FEKETE–SZEGÖ FUNCTIONAL FOR A GENERALIZED CLASS OF NON-BAZILEVIČ FUNCTIONS DEFINED BY USING A DIFFERENTIAL OPERATOR

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ABSTRACT

In this paper we obtain Fekete–Szegö inequalities for a new subclass $N^{n,b}_{\alpha,\mu,\lambda,\omega}(\varphi)$ of analytic functions defined by using a generalized differential operator. Some of our results generalize the related work of some authors.

Key words: a*nalytic, starlike, convex, Fekete–Szegö functional, Differential operator.*
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1. INTRODUCTION

Let $U = \{z : |z| < 1\}$ be the open unit disk and let $\mathcal{A}$ be the class of functions which are analytic in U given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U. \tag{1.1}$$

Also, let S denote the subclass of $\mathcal{A}$ consisting of univalent functions.
A function $f \in \mathcal{A}$ is said to be in the class $N_{\beta,\gamma,\delta}$ if it satisfies

$$\operatorname{Re}\left\{(1-\beta)\left[\frac{z}{f(z)}\right]^{\gamma} + \beta \frac{zf'(z)}{f(z)}\left[\frac{z}{f(z)}\right]^{\gamma}\right\} > \delta, \tag{1.2}$$

where $0 < \beta \le 1, 0 < \gamma < 1, 0 \le \delta < 1$ and $z \in U$.

The function class $N_{\beta,\gamma,\delta}$ was studied in [2, 11, 7]. We note that the class $N_{1,\gamma,0} \equiv N_{\gamma}$ was introduced by Obradovič [10] who called it the non-Bazilevič type class of functions.

If f and g are two analytic functions in U, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, for all $z \in U$.

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [9]).

Recently, for a function $f \in \mathcal{A}, \alpha, \mu, \lambda, \omega \ge 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, authors [3] defined the following differential operator, as follows:

$$D^0 f(z) = f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

$$D^1_{\alpha,\mu}(\lambda,\omega) f(z) = (\alpha - \mu\omega^{\lambda}) f(z) + (\mu\omega^{\lambda} - \alpha + 1) z f'(z)$$

$$= z + \sum_{k=2}^{\infty} [(k-1)(\mu\omega^{\lambda} - \alpha) + k] a_k z^k,$$

$$D^2_{\alpha,\mu}(\lambda,\omega) f(z) = D(D^1_{\alpha,\mu}(\lambda,\omega) f(z)),$$

$$\vdots$$

$$D^{n}_{\alpha,\mu}(\lambda,\omega)f(z)=D(D^{n-1}_{\alpha,\mu}(\lambda,\omega)f(z)), \quad (1.3)$$

with $D^{n}_{\alpha,\mu}(\lambda,\omega)f(0)=0$.

So, if f is given by (1), we see that

$$D^{n}_{\alpha,\mu}(\lambda,\omega)f(z)=z+\sum_{k=2}^{\infty}[(k-1)(\mu\omega^{\lambda}-\alpha)+k]^{n}a_k z^k. \quad (1.4)$$

Clearly:

(i) for $\alpha=\mu=0$, we obtain Sălăgean's differential operator [12],

(ii) for $\lambda=1,\omega=1$ and $\alpha=1$ we get the operator introduced by F. Al-Oboudi [1],

(iii) for $\lambda=1,\omega=1$ we obtain the operator introduced by M. Darus and R.W. Ibrahim [5],

(iv) for $\alpha=1$ we get the operator introduced by M. Darus and I. Faisal [4].

Next, using the differential operator $D^{n}_{\alpha,\mu}(\lambda,\omega)f(z)$, we introduce the following generalized non-Bazilevič class of functions as follows:

Definition 1.1 *Let φ be an univalent function which maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis, $\varphi(0)=1$ and $\varphi'(0)>0$. The class $N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$ consists of all functions $f\in\mathcal{A}$ satisfying the following subordination:*

$$(1-\beta)\left[\frac{z}{D^{n}_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}+\beta\frac{z(D^{n}_{\alpha,\mu}(\lambda,\omega)f(z))'}{D^{n}_{\alpha,\mu}(\lambda,\omega)f(z)}\left[\frac{z}{D^{n}_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}\prec\varphi(z), \quad (1.5)$$

for $\alpha,\mu,\lambda,\omega\geq 0, 0<\beta\leq 1, 0<\gamma<1, n\in\mathbb{N}_0$ and $z\in U$.

We remark that:

(i) $N^{0,\beta}_{\alpha,\mu,\lambda,\omega}(\frac{1+(1-2\delta)z}{1-z})\equiv N^{\beta}(\delta)$, where $N(\delta)$ is the class studied by Jiang and Guo in [7].

(ii) $N^{0,1}_{\alpha,\mu,\lambda,\omega}(\frac{1+(1-2\delta)z}{1-z})\equiv N(\delta)$, where $N(\delta)$ is the class of non-Bazilevič functions studied in [14].

(iii) $N^{0,1}_{\alpha,\mu,\lambda,\omega}(\frac{1+z}{1-z})\equiv N_{\gamma}$, where N_{γ} is the class of non-Bazilevič functions (see [10]).

In our paper, we obtain Fekete–Szegö inequalities for the subclass $N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$. Recently, Fekete–Szegö problem was studied by many authors [7, 6, 13], and some of them motivated us to come up with the class abovementioned.

Lemma 1.1 *[8] If $p(z)=1+c_1z+c_2z^2+\ldots, z\in U$ is a function with positive real part in U and ζ is a complex number, then*

$$|c_2-\zeta c_1^2|\leq 2\cdot\max\{1;|2\zeta-1|\}.$$

The result is sharp for the function given by

$$p(z)=\frac{1+z^2}{1-z^2}\text{ and } p(z)=\frac{1+z}{1-z}, \quad z\in U.$$

Lemma 1.2 *[8] If $p(z)=1+c_1z+c_2z^2+\ldots, z\in U$ is an analytic function with positive real part in U, then*

$$|c_2-\psi c_1^2|\leq\begin{cases}-4\psi+2 & \text{if } \psi\leq 0,\\ 2 & \text{if } 0\leq\psi\leq 1,\\ 4\psi-2 & \text{if } \psi\geq 1,\end{cases}$$

where :

(i) If $\psi<0$ or $\psi>1$, the equality holds if and only if $p(z)=\dfrac{1+z}{1-z}$ or one of its rotations.

(ii) If $0<\psi<1,$ the equality holds if and only if $p(z)=\dfrac{1+z^2}{1-z^2}$ or one of its rotations.

(iii) If $\psi=0$, the equality holds if and only if

$$p(z)=\left(\frac{1+b}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-b}{2}\right)\frac{1-z}{1+z},0\le b\le 1,$$

or one of its rotations.

(iv) If $\nu=1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $\psi=0$.

Also, the above upper bound is sharp and it can be improved as follows, when $0<\psi<1$:

$$|c_2-\psi c_1^2|+\psi|c_1|^2\le 2,\ \ 0\le\psi\le\frac{1}{2},$$

and

$$|c_2-\psi c_1^2|+(1-\psi)|c_1|^2\le 2,\ \ \frac{1}{2}\le\psi\le 1.$$

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that $n\in\mathbb{N}_0,\ \alpha,\mu,\lambda,\omega\ge 0, 0<\beta\le 1, 0<\gamma<1$ with $2\beta-\gamma\ne 0$ and $\beta-\gamma\ne 0, 0\le\delta<1$ and $z\in U$.
By using Lemma 1.1, we obtain the following theorem.

Theorem 2.1 *If function f given by (1) belongs to the class $N_{\alpha,\mu,\lambda,\omega}^{n,\beta}(\varphi)$ and ζ is a complex number, then*

$$|a_3-\zeta a_2^2|\le\frac{B_1}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|}$$
$$\cdot\max\left\{1,\left|\frac{B_2}{B_1}+\frac{(2\beta-\gamma)B_1}{(\beta-\gamma)^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\zeta\right)\right|\right\}.$$

The result is sharp.

Proof. Since $f\in N_{\alpha,\mu,\lambda,\omega}^{n,\beta}(\varphi),$ there exists a function w which is analytic in U with $w(0)=0, |w(z)|<1$ in U, and such that

$$(1-\beta)\left[\frac{z}{D_{\alpha,\mu}^n(\lambda,\omega)f(z)}\right]^\gamma+\beta\frac{z(D_{\alpha,\mu}^n(\lambda,\omega)f(z))'}{D_{\alpha,\mu}^n(\lambda,\omega)f(z)}\left[\frac{z}{D_{\alpha,\mu}^n(\lambda,\omega)f(z)}\right]^\gamma=\varphi(w(z)). \tag{2.1}$$

Letting the function p be defined by

$$p(z):=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\ldots \tag{2.2}$$

it is clear that p is an analytic function with $p(0)=1$ and $Re\{p(z)\}>0$ for all $z\in U$. Therefore,

$$\varphi(w(z))=\varphi\left(\frac{p(z)-1}{p(z)+1}\right)=\varphi\left(\frac{1}{2}[c_1z+(c_2-\frac{c_1^2}{2})z^2+(c_3-c_1c_2+\frac{c_1^3}{4})z^3+\ldots]\right)$$
$$=1+\frac{1}{2}B_1c_1z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2]z^2+\ldots. \tag{2.3}$$

Since

$$(1-\beta)\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}+\beta\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}=$$
$$1+(\beta-\gamma)[(\mu\omega^{\lambda}-\alpha)+2]^n a_2 z+$$
$$(2\beta-\gamma)\left\{[2(\mu\omega^{\lambda}-\alpha)+3]^n a_3-\frac{\gamma+1}{2}[(\mu\omega^{\lambda}-\alpha)+2]^{2n}a_2^2\right\}z^2+\ldots, \tag{2.4}$$

it follows by (2.1), (2.3) and (2.4) that

$$a_2=\frac{B_1c_1}{2(\beta-\gamma)[(\mu\omega^{\lambda}-\alpha)+2]^n},$$

and

$$a_3=\frac{1}{2(2\beta-\gamma)[2(\mu\omega^{\lambda}-\alpha)+3]^n}\left[B_1c_2+(B_2-B_1)\frac{c_1^2}{2}\right]+\frac{(\gamma+1)B_1^2c_1^2}{8(\beta-\gamma)^2[2(\mu\omega^{\lambda}-\alpha)+3]^n}.$$

Therefore,

$$a_3-\zeta a_2^2=\frac{B_1}{2(2\beta-\gamma)[2(\mu\omega^{\lambda}-\alpha)+3]^n}[c_2-c_1^2\cdot\eta],$$

where

$$\eta=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(2\beta-\gamma)B_1}{(\beta-\gamma)^2}\left(\frac{[2(\mu\omega^{\lambda}-\alpha)+3]^n}{[(\mu\omega^{\lambda}-\alpha)+2]^{2n}}\zeta-\frac{\gamma+1}{2}\right)\right].$$

It follows from Lemma 1.1 that

$$|a_3-\zeta a_2^2|\le\frac{B_1}{[2(\mu\omega^{\lambda}-\alpha)+3]^n\,|2\beta-\gamma|}$$
$$\cdot\max\left\{1,\left|\frac{B_2}{B_1}+\frac{(2\beta-\gamma)B_1}{(\beta-\gamma)^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^{\lambda}-\alpha)+3]^n}{[(\mu\omega^{\lambda}-\alpha)+2]^{2n}}\zeta\right)\right|\right\}.$$

This result is sharp for the function f given by

$$(1-\beta)\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}+\beta\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}=\varphi(z^2),$$

or

$$(1-\beta)\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}+\beta\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)}\right]^{\gamma}=\varphi(z).$$

This completes the proof.

Taking $n=0$ in Theorem 2.1 we obtain:

Corollary 2.1 *If function f given by (1) belongs to the class $N^{0,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$ and ζ is a complex number, then*

$$|a_3-\zeta a_2^2|\le\frac{B_1}{|2\beta-\gamma|}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{(2\beta-\gamma)(1+\gamma-2\zeta)B_1}{2(\beta-\gamma)^2}\right|\right\}.$$

The result is sharp.

Corollary 2.2 *If f given by (1) belongs to the class $N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\frac{1+Az}{1+Bz})$ then for any complex number ζ, we have*

$$|a_3-\zeta a_2^2|\le \frac{A-B}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|}$$
$$\cdot\max\left\{1,\left|\frac{(2\beta-\gamma)(A-B)}{(\beta-\gamma)^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\zeta\right)-B\right|\right\}.$$

The result is sharp.

Setting $n=0, A=1-2\delta, 0\le\delta<1$ and $B=-1$ in Corollary 2.2 we get the result of Jiang ang Guo [7], Theorem 2:

Corollary 2.3 *If f given by (1) belongs to the class $N^{\beta}(\delta)$ then for any complex number ζ, we have*

$$|a_3-\zeta a_2^2|\le \frac{2(1-\delta)}{|2\beta-\gamma|}\cdot\max\left\{1,\left|1+\frac{(1-\delta)(2\beta-\gamma)(1+\gamma-2\zeta)}{(\beta-\gamma)^2}\right|\right\}.$$

Also, setting $\beta=1$ in Corollary 2.3 we get the result of Tuneski and Darus [14], Theorem 1:

Corollary 2.4 *If f given by (1) belongs to the class $N(\delta)$ then for any complex number ζ, we have*

$$|a_3-\zeta a_2^2|\le \frac{2(1-\delta)}{2-\gamma}\cdot\max\left\{1,\left|1+\frac{(1-\delta)(2-\gamma)(1+\gamma-2\zeta)}{(1-\gamma)^2}\right|\right\}.$$

Corollary 2.5 *If f given by (1) belongs to the class $N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\frac{1+z}{1-z})$ then for any complex number ζ, we have*

$$|a_3-\zeta a_2^2|\le \frac{2}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|}$$
$$\cdot\max\left\{1,\left|1+\frac{2(2\beta-\gamma)}{(\beta-\gamma)^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\zeta\right)\right|\right\}.$$

This result is sharp.

Setting $n=0$, in Corollary 2.5 we get the result of Tuneski and Darus [14], Corollary 1:

Corollary 2.6 *If f given by (1) belongs to the class N_γ then for any complex number ζ, we have*

$$|a_3-\zeta a_2^2|\le \frac{2}{2-\gamma}\cdot\max\left\{1,\left|1+\frac{(2-\gamma)(1+\gamma-2\zeta)}{(1-\gamma)^2}\right|\right\}.$$

By using Lemma 1.2, we have the following theorem:

Theorem 2.2 *Let* $\varphi(z)=1+B_1z+B_2z^2+\ldots(B_i>0, i\in\mathbb{N})$,

$$\theta_1=\frac{[(\mu\omega^\lambda-\alpha)+2]^{2n}}{[2(\mu\omega^\lambda-\alpha)+3]^n}\left[\frac{\gamma+1}{2}+\frac{(B_2-B_1)(\beta-\gamma)^2}{(2\beta-\gamma)B_1^2}\right],$$

and

$$\theta_2=\frac{[(\mu\omega^\lambda-\alpha)+2]^{2n}}{[2(\mu\omega^\lambda-\alpha)+3]^n}\left[\frac{\gamma+1}{2}+\frac{(B_1+B_2)(\beta-\gamma)^2}{(2\beta-\gamma)B_1^2}\right].$$

If f given by (1) belongs to the class $N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$, then we have the following sharp results:

i) If $\psi \le \theta_1$, then

$$|a_3-\psi a_2^2| \le \frac{1}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|}\left\{B_2+\frac{(2\beta-\gamma)B_1^2}{(\beta-\gamma)^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\psi\right)\right\}. \tag{2.6}$$

ii) If $\theta_1 \le \psi \le \theta_2$, then

$$|a_3-\psi a_2^2| \le \frac{B_1}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|}. \tag{2.7}$$

iii) If $\psi \ge \theta_2$, then

$$|a_3-\psi a_2^2| \le \frac{1}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|}\left\{\frac{(\gamma-2\beta)B_1^2}{(\beta-\gamma)^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\psi\right)-B_2\right\}.$$

(12)

Proof. Let $f \in N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$ and p given by (2.2). Similar as in Theorem 2.1, it follows that

$$a_3-\psi a_2^2 = \frac{B_1}{2[2(\mu\omega^\lambda-\alpha)+3]^n(2\beta-\gamma)}(c_2-\rho c_1^2), \tag{2.8}$$

where

$$\rho = \frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(2\beta-\gamma)B_1}{(\beta-\gamma)^2}\left(\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\psi-\frac{\gamma+1}{2}\right)\right].$$

If $\psi \le \theta_1$, then we have $\rho \le 0$. The first inequality (2.5) is established as follows by an application of Lemma 1.2 to equality (2.8). If $\psi = \theta_1$, we have $\rho = 0$, therefore equality in (2.5) holds if and only if

$$p(z) = \left(\frac{1+b}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-b}{2}\right)\frac{1-z}{1+z},\ 0 \le b \le 1 \text{andz} \in U.$$

It can be easily verified that, for $\theta_1 \le \psi \le \theta_2$,

$$\max\left\{\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(2\beta-\gamma)B_1}{(\beta-\gamma)^2}\left(\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\psi-\frac{\gamma+1}{2}\right)\right]\right\} \le 1.$$

Next, applying Lemma 1.2 to equality (2.8), we obtain

$$|a_3-\psi a_2^2| \le \frac{B_1}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|},$$

which is the inequality (2.6). In this case, the equality take place for

$$p(z) = \frac{1+z^2}{1-z^2},\ z \in U.$$

For $\psi \ge \theta_2$ we obtain $\rho \ge 1$. Hence, applying Lemma 1.2 to (2.8), we get the inequality (2.7). If $\psi = \theta_2$ we have $\rho = 1$, therefore equality holds if and only if

$$\frac{1}{p(z)} = \left(\frac{1+b}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-b}{2}\right)\frac{1-z}{1+z},\ 0 \le b \le 1 \text{andz} \in U.$$

Now, we will show that the bounds are sharp.

Let the function $K^s\ (s = 2,3,\ldots)$, be defined by

$$(1-\beta)\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)K^s(z)}\right]^\gamma+\beta\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)K^s(z))'}{D^n_{\alpha,\mu}(\lambda,\omega)K^s(z)}\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)K^s(z)}\right]^\gamma=\varphi(z^{s-1}),$$

with $K^s(0)=0=(K^s(0))'-1.$ Also, let the functions F_t and $H_t(0\le t\le 1)$ be defined by

$$(1-\beta)\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)F_t(z)}\right]^\gamma+\beta\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)F_t(z))'}{D^n_{\alpha,\mu}(\lambda,\omega)F_t(z)}\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)F_t(z)}\right]^\gamma=\varphi\left(\frac{z(z+t)}{1+tz}\right),$$

with $F_t(0)=0=F_t'(0)-1,$ and

$$(1-\beta)\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)H_t(z)}\right]^\gamma+\beta\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)H_t(z))'}{D^n_{\alpha,\mu}(\lambda,\omega)H_t(z)}\left[\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)H_t(z)}\right]^\gamma=\varphi\left(-\frac{z(z+t)}{1+tz}\right)+\delta,$$

with $H_t(0)=0=H_t'(0)-1.$

We note that the functions $K^s,F_t,G_t\in N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi).$

For $\psi<\theta_1$ or $\psi>\theta_2$, the equality holds if and only if $f\equiv K^2$ or one of its rotations. Also, if $\theta_1<\psi<\theta_2$, the equality holds if and only if $f\equiv K^3$ or one of its rotations. If $\psi=\theta_1$, the equality holds if and only if $f\equiv F_t$ or one of its rotations. If $\psi=\theta_2$, the equality holds if and only if $f\equiv H_t$ or one of its rotations.

Example 2.3 *Let* $\varphi(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$

$$\theta_1=\frac{[(\mu\omega^\lambda-\alpha)+2]^{2n}}{2[2(\mu\omega^\lambda-\alpha)+3]^n}\left[\gamma+1-\frac{(\beta-\gamma)^2\pi^2}{6(2\beta-\gamma)}\right],$$

and

$$\theta_2=\frac{[(\mu\omega^\lambda-\alpha)+2]^{2n}}{2[2(\mu\omega^\lambda-\alpha)+3]^n}\left[\gamma+1+\frac{5(\beta-\gamma)^2\pi^2}{6(2\beta-\gamma)}\right].$$

If f given by (1) belongs to the class $N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$, then we have the following sharp results:

i) If $\psi\le\theta_1$, then

$$|a_3-\psi a_2^2|\le\frac{4}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|\,\pi^2}\left\{\frac{2}{3}+\frac{4(2\beta-\gamma)}{(\beta-\gamma)^2\pi^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\psi\right)\right\}.$$

ii) If $\theta_1\le\psi\le\theta_2$, then

$$|a_3-\psi a_2^2|\le\frac{4}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|\,\pi^2}.$$

iii) If $\psi\ge\theta_2$, then

$$|a_3-\psi a_2^2|\le\frac{4}{[2(\mu\omega^\lambda-\alpha)+3]^n\,|2\beta-\gamma|\,\pi^2}\left\{\frac{4(\gamma-2\beta)}{(\beta-\gamma)^2\pi^2}\left(\frac{\gamma+1}{2}-\frac{[2(\mu\omega^\lambda-\alpha)+3]^n}{[(\mu\omega^\lambda-\alpha)+2]^{2n}}\psi\right)-\frac{2}{3}\right\}.$$

For $\theta_1\le\psi\le\theta_2$, Theorem 2.2 becomes:

Corollary 2.7 *Let* $f \in N^{n,\beta}_{\alpha,\mu,\lambda,\omega}(\varphi)$ *given by (1) where*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \dots \quad (B_i > 0, i \in \mathbb{N}).$$

Also let $\theta_1 \le \psi \le \theta_2$ and

$$\theta_3 = \frac{[(\mu\omega^\lambda - \alpha) + 2]^{2n}}{[2(\mu\omega^\lambda - \alpha) + 3]^n} \left[\frac{\gamma + 1}{2} + \frac{(\beta - \gamma)^2 B_2}{(2\beta - \gamma) B_1^2} \right].$$

i)If $\theta_1 \le \psi \le \theta_3$, then

$$|a_3 - \psi a_2^2| + |\psi - \theta_1| \cdot |a_2|^2 \le \frac{B_1}{|2\beta - \gamma| [2(\mu\omega^\lambda - \alpha) + 3]^n}. \tag{2.9}$$

ii)If $\sigma_3 \le \nu \le \sigma_2$, then

$$|a_3 - \psi a_2^2| + |\theta_2 - \psi| \cdot |a_2|^2 \le \frac{B_1}{|2\beta - \gamma| [2(\mu\omega^\lambda - \alpha) + 3]^n}. \tag{2.10}$$

Proof. If $\theta_1 \le \psi \le \theta_3$, we have

$$|a_3 - \psi a_2^2| + |\psi - \theta_1| \cdot |a_2|^2 \le \frac{B_1}{2|2\beta - \gamma| [2(\mu\omega^\lambda - \alpha) + 3]^n} \cdot \left(|c_2 - k c_1^2| + k |c_1|^2 \right),$$

so, applying Lemma 1.2, we obtain

$$|a_3 - \psi a_2^2| + |\psi - \theta_1| \cdot |a_2|^2 \le \frac{B_1}{|2\beta - \gamma| [2(\mu\omega^\lambda - \alpha) + 3]^n},$$

which is the inequality (2.9). For $\theta_3 \le \psi \le \theta_2$, we have

$$|a_3 - \psi a_2^2| + |\theta_2 - \psi| \cdot |a_2|^2 \le \frac{B_1}{2|2\beta - \gamma| [2(\mu\omega^\lambda - \alpha) + 3]^n} \cdot \left(|c_2 - k c_1^2| + (1-k) |c_1|^2 \right)$$

and, applying Lemma 1.2, we obtain

$$|a_3 - \psi a_2^2| + |\theta_2 - \psi| \cdot |a_2|^2 \le \frac{B_1}{|2\beta - \gamma| [2(\mu\omega^\lambda - \alpha) + 3]^n},$$

which is the inequality (2.10). This completes the proof.

Remark 2.1 *By specializing the parameters* α, μ, λ *and* ω *in the above results we obtain Fekete-Szegö inequalities for the subclasses involving the operators mentioned in introduction.*

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