SIMPSON'S RULE AND ROMBERG INTEGRATION TO SOLVE LINEAR FREDHOLM INTEGRAL EQUATION WITH CONTINUOUS KERNEL

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ABSTRACT
In this paper, we discussed Simpson's Rule (S.R) and Romberg Integration (R.I) are used to solve linear Fredholm integral equation of the second kind with continuous kernel. We have implemented some examples using these methods and we had a numerical comparison of the results to prove the effectiveness of these methods in the solution of the integral equations.

Keywords: *linear Fredholm Integral Equation; Simpson's Rule; Romberg Integration.*

1. Introduction

Linear Fredholm Integral Equations of the second kind draws great attention because of the importance of these equations in matters related to the physical issues and dynamics of various kinds studies [1]. There are many references that dealt with the fundamental concepts of integral equations [2], [3]. Also proved a lot of numerical methods to solve the kinds of integral equations, such as the Trapezoidal Rule, Nyström method, the Galerkin method; Simpson's Rule and Romberg Integration (see [4], [5], [6]). Mirzaee and Piroozfar in [7], used modified Simpson's quadrature rule for solving linear Fredholm integral equations of the second kind. In this paper, we use Simpson's Rule and Romberg Intrgration to discuss numerically the solution of the Linear Fredholm Integral Equation of the second kind with continuous kernel of the form

$$\mu\phi(x) = f(x) + \lambda\int_a^b k(x,t)\phi(t)dt \qquad (1)$$

where μ is a constant defines the kind of the integral equation, $\phi(x)$ is an unknown function, will be determined, the functions $f(x)$ and $k(x,t)$ are given analytical functions defined, respectively, $\{a \le x \le b, \quad a \le t \le b\}$, μ and λ are constants that have many physical meanings.

2. Simpson's Rule

Consider the Fredholm integral equation of the second kind

$$\phi(x) = f(x) + \int_a^b k(x,t)\phi(t)dt \qquad (2)$$

The function $f(x)$ and $k(x,t)$ are given functions. We shall assume that $f(x)$ is continuous on some interval $[a,b]$ say, and $k(x,t)$ is continuous in $a < t < x < b$ and that satisfies a uniform Lipschitz condition in ϕ.

We will subdivide the interval of integration $[a,b]$ in to n equal subintervals of width $h = \frac{x_n}{n}; n \ge 1, x_n = b.$

We shall set $x_i = ih, \quad 0 \le i \le n$ then, we can rewrite the integral part in (2) as:

$$\int_a^{x_i} k(x,t)\phi(t)dt \approx h\sum_{j=0}^{i} w_{ij}\, k\left(x_i, t_j\right)\phi\left(t_j\right)$$

$$= h\sum_{j=0}^{i} w_{ij} k_{ij} \phi\left(t_j\right) + E_{i,t}\left(k\left(x_i, t\right)\phi(t)\right) \tag{3}$$

Such that, $x = x_i = ih, \quad h = {b-a}/{n}, \quad x_i = t_i, i = 2,3,\ldots,n.$

Equation (2) represent an equal interval quadrature formula with remainder $E_{i,j}$, the weights $\{w_{ij}\}$ being supposed given or chosen.

The general form of Simpson's rule is:

$$A = \int_a^b g(x)dx = \frac{h}{3}\left[g(a) + 2\sum_{i=1}^{(n/2)-1} g\left(x_{2i}\right) + 4\sum_{i=1}^{n/2} g\left(x_{2i-1}\right) + g(b)\right] \tag{4}$$

We can apply the Simpson's rule on Fredholm integral equation by using Day's starting procedure.

Day's starting procedure:

Define:

$$\begin{aligned}
\phi_{11} &= f_1 + hk(h,0)f_0 \\
\phi_{12} &= f_1 + \frac{h}{2}\left[k\left(\mathrm{h},0\right)f_0 + k(h,h)\phi_{11}\right] \\
\phi_{13} &= f_{1/2} + \frac{h}{4}\left[k\left(\frac{h}{2},0\right)f_0 + k\left(\frac{h}{2},\frac{h}{2}\right)\left(\frac{f_0}{2} + \frac{\phi_{12}}{2}\right)\right]
\end{aligned} \tag{5}$$

Then:

$$\phi_1 = f_1 + \frac{h}{6}\left[k(h,0)f_0 + 4k\left(h,\frac{h}{2}\right)\phi_{13} + k(h,h)\phi_{12}\right] \tag{6}$$

Next let:

$$\phi_{21} = f_2 + 2hk(2h,\mathrm{h})\phi_1$$

Then:

$$\phi_2 = f_2 + \frac{h}{3}\left[k(2h,0)f_0 + 4k\left(2h,\mathrm{h}\right)\phi_1 + k(2h,2h)\phi_{21}\right] \tag{7}$$

Finally with,

$$\phi_{31} = f_3 + \frac{3h}{2}\left[k\left(3\mathrm{h},\mathrm{h}\right)\phi_1 + k(3h,2h)\phi_2\right]$$

We obtain:

$$\phi_3 = f_3 + \frac{3h}{8}\left[k(3h,0)f_0 + 3k\left(3h,\mathrm{h}\right)\phi_1 + 3k(3h,2h)\phi_2 + k(3\mathrm{h},3\mathrm{h})\phi_{31}\right]$$

A convenient and simple continuation of Day's starting procedure can be based on Simpson's rule in the following manner.(For only ϕ_0 and ϕ_1 are required) we can use Simpson's rule to give

$$\phi_r = f_r + \frac{h}{3}\sum_{j=0}^{r} w_{rj}\, k(rh, jh)\phi_j\,, \quad r = 2,3,\ldots \tag{8}$$

Where r is even and the weights are given by:

$$w_{r0} = w_{rr} = 1, \quad w_{rj} = 3-(-1)^j\,, \quad 1 \le j \le r-1$$

Since the integral in (2) vanishes for $r = 0,$ then:

$$\phi_0 = f_0 \tag{9}$$

The equations (6),(8) and (9) (Which are $r+1$ equations in ϕ_i , $0 \le i \le r$ represented the approximation to the solution $\phi(x)$ of (2) at $rh,\ r = 0,1,\ldots$ and can be written as the following system:

$$\phi_0 = f_0$$

$$\phi_1 = f_1 + \frac{h}{6}\left[k(h,0)f_0 + 4k\left(h,\frac{h}{2}\right)\phi_{13} + k(h,h)\phi_{12}\right]$$

$$\phi_2 = f_2 + \frac{h}{3}\left[k(2h,0)\phi_0 + 4k(2h,h)\phi_1 + k(2h,2h)\phi_2\right]$$

$$\phi_3 = f_3 + \frac{h}{3}\left[k(3h,0)\phi_0 + 4k(3h,h)\phi_1 + 2k(3h,2h)\phi_2 + k(3h,3h)\phi_3\right]$$

$$\phi_n = f_n + \frac{h}{3}\left[k(nh,0)\phi_0 + 4k(nh,h)\phi_1 + \cdots + 4k(nh,(n-1)h)\phi_{n-1} + k(nh,nh)\phi_n\right]$$

3. Romberg Integration

The Romberg integration is depend on the Trapezoidal rule for integer the function $f(x)$ in the interval $[a,b]$.

Consider the intervals $[x_{i-1}, x_i]$ where $x_i - x_{i-1} = h,\ i = 1,2,\ldots,n$ and put $x_0 = a, x_n = b$ then we get

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] - \frac{b-a}{12}h^2 f''(\zeta) \tag{10}$$

$a<\zeta<b, h=\dfrac{b-a}{n}, x_i=a+ih, i=0,1,\ldots,n$ if $h_k=\dfrac{b-a}{n_k}=\dfrac{b-a}{2^{k-1}}$, the Trapezoidal rule is become in the form

$$\int_a^b f(x)dx=\frac{h_k}{2}\left[f(a)+f(b)+2\sum_{i=1}^{2^{k-1}-1} f(n+ih_k)\right]-\frac{b-a}{12}h_k{}^2 f''(\zeta_k) \qquad (11)$$

Let

$$R_{1,1}=\frac{h_1}{2}\left[f(a)+f(b)\right]=\frac{b-a}{2}\left[f(a)+f(b)\right] \qquad (12)$$

$$R_{2,1}=\frac{h_2}{2}\left[f(a)+f(b)+2\text{f}(a+h_2)\right]=\frac{1}{2}\left[R_{1,1}+h_1 f(n+h_1/2)\right] \qquad (13)$$

$$R_{3,1}=\frac{h_3}{2}\left[f(a)+f(b)+2\left[f(a+h_3)+f(a+2h_3)+f(a+3h_3)\right]\right]$$

$$R_{3,1}=\frac{1}{2}\left[R_{2,1}+h_2\left[f(a+h_2/2)+f(a+3h_2/2)\right]\right] \qquad (14)$$

In general, we get

$$R_{k,1}=\frac{1}{2}\left[R_{k-1,1}+h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(i-1/2)h_{k-1})\right], \quad k=2,3,\ldots,n \qquad (15)$$

Now, consider the Fredholm integral equation of the second kind

$$\phi(x)=f(x)+\int_a^b k(x,t)\phi(t)dt, \quad \lambda=1 \qquad (16)$$

Using equation (12) in (16), we get

$$\phi(x)=f(x)+\int_a^b k(x,t)\phi(t)dt \;\square\; f(x)+$$

$$+\frac{h_k}{2}\left[k(x,t_0)\phi(t_0)+2\sum_{i=1}^{2^{k-1}-1} k(x,t_i)\phi(t_i)+k(x,t_n)\phi(t_n)\right] \qquad (17)$$

or

$$\phi(x)=f(x)+\frac{h_k}{2}\left[k(x,t_0)\phi_0+2\sum_{i=1}^{2^{k-1}-1} k(x,t_i)\phi_i+k(x,t_n)\phi_n\right] \qquad (18)$$

where equations (17) and (18) are the approximate solutions for equation (16) and we can found the error if we solve $n+1$of the values $\phi_i=\phi(x_i)=\phi(t_i), \quad i=0,1,\ldots,n$. Then, equation (18) becomes the system of $n+1$ of equations

$$\phi_j=f_j+\frac{h_k}{2}\left[k_{j0}\phi_0+2\sum_{i=1}^{2^{k-1}-1} k_{ji}\phi_i+k_{jn}\phi_n\right], \quad j=0,1,\ldots,n, \quad k=1,2,\ldots,n$$

or

$$\phi_j=f_j=\frac{h_k}{2}\left[k_{j0}\phi_0+2(k_{j1}\phi_1+k_{j2}\phi_2+\ldots+k_{j,2^{k-1}-1}\phi_{2^{k-1}-1})+k_{jn}\phi_n\right]$$

or in the matrix form

$$K\Phi=F$$

4. Numerical Experiments and Discussions

Example 1:

Consider the linear Fredholm integral equation:

$$\phi(x) = \frac{2}{3}x + \int_0^1 xt\,\phi(t)dt \qquad (19)$$

where the exact solution is $\phi(x) = x$, here $\lambda = 1$, $\mu = 1$. In table (4.1)-(4.2) we present the exact solution, the approximate numerical solutions and their corresponding errors for some points, we suppose that $N = 20, 50.$

In tables (4.1)-(4.4):

$\phi^{S.R} \rightarrow$ approximate solution of S.R, $E^{S.R} \rightarrow$ the error of S.R, $\phi^{R.I} \rightarrow$ approximate solution of R.I and $E^{R.I} \rightarrow$ the error of R.I .

Case 1: $N = 20$,

x	Exact. sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.1	0.10000000	0.1000375102	3.75102×10^{-5}	0.1000333395	3.33395×10^{-5}
0.2	0.20000000	0.2005508573	5.508573×10^{-4}	0.2005340506	5.340506×10^{-4}
0.3	0.30000000	0.30275079664	2.75079664×10^{-3}	0.3027122058	2.7122058×10^{-3}
0.4	0.40000000	0.4086964066	8.6964066×10^{-3}	0.4086250544	8.6250544×10^{-3}
0.5	0.50000000	0.5089923715	8.9923715×10^{-3}	0.5088735530	8.8735530×10^{-3}
0.6	0.60000000	0.6185179655	1.85179655×10^{-2}	0.6183105968	1.8310596×10^{-2}
0.7	0.70000000	0.7528666308	5.28666308×10^{-2}	0.7526400968	5.2640096×10^{-2}
0.8	0.80000000	0.8781703379	7.81703379×10^{-2}	0.8778693617	7.7869361×10^{-2}
0.9	0.90000000	0.9803701723	8.03701723×10^{-2}	0.9800062595	8.0006259×10^{-2}
1.0	1.00000000	1.101132479	1.01132479×10^{-1}	1.100553106	1.0055310×10^{-1}

Table(4.1)

Case 2: $N = 50$,

x	Exact.sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.1	0.10000000	0.1000340063	3.40063×10^{-5}	0.100033390	3.33390×10^{-5}
0.2	0.20000000	0.2005367349	5.36734×10^{-4}	0.2005340453	5.340453×10^{-4}
0.3	0.30000000	0.3019860384	1.98603×10^{-3}	0.3019841735	1.984173×10^{-3}
0.4	0.40000000	0.4071815748	7.18157×10^{-3}	0.4071769229	7.176922×10^{-3}
0.5	0.50000000	0.5084157696	8.41576×10^{-3}	0.508414589	8.414589×10^{-3}
0.6	0.60000000	0.6090730983	9.07309×10^{-3}	0.6090584160	9.058416×10^{-3}
0.7	0.70000000	0.7138041324	1.38041×10^{-2}	0.7137672528	1.376725×10^{-2}
0.8	0.80000000	0.8471133077	4.71133×10^{-2}	0.8470600905	4.706009×10^{-2}
0.9	0.90000000	0.9734641078	7.34641×10^{-2}	0.9733081881	7.330818×10^{-2}
1.0	1.00000000	1.081324772	8.13247×10^{-2}	1.081151062	8.115106×10^{-2}

Table(4.2)

Example 2:

Consider the linear Fredholm integral equation:

$$\phi(x) = \frac{x}{4} + \frac{1}{6} + \int_0^1 (x - t)\phi(t)dt \qquad (20)$$

where the exact solution is $\phi(x)=\frac{x}{2}$, here $\lambda=1,\ \mu=1$. In table (4.3)-(4.4) we present the exact solution, the approximate numerical solutions and their corresponding errors for some points, we suppose that $N=20,50.$

Case 1: $N=20$,

x	Exact.sol.	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.1	0.050000000	0.0500833780	8.3378×10^{-5}	0.05006250000	6.250000×10^{-5}
0.2	0.100000000	0.1006689872	6.6898×10^{-4}	0.1006259379	6.259379×10^{-4}
0.3	0.150000000	0.1522601252	2.26012×10^{-3}	0.152196241	2.1962624×10^{-3}
0.4	0.200000000	0.2053761335	5.376133×10^{-3}	0.2052895041	5.2895041×10^{-3}
0.5	0.250000000	0.255476148	5.476148×10^{-3}	0.2554369223	5.4369223×10^{-3}
0.6	0.300000000	0.3063344099	6.334409×10^{-3}	0.3063132812	6.3132812×10^{-3}
0.7	0.350000000	0.3566867589	6.686758×10^{-3}	0.3566415785	6.6415785×10^{-3}
0.8	0.400000000	0.4136010212	1.360102×10^{-2}	0.4135285143	1.3528514×10^{-2}
0.9	0.450000000	0.4756547389	2.565473×10^{-2}	0.4755533198	2.5553319×10^{-2}
1.0	0.500000000	0.5424689684	4.2468968×10^{-2}	0.5423366308	4.2336630×10^{-2}

Table(4.3)

Case 2: $N=50$,

x	Exact.sol	$\phi^{S.R}$	$E^{S.R}$	$\phi^{R.I}$	$E^{R.I}$
0	0	0	0	0	0
0.1	0.0500000000	0.05008336470	8.336470×10^{-5}	0.05006003361	6.003361×10^{-5}
0.2	0.1000000000	0.1006670009	6.670009×10^{-4}	0.1006212683	6.212683×10^{-4}
0.3	0.1500000000	0.152066012	2.066012×10^{-3}	0.152041286	2.041286×10^{-3}
0.4	0.2000000000	0.2022737124	2.273712×10^{-3}	0.2022686570	2.268657×10^{-3}
0.5	0.2500000000	0.2532118983	3.2118983×10^{-3}	0.253093911	3.093911×10^{-3}
0.6	0.3000000000	0.3038981893	3.8981893×10^{-3}	0.3038929918	3.892991×10^{-3}
0.7	0.3500000000	0.3553171953	5.3171953×10^{-3}	0.3553146536	5.314653×10^{-3}
0.8	0.4000000000	0.4055226662	5.5226662×10^{-3}	0.4055173265	5.517326×10^{-3}
0.9	0.4500000000	0.4567224924	6.7224924×10^{-3}	0.4567199161	6.719916×10^{-3}
1.0	0.5000000000	0.5071471432	7.1471432×10^{-3}	0.5071416613	7.141661×10^{-3}

Table(4.4)

5. The Conclusion

From the previous discussions we conclude the following:

1) The error in the estimation of the approximate solution, by using the Romberg Integration is less than the error in the estimation of the approximate solution, using the Simpson's Rule, in all cases in the examples.

2) If n is increased, approximate solution better and less error, when $n=50$ approximate solution is the best and values of the error is the less than $n=20.$

So, the stability of the Romberg Integration more than the Simpson's Rule and then the Romberg Integration is better to evaluate the approximate solution than the Simpson's Rule.

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