appropriate formulated. Dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems in which usually by employing differential equations or difference equations. The Frenet-Serret formulas describe the kinematic properties of a particle which moves along a continuous, differentiable curve in Euclidean space three-dimensional real space or the geometric properties of the curve itself in any case of any motion. The Frenet-Serret trihedron plays a key role in the differential geometry of curves such that its shows ultimately leading to a more or less complete classification of smooth curves in Euclidean space up to congruence. In this paper, we will establish mechanics equations of Frenet-Serret frame on Minkowski space and we will consider a relativistic for an electromagnetic field that it is moving under the influence of its own Frenet-Serret curvatures. Also, we will get the mechanical equations of motion for several curvatures dependent actions of interest in physics.

Keywords: *Frenet-Serret Curvature, Mechanical System, Minkowski Space, Lagrangian Equation.*
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1. INTRODUCTION

There are many applications of differential geometry and mathematical physics. These applications are used in many areas. These applications is related to the geodesic. It is well known that the shortest path between two points is called a geodesic. Time-dependent equations of geodesics for an object can be found with the help of the Euler-Lagrange equations. The basic information about them can be seen in many mechanical and geometry books. We can say that differential geometry provides a good workplace for studying Lagrangians of classical mechanics and field theory. The dynamic equations for moving bodies are obtained for Lagrangian mechanics by many authors in many areas [1-4]. *Kasap* introduced Weyl-Euler-Lagrange equations of motion on flat manifold [5]. *Kasap* and *Tekkoyun* examined Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research [6]. *Kasap* showed that Weyl-Euler-Lagrange and Weyl–Hamilton equations on $\mathbb{R}_n^{2n}$ which is a model of tangent manifolds of constant W-sectional curvature [7]. Para-complex analogue of the Euler-Lagrange and the Hamilton equations was obtained in the framework of para-Kählerian manifold and the geometric results on a para-complex mechanical systems were found by Tekkoyun [8]. *Yildirim* and *Ilarslan* investigated the same problem for null Bertrand type curves in Minkowski 3-space E_1^3 by using the well known algorithm given by Chao and Gao, and obtained new results for null Bertrand type curves in Minkowski $3-$space E_1^3 [9]. *Arreaga-Garcia* and *Morales* presented an introduction to the study of a relativistic particle moving under the influence of its own Frenet-Serret curvatures [10]. *Arreaga* et al. considered the motion of a particle described by an action that is a functional of the Frenet–Serret (FS) curvatures associated with the embedding of its worldline in Minkowski space [11]. *Capovilla* et al. demonstrated how reparametrization covariant dynamical variables and their projections onto the Frenet-Serret frame can be exploited to provide not only a significant simplification of but also novel insights into the canonical analysis [12]. *Capovilla* et al. considered the Frenet-Serret geometry of null curves in a three and a four-dimensional Minkowski background. They developed a theory of deformations adapted to the Frenet-Serret frame. They exploited it to provide a Lagrangian description of the dynamics of geometric models for null curves [13]. *Ali* and *Sarkar* established the Serret-Frenet equations in Minkowski space. These equations originally formulated in Euclidean space in $\mathbb{R}^3$, constitute a beautiful set of vector differential equations which contains all intrinsic properties of parameterized curve [14]. *Liang* and *Lee* calculated the vorticity vector in Godel, Kerr, Lewis, Schwarzschild, and Minkowski metrics and find that the vorticity vector of the specific observers is the angular velocity of the gyroscopic precession [15]. *Yilmaz et al.* examined that tangent and trinormal spherical images of a time-like curve lying on the pseudohyperbolic space H_0^3 in Minkowski space-time are investigated. They observed that the mentioned spherical images are space-like curves. Besides, they determined relations between Frenet–Serret invariants of spherical images and the base curve [16]. *Iyigun* defined the tangent spherical image of a unit speed timelike curve lying on the pseudohyperbolic space $H_0^2(r)$ in L^3 [17].

2. PRELIMINARIES

In this study, all the manifolds and geometric objects are $\mathbb{C}^{\infty}$. Also, in this paper, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted such that denote by $\mathbb{R}_1^3$ an on Minkowski space. Then $\mathcal{F}(\mathbb{R}_1^3)$, $\chi(\mathbb{R}_1^3)$ and $\Lambda^1(\mathbb{R}_1^3)$ are the set of functions on $\mathbb{R}_1^3$, the set of vector fields on $\mathbb{R}_1^3$ and the set of 1-forms on $\mathbb{R}_1^3$, respectively.

3. THE THEORY OF J-HOLOMORPHIC CURVES

Definition 1. J-holomorphic curve is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy–Riemann equations.

Pseudoholomorphic curves have since revolutionized the study of symplectic manifolds. The theory of $J-$holomorphic curves is one of the new techniques which have recently revolutionized the study of symplectic geometry, making it possible to study the global structure of symplectic manifolds. The methods are also of interest in the study of Kähler manifolds, since often when one studies properties of holomorphic curves in such manifolds it is necessary to perturb the complex structure to be generic [18].

4. SYMPLECTIC AND PSEUDO-RIEMANNIAN MANIFOLD

Definition 2. A **symplectic manifold** is a smooth manifold (M) equipped with a closed nondegenerate differential $2-$form (ω) called the symplectic form.

Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds, e.g., in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field: The set of all possible configurations of a system is modelled as a manifold, and this manifold's cotangent bundle describes the phase space of the system. The basic example of an almost complex symplectic manifold is standard Euclidean space $(\mathbb{R}^{2n},\omega_0)$ with its standard almost complex structure J_0 obtained from the usual identification with $\mathbb{C}^n$. Thus, one sets $z_j = x_{2j-1} + ix_{2j}$ for $j=1,\ldots,n$ and defines J_0 by

$$J_0(\partial_{2j-1}) = \partial_{2j}, \qquad J_0(\partial_{2j}) = -\partial_{2j-1}. \tag{1}$$

Where $\partial_j = \partial/\partial x_j$ is the standard basis of $T_x\mathbb{R}^{2n}$. Kähler manifolds give another basic example [18].

Definition 3. A pseudo-**Riemannian manifold** (M,g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g which, unlike a Riemannian metric, need not be positive-definite, but must be non-degenerate. Such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero.

5. (PARA)COMPLEX DIFFERENTIAL GEOMETRY

Definition 4. A **(para)complex structure** on a real finite dimensional vector space V is an endomorphism $J \in End(V)$ such that $J^2 = \pm I$, $J = \pm Id$ and the two eigenspaces $V^{\pm} := ker(Id \mp J)$ to the eigenvalues ± 1 of J have the same dimension.

We call the pair (V,J) a para-complex vector-space. An almost para-complex structure on a smooth manifold M is an endomorphism field $J \in \Gamma(End(TM))$ such that, for all $p \in M$, J_p is a para-complex structure on T_pM. It is called integrable if the distributions $T^{\pm}M = ker(Id \mp J)$ are integrable. An integrable almost paracomplex structure on M is called a paracomplex structure on M and a manifold M endowed with a paracomplex structure is called a para-complex manifold. Like in the complex case we now define the Nijenhuis tensor N of an almost paracomplex structure J by

$$N_J[X,Y] := [X,Y] + [JX,JY] - J[X,JY] - J[JX,Y], \tag{2}$$

for all vector fields X and Y on M [19].

6. MINKOWSKI SPACE AND CURVE TYPES

In this setting the three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional manifold for representing a spacetime. In theoretical physics, Minkowski space is often contrasted with Euclidean space. The concept of a Euclidean space cover Euclidean plane and the three-dimensional space of

Euclidean geometry as spaces of dimensions 2 and 3 respectively. While a Euclidean space has only spacelike dimensions, a Minkowski space also has one timelike dimension. Therefore the isometry group of a Euclidean space is the Euclidean group and for a Minkowski space it is the Poincaré group. Minkowski space is a four-dimensional real vector space equipped with a nondegenerate, symmetric bilinear form with signature $(-,+,+,+)$. Minkowski space is a pseudo-Euclidean space with $n=4$ and $n-k=1$. Elements of Minkowski space are called events or four-vectors. Minkowski space is often denoted $\mathbb{R}_1^3$ to emphasize the signature, although it is also denoted M^4 or simply M. It is perhaps the simplest example of a pseudo-Riemannian manifold.

Definition 5. Let $X=(x_i)$, $Y=(y_i)\in\mathbb{R}^3$ be any two vectors. As follows

$$<,>:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^1,\ <X,Y>_L=-x_1y_1+x_2y_2+x_3y_3 \tag{3}$$

in the form of a function. This function are bilinear and symmetric. This the inner product function $<X,Y>_L$ along with $\mathbb{R}^3$ is called Minkowski space or the Lorenz space and it is been shown $\mathbb{R}_1^3$.

Theorem 1. Let $X\in\mathbb{R}_1^3$ be any one vector.

$$\begin{aligned}&1.\ \text{If } <X,X>_L>0 \ \text{ or } \ \vec{X}=0, \quad \vec{X} \ \text{is spacelike},\\ &2.\ \text{If } <X,X>_L<0, \quad \vec{X} \ \text{is timelike},\\ &3.\ \text{If } <X,X>_L=0, \quad \vec{X} \ \text{is lightlike (isotropic, null)}.\end{aligned} \tag{4}$$

Theorem 2. $\alpha: I\to\mathbb{R}_1^3$ given regular curve. Each $t\in I$ for α velocity vector,

$$\begin{aligned}&1.\ \text{If } <\alpha'(t),\alpha'(t)>_L>0, \quad \alpha \ \text{curve is spacelike},\\ &2.\ \text{If } <\alpha'(t),\alpha'(t)>_L<0, \quad \alpha \ \text{curve is timelike},\\ &3.\ \text{If } <\alpha'(t),\alpha'(t)>=0, \quad \alpha \ \text{curve is lightlike}.\end{aligned} \tag{5}$$

7. FRENET-SERRET FORMULAS

Definition 6. Frenet-Serret formulas describe the derivatives of the so-called tangent, normal, and binormal unit vectors in terms of each other.

Vector notation and linear algebra currently used to write these formulas were not yet in use at the time of their discovery. The tangent, normal, and binormal unit vectors, often called **T**, **N**, and **B**, or collectively the Frenet–Serret trihedron or **TNB** trihedron, together form an orthonormal basis spanning $\mathbb{R}^3$, and are defined as follows: **T** is the unit vector tangent to the curve, pointing in the direction of motion. **N** is the normal unit vector, the derivative of **T** with respect to the arclength parameter of the curve, divided by its length. **B** is the binormal unit vector, the cross product of **T** and **N**. The Frenet–Serret formulas are

$$\begin{bmatrix}\frac{d\mathbf{T}}{ds}\\ \frac{d\mathbf{N}}{ds}\\ \frac{d\mathbf{B}}{ds}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \kappa & \mathbf{0}\\ -\kappa & \mathbf{0} & \tau\\ \mathbf{0} & -\tau & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}. \tag{6}$$

Here d/ds is the derivative with respect to arclength, κ is the curvature and τ is the torsion of the curve. The two scalars κ and τ effectively define the curvature and torsion of a space curve. The associated collection, $\mathbf{T},\mathbf{N},\mathbf{B},\kappa$, and τ is called the Frenet–Serret tools.

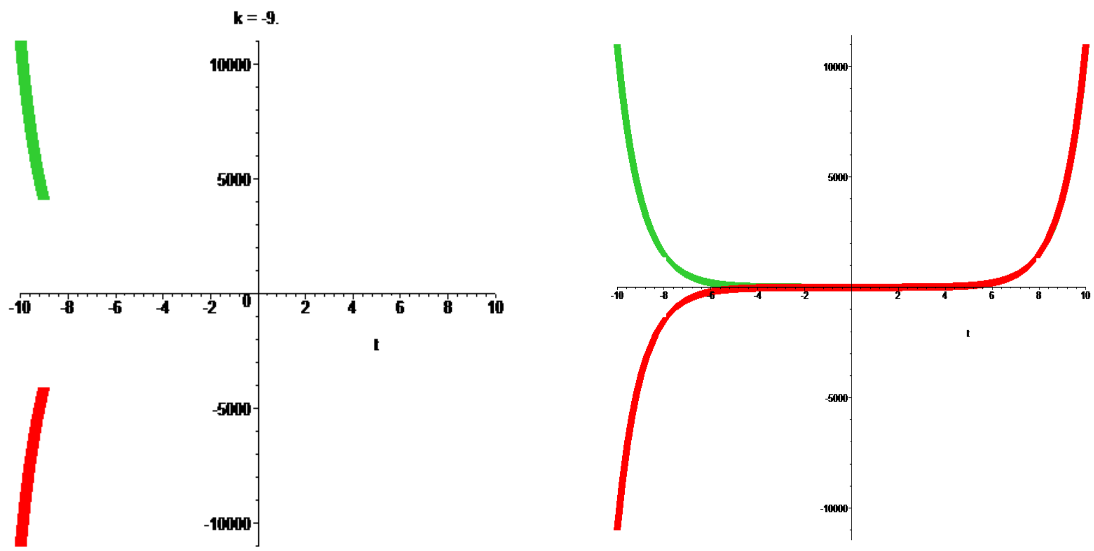
The Frenet-Serret formulas show that there is a pair of functions defined on the curve, the torsion and curvature, which are obtained by differentiating the trihedron, and which describe completely how the trihedron evolves in time along the curve. A key feature of the general method is that a preferred moving trihedron, provided it can be found, gives a complete kinematic description of the curve. It is well known that a Darboux trihedron is a natural moving trihedron constructed on a surface. It is the analog of the Frenet–Serret trihedron as applied to surface geometry. A Darboux trihedron exists at any non-umbilic point of a surface embedded in Euclidean space.

8. HOLOMORPHIC STRUCTURES OF FRENET-SERRET TRIHEDRON

Darboux invention that the introduction of the trihedron allows for a conceptual simplification of the problem of moving trihedron on curves and surfaces by treating the coordinates of the point on the curve and the trihedron vectors in a uniform manner. A trihedron consists of a point P in Euclidean space, and three orthonormal vectors $e_i = \frac{\partial}{\partial x_i}$, $i = 1,2,3$ based at the point P. A moving trihedron is a trihedron whose components depend on one or more parameters.

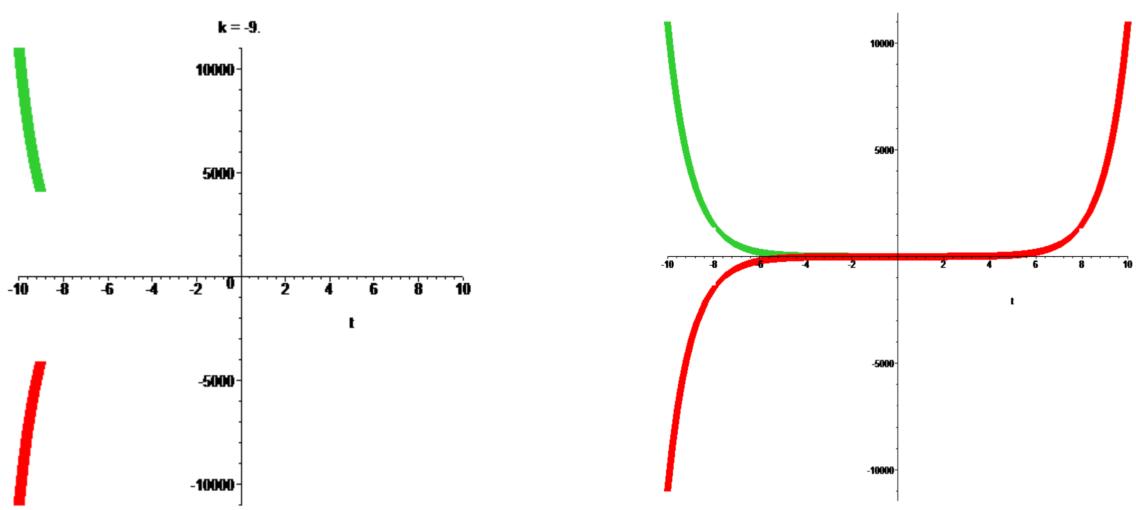
Proposition 1. J are paracomplex or tangent structures on Minkowski space. We consider the following hyperbolic curve types $\alpha_1(t)$ and $\alpha_2(t)$.

$$1.\ \alpha_1(t) = (\cosh t, \sinh t, 0) \ \ is \ \ hyperbola \ \ such \ \ that \ \ x_1^2 = x_2^2 + 1, \qquad x^3 = 0.$$



$(Graph \ \ 1)$

$$2.\ \alpha_1(t) = (-\sinh t, -\cosh t, 0) \ \ is \ \ hyperbola \ \ such \ \ that \ \ x_1^2 = x_2^2 - 1, \qquad x_3 \neq 0.$$



$(Graph \ \ 2)$

(7)

These curves $(\alpha_1(t)$ and $\alpha_2(t))$ in Minkowski space are spacelike and timelike.

Proof. We can be seen below.

$$\begin{aligned} &1.\ \alpha_1'(t) = (\sinh t, \cosh t, 0),\\ &\qquad < \alpha_1'(t), \alpha_1'(t) >= -\sinh^2 t + \cosh^2 t = 1 > 0 \ \text{ and } \ \alpha_1(t) \ \text{ is spacelike.}\\ &2.\ \alpha_2'(t) = (-\cosh t, -\sinh t, 0),\\ &\qquad < \alpha_2'(t), \alpha_2'(t) >= \sinh^2 t - \cosh^2 t = -1 < 0 \ \text{ and } \ \alpha_2(t) \ \text{ is timelike.} \end{aligned} \tag{8}$$

Definition 7. In three dimensions, the vector from the origin to the point with Cartesian coordinates (x, y, z) can be written as [20]:

$$r = x\vec{i} + y\vec{j} + z\vec{k} = x\left(\frac{\partial}{\partial x}\right) + y\left(\frac{\partial}{\partial y}\right) + z\left(\frac{\partial}{\partial z}\right). \tag{9}$$

Let M be a smooth manifold. If M admits a complex structure A , then M admits an almost complex structure J. Let $dim_{\mathbb{C}} M = m$ and (z, U) be any holomorphic chart inducing a coordinate frame $\partial x_1, \partial y_1, \ldots, \partial x_m, \partial y_m$. Then J is given locally as

$$J_p(\partial x_i|p) = \partial y_i|p \quad , \qquad J_p(\partial y_i|p) = -\partial x_i|p, \tag{10}$$

where $1 \le i \le m$ and $p \in U$ [21].

Proposition 2. Let $\alpha_1(t)$ and $\alpha_2(t)$ be curves taken as a vector and they are transferred holomorphic structures such that its are $J\left(\frac{\partial}{\partial x_1}\right), J\left(\frac{\partial}{\partial x_2}\right), J\left(\frac{\partial}{\partial x_3}\right)$ and $J\left(\frac{\partial}{\partial t}\right)$;

$$\begin{aligned} &1.\ J\left(\frac{\partial}{\partial x_1}\right) = \cosh t \frac{\partial}{\partial x_1} + \sinh t \frac{\partial}{\partial x_2},\\ &2.\ J\left(\frac{\partial}{\partial x_2}\right) = -\sinh t \frac{\partial}{\partial x_1} - \cosh t \frac{\partial}{\partial x_2},\\ &3.\ J\left(\frac{\partial}{\partial x_3}\right) = 0, \qquad 4.\ J\left(\frac{\partial}{\partial t}\right) = 0. \end{aligned} \tag{11}$$

Proof. Let’s we take, using Definition 4**,** the combination of the above structures (11):

$$\begin{aligned} &1.\ J^2 \frac{\partial}{\partial x_1} = \cosh t \left(\cosh t \frac{\partial}{\partial x_1} + \sinh t \frac{\partial}{\partial x_2}\right) + \sinh t \left(-\sinh t \frac{\partial}{\partial x_1} - \cosh t \frac{\partial}{\partial x_2}\right) = (\cosh^2 t - \sinh^2 t) \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1}.\\ &2.\ J^2 \frac{\partial}{\partial x_2} = -\sinh t \left(\cosh t \frac{\partial}{\partial x_1} + \sinh t \frac{\partial}{\partial x_2}\right) - \cosh t \left(-\sinh t \frac{\partial}{\partial x_1} - \cosh t \frac{\partial}{\partial x_2}\right) = (\cosh^2 t - \sinh^2 t) \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2}.\\ &3.\ J^2 \frac{\partial}{\partial x_3} = 0, \qquad 4.\ J^2 \frac{\partial}{\partial t} = 0. \end{aligned} \tag{12}$$

As can be seen from above, J^2 is I (paracomplex) or 0 (tangent).

9. (EULER)LAGRANGE DYNAMICS EQUATIONS

Lemma 2. The function that reduces the closed 2-form on a vector field to 1-form on the phase space defined of a mechanical system is equal to the differential of the energy function 1-form of the Lagrangian or the Hamiltonian mechanical systems [22, 23].

Theorem 3. If α and β are 1-forms, then $\alpha \wedge \beta$ is a 2-forms.

Definition 8. Let M be an n-dimensional manifold and TM its tangent bundle with canonical projection $\tau_M: TM \to M$. TM is called the phase space of velocities of the base manifold M. Let $L: TM \to R$ be a differentiable function on TM called the **Lagrangian function**. Here, $L = T - V$ such that T is the kinetic energy and V is the potential energy of a mechanical system. In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$, so we have $L = m\dot{x}^2/2 - kx^2/2$. We consider the closed 2-form and base space (J) on TM given by $\Phi_L = -d\mathbf{d}_J L = -d(J(\mathbf{d})L)$. Consider the equation

$$i_\xi \Phi_L = dE_L. \tag{13}$$

Where i_ξ is reduction function and $i_\xi \Phi_L = \Phi_L(\xi)$ is defined in the form. Then ξ is a vector field, we shall see that

(13) under a certain condition on ξ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation (13) is named as **Lagrange dynamical equation** [24, 25].

Definition 9. We shall see that for motion in a potential, $E_L = VL - L$ is an energy function and $V = J\xi$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, ξ) is known as **Lagrangian system** on the tangent bundle TM. If it is continued the operations on (13) for any coordinate system then infinite dimension **Lagrange's equation is** obtained the form below. The equations of motion in Lagrangian mechanics are the Lagrange equations of the second kind, also known as the Euler–Lagrange equations;

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \tag{14}$$

Definition 10. We have $\partial L/\partial \dot{x} = mx$ and $\partial L/\partial x = -kx$, so eq. (14) gives $m\ddot{x} = -kx$ which is exactly the result obtained by using $F = ma$ at Newton's second law for the mechanical problem. The Euler-Lagrange equation (14) gives $m\ddot{x} = -dV/dx$. In a three-dimensional setup written in terms of Cartesian coordinates, the potential takes the form $V(x_1, x_2, x_3)$, so the Lagrangian is $L = m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)/2 - V(x_1, x_2, x_3)$. So, the three Euler-Lagrange equations may be combined into the vector statement $m\ddot{x} = -\nabla V$.

Definition 11. Let $\mathbb{R}_1^3$ be an $(3,1)$ real-dimensional manifold and $\mathbb{R}_1^3$ its tangent bundle with canonical projection $\tau_M: \mathbb{R}_1^3 \to \mathbb{R}$. $\mathbb{R}_1^3$ is called the phase space of velocities of the base manifold $\mathbb{R}$. Let $L: \mathbb{R}_1^3 \to \mathbb{R}$ be a differentiable function on TM called the **Lagrangian function**.
Here, $L = T - V$ such that T is the kinetic energy and V is the potential energy of a mechanical system. We consider the closed 2-form on TM given by

$$\Phi_L = -d(\mathbf{d}_J L), \qquad \mathbf{d} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} dx_i, \qquad \mathbf{d}_J = \sum_{i=1}^{3} J(\frac{\partial}{\partial x_i}) dx_i + J(\frac{\partial}{\partial t}) dt. \tag{15}$$

10. EULER-LAGRANGE EQUATIONS

We, using Theorem 3 and (13)**,** is to obtain paracomplex Euler-Lagrange equations for relativistic, quantum and classical mechanics on Minkowski space $\mathbb{R}_1^3$. Let $\mathbb{R}_1^3$ be a on Minkowski space and $\{x_1, x_2, x_3\}$ be its coordinate functions.

Proposition 3. The holomorphic base structures vectors can be selected as orthonormal bases Frenet-Serret unit vectors. Then Frenet-Serret trihedron as follows;

$$\mathbf{T} = \frac{\partial}{\partial x_1}, \mathbf{N} = \frac{\partial}{\partial x_2}, \mathbf{B} = \frac{\partial}{\partial x_3}, K = \frac{\partial}{\partial t}, \mathbf{T}^2 = \frac{\partial^2}{\partial x_1 \partial x_1}, \mathbf{TN} = \frac{\partial^2}{\partial x_1 \partial x_2}, \mathbf{T}L = \frac{\partial L}{\partial x_1}. \tag{16}$$

Let the semispray (ξ) be the vector field determined by

$$\xi = X^1\mathbf{T} + X^2\mathbf{N} + \mathbf{X}^3\mathbf{B} + TK, \tag{17}$$

where $X^1 = \dot{x}_1$, $X^2 = \dot{x}_2$, $X^3 = \dot{x}_3$, $T = \dot{t}$ and the dot indicates the derivative with respect to time t (similarly [8]). By means of the proper almost paracomplex structure J given by (11), the vector field is defined by

$$V = J(\xi) = X^1[\cosh t\mathbf{T}L + \sinh t\mathbf{N}L] + X^2[-\sinh t\mathbf{T}L - \cosh t\mathbf{N}L] + X^3 0, \tag{18}$$

which is named *Liouville vector field* on Minkowski space $\mathbb{R}_1^3$. The maps given by $T, P: \mathbb{R}_1^3 \to \mathbb{R}$ such that $T = \frac{1}{2} m_i (\dot{x}_1^{2} + \dot{x}_2^2 + \dot{x}_3^2)$, $P = m_i g h$ are said to be *the kinetic energy* and *the potential energy of the system*, respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on Minkowski space $\mathbb{R}_1^3$, respectively. Then $L: \mathbb{R}_1^3 \to \mathbb{R}$ is a map that satisfies the conditions; (i) $L = T - P$ is a *Lagrangian function*, (ii) the function determined by $E_L = V(L) - L$, is *energy function*. The function i_J induced by J and denoted by $i_J \omega(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} \omega(X_1, \ldots, JX_i, \ldots, X_n)$, is called *vertical derivation*, where $\omega \in \wedge^1 (\mathbb{R}_1^3)$, $X_i \in \chi(\mathbb{R}_1^3)$. The *vertical differentiation* $\mathbf{d}_J$ is given by $\mathbf{d}_J = [i_J, d] = i_J d - d i_J$, where d is the usual exterior derivation. For the almost paracomplex structure J given by (11), the form on Minkowski space $\mathbb{R}_1^4$ is the closed 2-form determined by $\Phi_L = -d(\mathbf{d}_J L)$ such that $\mathbf{d}_J: \mathcal{F}(\mathbb{R}_1^3) \to \wedge^1 \mathbb{R}_1^3$,

$$\begin{aligned}
&\mathbf{d} = \mathbf{T}dx_1 + \mathbf{N}dx_2 + \mathbf{B}dx_3 + Kdt,\\
&\mathbf{d}_J = J\mathbf{T}dx_1 + J\mathbf{N}dx_2 + J\mathbf{B}dx_3 + JKdt,\\
&\mathbf{d}_J L = [\cosh ht\mathbf{T}L + \sinh ht\mathbf{N}L]dx_1 + [-\sinh t\mathbf{T}L - \cosh t\mathbf{N}L]dx_2.
\end{aligned} \tag{19}$$

Now, we will calculate the first part of (13). Through a direct computation using (19), the closed 2-form Φ_L is seen to be as follows:

$$\begin{aligned}
\Phi_L = &(\cosh ht\mathbf{T}^2 + \sinh ht\mathbf{TN})dx_1 \wedge dx_1 + (-\sinh t\mathbf{T}^2 - \cosh t\mathbf{TN})dx_2 \wedge dx_1\\
&+(\cosh ht\mathbf{NT} + \sinh ht\mathbf{N}^2)dx_1 \wedge dx_2 + (-\sinh t\mathbf{NT} - \cosh t\mathbf{N}^2)dx_2 \wedge dx_2\\
&+(\cosh ht\mathbf{BT} + \sinh ht\mathbf{BN})dx_1 \wedge dx_3 + (-\sinh t\mathbf{BT} - \cosh t\mathbf{BN})dx_2 \wedge dx_3\\
&+(\sinh ht\mathbf{T}dt + \cosh htK\mathbf{T}dt + \cosh ht\mathbf{N}dt + \sinh htK\mathbf{N}dt)dx_1 \wedge dt\\
&+[-\cosh t\mathbf{T}dt - \sinh htK\mathbf{T}dt - \sinh t\mathbf{N}dt - \cosh htK\mathbf{N}dt]dx_2 \wedge dt
\end{aligned} \tag{20}$$

and then

$$\begin{aligned}
\Phi_L(\xi) = &X^1(\cosh ht\mathbf{T}^2L + \sinh ht\mathbf{TN}L)dx_1 - X^1(\cosh ht\mathbf{T}^2L + \sinh ht\mathbf{TN}L)dx_1\\
&-X^1(-\sinh t\mathbf{NT}L - \cosh t\mathbf{N}^2L)dx_2 + X^1(\cosh ht\mathbf{NT}L + \sinh ht\mathbf{N}^2L)dx_2\\
&+X^1(\cosh ht\mathbf{BT}L + \sinh ht\mathbf{BN}L)dx_3 + X^1(\sinh ht\mathbf{T}Ldt + \cosh htK\mathbf{T}Ldt + \cosh ht\mathbf{N}Ldt + \sinh htK\mathbf{N}Ldt)dt\\
&+X^2(-\sinh t\mathbf{T}^2L - \cosh t\mathbf{TN}L)dx_1 - X^2(\cosh ht\mathbf{T}^2L + \sinh ht\mathbf{TN}L)dx_1\\
&+X^2(-\sinh t\mathbf{NB}L - \cosh t\mathbf{N}^2L)dx_2 - X^2(-\sinh t\mathbf{NB}L - \cosh t\mathbf{N}^2L)dx_2\\
&+X^2(-\sinh t\mathbf{BT}L - \cosh t\mathbf{BN}L)dx_3 + X^2[-\cosh t\mathbf{T}Ldt - \sinh htK\mathbf{T}Ldt - \sinh t\mathbf{N}Ldt - \cosh htK\mathbf{N}Ldt]dt\\
&-X^3(\cosh ht\mathbf{BT}L + \sinh ht\mathbf{BN}L)dx_1 - X^3(-\sinh t\mathbf{BT}L - \cosh t\mathbf{BN}L)dx_2\\
&-T(\sinh ht\mathbf{T}Ldt + \cosh htK\mathbf{T}Ldt + \cosh ht\mathbf{N}Ldt + \sinh htK\mathbf{N}Ldt)dx_1\\
&-T[-\cosh t\mathbf{T}Ldt - \sinh htK\mathbf{T}Ldt - \sinh t\mathbf{N}Ldt - \cosh htK\mathbf{N}Ldt]dx_2.
\end{aligned} \tag{21}$$

Then the energy function E_L is found as follows:

$$E_L = V(L) - L = X^1[\cosh ht\mathbf{T}L + \sinh ht\mathbf{N}L] + X^2[-\sinh t\mathbf{T}L - \cosh t\mathbf{N}L] - L. \tag{22}$$

Now, we will calculate the second part of (13). Thus, the differential energy function is as follows:

$$\begin{aligned}
dE_L = &X^1(\cosh ht\mathbf{T}^2Ldx_1 + \sinh ht\mathbf{TN}Ldx_1) + X^2[-\sinh t\mathbf{T}^2Ldx_1 - \cosh t\mathbf{TN}Ldx_1] - \mathbf{T}Ldx_1\\
&+X^1(\cosh ht\mathbf{NT}Ldx_2 + \sinh ht\mathbf{N}^2Ldx_2) + X^2[-\sinh t\mathbf{NT}Ldx_2 - \cosh t\mathbf{N}^2Ldx_2] - \mathbf{N}Ldx_2\\
&+X^1(\cosh ht\mathbf{BT}Ldx_3 + \sinh ht\mathbf{BN}Ldx_3) + X^2[-\sinh t\mathbf{BT}Ldx_3 - \cosh t\mathbf{BN}Ldx_3] - \mathbf{B}Ldx_3\\
&+X^1(\sinh ht\mathbf{T}Ldt + \cosh htK\mathbf{T}Ldt + \cosh ht\mathbf{N}Ldt + \sinh htK\mathbf{N}Ldt)\\
&+X^2[-\cosh t\mathbf{T}Ldt - \sinh htK\mathbf{T}Ldt - \sinh t\mathbf{N}Ldt - \cosh htK\mathbf{N}Ldt] - KLdt.
\end{aligned} \tag{23}$$

According to (13), using (21) and (23) then we find the following equations.

$$\begin{aligned}
&1. -(X^1\mathbf{T} + X^2\mathbf{N} + \mathbf{X}^3\mathbf{B} + TK)(\cosh ht\mathbf{T}L) - (X^1\mathbf{T} + X^2\mathbf{N} + \mathbf{X}^3\mathbf{B} + TK)(\sinh ht\mathbf{N}L) + \mathbf{T}L = 0,\\
&\qquad -\xi(\cosh ht\mathbf{T}L) - \xi(\sinh ht\mathbf{N}L) + \mathbf{T}L = 0,\\
&2. \quad (X^1\mathbf{T} + X^2\mathbf{N} + \mathbf{X}^3\mathbf{B} + TK)(\sinh ht\mathbf{T}L) + (X^1\mathbf{T} + X^2\mathbf{N} + \mathbf{X}^3\mathbf{B} + TK)(\cosh ht\mathbf{N}L) + \mathbf{N}L = 0,\\
&\qquad \xi(\sinh t\mathbf{T}L) + \xi(\cosh t\mathbf{N}L) + \mathbf{N}L = 0.
\end{aligned} \tag{24}$$

Suppose that a curve $\beta: \mathbb{R}_1^3 \to \mathbb{R}$ be an integral curve and $\xi(\beta) = \frac{\partial}{\partial \mathbf{t}}(\beta)$. Where $\mathbf{t}$ is the time variable. Thus equations are as show Frenet-Serret trihedron;

$$\begin{aligned}
&dif1: \quad -\frac{\partial}{\partial \mathbf{t}}(\cosh ht\mathbf{T}L) - \frac{\partial}{\partial \mathbf{t}}(\sinh ht\mathbf{N}L) + \mathbf{T}L = 0,\\
&dif2: \quad \frac{\partial}{\partial \mathbf{t}}(\sinh t\mathbf{T}L) + \frac{\partial}{\partial \mathbf{t}}(\cosh t\mathbf{N}L) + \mathbf{N}L = 0.
\end{aligned} \tag{25}$$

The equations calculated in (25) are named paracomplex *Euler-Lagrange equations* constructed of the Frenet-Serret trihedron on Minkowski space. Thus the triple $(\mathbb{R}_1^3, \Phi_L, \xi)$ is named a **Euler-Lagrange mechanical system** on

Minkowski space.

11. COMPUTER SOLUTION OF EQUATIONS

The location of each object in space represented by the three dimensions. Next dimensions of objects in three dimensions shows time, position or mass. Examples of these, we can be time, position and mass. These found (25) are partial differential equation. We can solve these equations using Maple computer algebra-geometry program. The implicit solution of the above equation (25) is as follows:

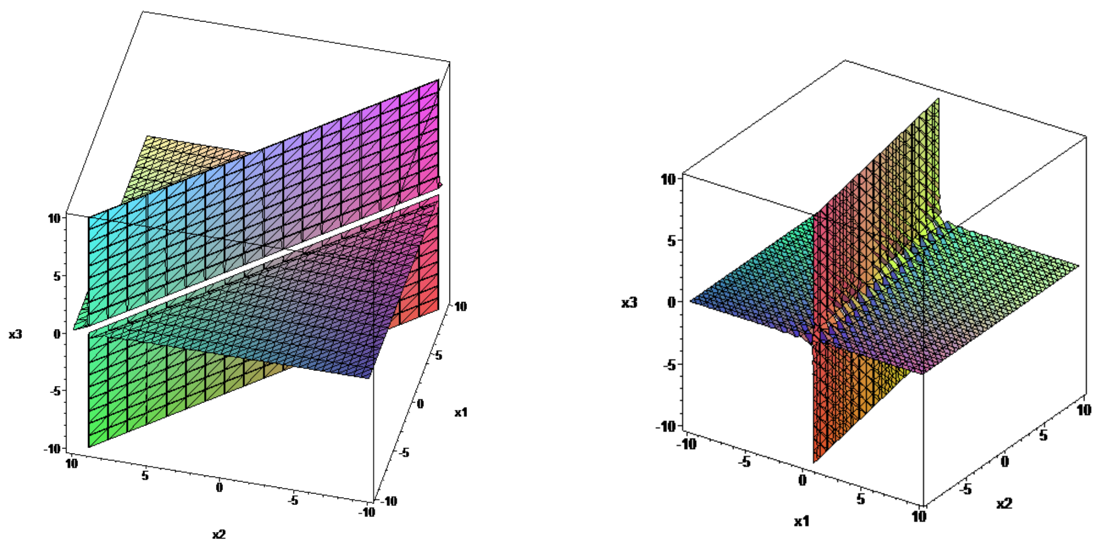
$$L(x_1,x_2,x_3,t,\mathbf{t}) = F_1(\mathbf{t},x_3) + exp(\mathbf{t}) * F_2(x_3,x_1) + F_3(\mathbf{t},x_3) + exp(-\mathbf{t}) * F_4(x_3,x_2). \qquad (26)$$

Here, $\mathbf{t}$ represents the time dimension. In addition, t be taken position dimension. The following graphs are obtained and for some value t and $\mathbf{t}$.

Example 1. Closed functions specially selected and $L(x_1,x_2,x_3,t,\mathbf{t})$ is obtained.

$$L(x_1,x_2,x_3,t,\mathbf{t}) = x_3 + x_3 * x_1 + x_2 * x_3, \quad for \ \ t = 0, \ \mathbf{t} = \mathbf{0}.$$

$$L(x_1,x_2,x_3,t,\mathbf{t}) = x_3 + \exp(1) * x_3 * x_1 + \exp(-1) * x_2 * x_3 \quad for \ \ t = 0, \ \mathbf{t} = \mathbf{1}.$$



(Graph 3)

(27)

12. DISCUSSION

Three spatial and one temporally unlimited space to be four-dimensional space is defined as a Minkowski space. Minkowski space with the help of the curves becomes understandable. Curves shows the path of in the space the moving object. Nonzero velocity vector at each point of the curve is defined regular curve. A regular space curve which carries on orthonormal trihedron examined by Frenet trihedron. It may also be taken on curved surfaces. This curve may also be taken on the surface and it can be defined the Darboux trihedron. Movement of a particle can be considered as a regular curve. On this curve, which is formed by the tangent, tangent vector area is also available. Movements of objects in space is modeled by the Euler-Lagrange equations. In this study, on Frenet-Serret trihedron of regular curves equation, representing a body motion, determined by the Euler-Lagrange equations.

Here obtained equations (25) are very important to explain the space-time mechanical and physical problems. Minkowski space has been used in solving problems in different physical areas. The paracomplex Euler-Lagrangian mechanical equations derived here are suggested to deal with problems in electrical, magnetical and gravitational fields of relativistic, quantum and classical mechanics of physics on Minkowski space [27, 28].

It is well-known that a classical field theory explain the study of how one or more physical fields interact with matter. Also, it is used quantum and classical mechanics of physics branches. In this study, the Euler-Lagrange mechanical equations (25) derived on a generalized on Minkowski space may be suggested to deal with problems in electrical, magnetical and gravitational fields for the path of movement (27) of defined space moving objects [29, 30].

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