HYPERCHAOTIC BEHAVIOR IN A NEW 3-D DISCRETE-TIME SYSTEMS

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ABSTRACT
In this paper we give a strict proof of hyperchaos in a new 3-D discrete-time systems using the standard definition of the largest Lyapunov exponent as the usual test for chaos and hyperchaos.

Keywords: *Lyapunov exponents, hyperchaos, three-dimensional discrete-time systems.*

1. INTRODUCTION

In the theoretical research of the chaotic dynamical systems, sensitivity of initial conditions is quantified by the Lyapunov exponents. For example, consider two trajectories with nearby initial conditions on an attracting manifold, when the attractor is chaotic, the trajectories diverge, on average, at an exponential rate characterized by the largest Lyapunov exponent [1-9-10-12].

The term hyperchaos was introduced by Rossler [2], hyperchaos generation guaranteed by two or more positive Lyapunov exponents. For continuous-time systems, hyperchaos exists only in higher than or equal to four-dimensional autonomous systems, while in discrete-time systems only in higher than or equal to two-dimensional maps [5-6-7]. Hyperchaotic behaviour has attracted considerable attention because of its theoretical and practical applications in various fields [3-4].

The purpose of this paper is to prove strictly the existence of hyperchaos in a three dimensional discrete-time systems using the standard definition of the largest Lyapunov exponent [8-11-13]. Lyapunov exponents as the usual test for chaos and hyperchaos.

2. A NEW 3-D DISCRETE-TIME DYNAMICAL SYSTEMS

Let us consider the following new 3-dimensional discrete-time dynamical systems defined by:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} ax_n - y_n^2 - z_n \\ \alpha + by_n - z_n^2 \\ \beta + cz_n \end{pmatrix} \tag{1}$$

Where a , b , c , α and β are constant parameters, and $X_n = (x_n, y_n, z_n) \in \mathrm{R}^3$ is the state variable, X_0 is the initail state, and $n = 0,1,2,\ldots$ is the discrete time.

DEFINITION 1 (Lyapunov exponents for a discrete dynamical system [9]): Consider the following 3-dimensional discrete-time dynamical system:

$$X_{n+1} = f(X_n), X_n \in \mathrm{R}^3, n = 0,1,2,\ldots \tag{2}$$

where the function $f : \mathrm{R}^3 \to \mathrm{R}^3$ is the map associated with system (1). Let $J(X_n)$ be its Jacobian evaluated at $X_n \in \mathrm{R}^3, n = 0,1,2,\ldots$, and define the matrix:

$$T_N(X_0) = J(X_{N-1})J(X_{N-2})\ldots J(X_1)J(X_0) \tag{3}$$

Moreover, Let $J_i(X_0, N)$ be the module of the i^{th} eigenvalue of the N^{th} matrix $T_N(X_0)$, where $i = 1,2,3$ and $N = 0,1,2,\ldots$

Now, the Lyapunov exponents for a three-dimensional discrete-time system are defined by:

$$l_i(X_0) = \ln(\lim_{N \to +\infty} J_i(X_0, N)^{\frac{1}{N}}), i = 1,2,3. \tag{4}$$

3. PROOF OF HYPERCHAOS USING THE LARGEST LYAPUNOV EXPONENTS

The goal of this paper is the strict proof in which the 3-D discret-time dynamical systems given by equation (1) has two or more positive Lyapunouv exponents for some ranges of parameters space. We use the method of the calculation of the largest Lyapunouv exponent.

The Jacobian matrix of the system (1) is given by:

$$J(X_n) = \begin{pmatrix} a & -2y_n & -1 \\ 0 & b & -2z_n \\ 0 & 0 & c \end{pmatrix} \tag{5}$$

The matrix $T_N(X_0)$ is given by:

$$T_N(X_0) = \begin{pmatrix} a & \frac{-2\partial y_N}{\partial y} & -1 \\ 0 & b & \frac{-2\partial z_N}{\partial z} \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} a & \frac{-2\partial y_{N-1}}{\partial y} & -1 \\ 0 & b & \frac{-2\partial z_{N-1}}{\partial z} \\ 0 & 0 & c \end{pmatrix} \cdots \begin{pmatrix} a & \frac{-2\partial y_0}{\partial y} & -1 \\ 0 & b & \frac{-2\partial z_0}{\partial z} \\ 0 & 0 & c \end{pmatrix} \tag{6}$$

Now, some algebra leads to the follwing simplified from of the matrix $T_N(X_0)$

$$T_N(X_0) = \begin{pmatrix} a^{N+1} & A & B \\ 0 & b^{N+1} & C \\ 0 & 0 & c^{N+1} \end{pmatrix} \tag{7}$$

where A, B and C are elements in the above upper triangular matrix are not important for calculing, the eigenvalues of $T_N(X_0)$ are defined by:

$$\delta_1 = a^{N+1} \tag{8}$$

$$\delta_2 = b^{N+1} \tag{9}$$

$$\delta_3 = c^{N+1} \tag{10}$$

and therefore one has:

$$J_1(X_0, N) = |\delta_1| = |a^{N+1}| \tag{11}$$

$$J_2(X_0, N) = |\delta_2| = |b^{N+1}| \tag{12}$$

$$J_3(X_0, N) = |\delta_3| = |c^{N+1}| \tag{13}$$

and it follow from the definition of the Lyapunov exponent (4) that:

$$l_1(X_0) = \ln(\lim_{N \to +\infty} J_1(X_0, N)^{\frac{1}{N}}) = \ln(\lim_{N \to +\infty} |a^{N+1}|^{\frac{1}{N}}) = \ln|a| \tag{14}$$

$$l_2(X_0) = \ln(\lim_{N \to +\infty} J_2(X_0, N)^{\frac{1}{N}}) = \ln(\lim_{N \to +\infty} |b^{N+1}|^{\frac{1}{N}}) = \ln|b| \tag{15}$$

$$l_3(X_0) = \ln(\lim_{N \to +\infty} J_3(X_0, N)^{\frac{1}{N}}) = \ln(\lim_{N \to +\infty} |c^{N+1}|^{\frac{1}{N}}) = \ln|c| \tag{16}$$

Finally, then the systems (1) is hyperchaotic in the sense that the largest Lyapunov exponent if more than one Lyapunov exponents of system (1) are positive.

Hence we have proved the following theorem:

THEOREM 2: The systems (1) is hyperchaotic in the sense that the largest Lyapunov exponent if one of the following conditions is satisfied:

$$|a|>1,|b|>1 \quad (17)$$

$$|a|>1,|c|>1 \quad (18)$$

$$|b|>1,|c|>1 \quad (19)$$

$$|a|>1,|b|>1,|c|>1 \quad (20)$$

4. EXAMPLE

In this section, we give an elementary example of the above analytical analysis. Indeed, we choose the parameters $a=2, b=5, c=\frac{1}{2}, \alpha=1$ and $\beta=\frac{1}{2}$. The system (1) exhibits hyperchaotic behavior, in this case, the system has two positive Lyapunov exponents, $l_1=0.693, l_2=1.609$ and single negative Lyapunov exponent, $l_3=-0.693$. On the other hand, the one fixed point of the system (1) is $E=(1,0,1)$ and the Jacobian matrix of (1) evaluated at a point E is given by:

$$J_E=\begin{pmatrix} 2 & 0 & -1 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad (21)$$

and its characteristic polynomial is $(2-\lambda)(5-\lambda)(\frac{1}{2}-\lambda)=0$. Hence, the eigenvalues at $E=(1,0,1)$, are $\lambda_1=2, \lambda_2=5$ and $\lambda_3=\frac{1}{2}$. Then E is a saddle fixed point under these parameters.

5. CONCLUSION

Analytic evaluation of the Lyapunov exponents for dynamical systems is available rarely, so, usually we need to turn to the computational estimates. In this paper, we show that there exist hyperchaotic behavior in a three dimensional discrete-time systems using the standard definition of Lyapunov exponents.

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