APROXIMATE SOLUTION FOR A COUPLED SYSTEM OF DIFFERENTIAL EQUATIONS ARISING FROM A THERMAL IGNITION PROBLEM

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ABSTRAT

This article presents the approximate series solution of a coupled system of a partial differential equations and ordinary differential equations. The couples system arises from fast ignitig catalytic converters in automobile engineering. The homotopy analysis method (HAM) is applied to obtain the series solution. The HAM solution contains an auxiliary parameter which provides a convenient way of controlling the convergence region of series solution. Results are presented for different case studies and is compared for a case where the continuous solution is explicitly known.

Keywords: *Automobile engineering, approximate solution to system of PDEs and ODEs.*

1. INTRODUCTION

The exhaust system of the automobile is located between the engine outlet and exhaust pipe. This system has a device that is known as catalytic converter. Its function is to convert the pollutant gasses flowing out of the engine into harmless gases. As the process initiates, the pressure of hot inlet gas heats up the converter. There is no significant chemical reaction at this stage. After some passage of time, the temperature inside the converter raises and reach a significant high level for the reaction to occur. As reaction starts up, more heat generates. This process divides the converter into two regions. A region with low temperature and less reaction rate while the other region with high temperature and large reaction rates. The transition stage between these two regions is called light-off front. This light-off front moves towards the converter inlet. This whole process affects the pollution control of an automobile. Hence understanding of the motor vehicle exhaust emission is an important aspect in automobile engineering.

The converting phenomena of catalyst can be described in terms of interface heat-transfer models [1,10,4,5]. This study is based on the mathematical model presented by Leighton and Chang [4]. The model consists of a coupled system of partial differential equations and an ordinary differential equation. Let T_g and T_s be the vehicle temperature and converter temperature respectively. The system of equations is given as follows as presented in [4]

$$\pi a^{1262}(\rho c_p)_g[\frac{\partial T_g}{\partial t}+U\frac{\partial T_g}{\partial x}]=2\pi a^{126}h(T_s-T_g) \tag{1.1}$$

$$2\pi a^{126}\Delta r(\rho c_p)_s\frac{\partial T_s}{\partial t}=2\pi a^{126}h_0(T_g-T_s)+(2\pi a^{126}\Delta r+\pi a^{1262})Ae^{\beta(T_s-T_g^{in})},$$

$$t>0, 0<x\le 1$$

$$T_g(0,t)=T_g^{in},\quad t>0$$

$$T_g(x,0)=T_s(x,0)=T_s^0,\quad 0<x\le l.$$

The physical meanings of the functions and other terms in the above system can be found in [7]. In order to simplify the model, we let

$$u=\beta(T_g-T_g^{in})+\eta, v\beta(T_s-T_g^{in})+\eta, a=U,1.2 \tag{1.2}$$

$$c=\frac{2h}{a^{126}(\rho c_p)_g}, b=\frac{h}{\Delta r(\rho c_p)_s}, \lambda=\frac{(2\Delta r+a^{126})A\beta}{2\Delta r(\rho c_p)_s e^{\eta}},\ \text{where}\ \eta=\beta(T_g^{in}-T_s^0).$$

Which gives,

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+cu=cv,\quad t>0, 0<x\le l \tag{1.3}$$

$$\frac{\partial v}{\partial t}+bv=bu+\lambda e^{v},\quad t>0,0<x\le l$$
$$u(0,t)=\eta,\quad t>0$$
$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad 0<x\le l.$$

Several mathematical models have been presented for the automobile emission problems. However, most of the literature is related with mechanical experimentation and numerical simulation (10, 7, 11). Another study [11] is based on the simplified version, which is a scalar initial boundary value problem. A different mathematical model consisting of the coupled system of heat equation and an ordinary differential equation is presented in [12]. The authors investigated the transient behavior of a catalytic converter in first few minutes of operations. A mathematical model that describes the electrically heated monolith converter was proposed in [3] in order to study the cold start emission performance.

In this article, we consider the coupled system (1.1). The paper presents the series analytic solutions for the model (1.1). The homotopy analysis method (HAM) is applied to the problem. This method was first proposed by Liao in his PH. D. thesis [13]. A systematic description on HAM is presented in [14]. In recent years, this method has been successfully employed to solve many types of nonlinear, homogeneous or nonhomogeneous, equations and system of equations occurred in science and engineering problems [9, 15, 16]. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. In this paper, an alternative approach based on HAM is presented to approximate the solution on nonlinear coupled system of PDEs.

The article is organized as follows: Section II presents a brief introduction to the homotopy analysis method. In section III, the series analytic solution is presented for the coupled system. Section IV comprises of the examples and surface plots. A concluding remark is given in the final section.

1. BASIC IDEAS OF HAM

We consider the following differential equations,

$$N_i[z_i(x,t)]=0,\quad i=1,2,...,n,$$

where N_i are nonlinear operators that represent the whole equations, x and t denote the independent variables and $z_i(x,t)$ be the unknown functions respectively. By means of generalizing the traditional homotopy method, Liao [13] constructed the so-called zero-order deformation equations

$$(1-q)L[\phi_i(x,t:q)-z_{i,0}(x,t)]=qh_iN_i[\phi_i(x,t:q)], \tag{2.1}$$

where $q\in 0,1]$ is an embedding parameter, h_i are nonzero auxiliary functions, L is an auxiliary linear operator, $z_{i,0}(x,t)$ are initial guesses of $z_i(x,t)$ and are $\phi_i(x,t;q)$ unknown functions. It is important to note that, one has great freedom to choose auxiliary objects in HAM. Obviously, when $q=0$ and $q=1$, both $\phi_i(x,t;0)=z_{i,0}(x,t)$ and $\phi_i(x,t;1)=z_i(x,t)$ hold. Thus as q increases from 0 to 1, the solution varies from the initial guess to the solutions . Expanding in Taylor series with respect to q , one has

$$\phi_i(x,t;q)=z_{i,0}(x,t)+\sum_{m=1}^{+\infty}z_{i,m}(x,t)q^{m}, \tag{2.2}$$

where

$$z_{i,m}=\frac{i}{m!}\frac{\partial^m\phi_i(x,t;q)}{\partial q^m}|_{q=0} \tag{2.3}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameters and the auxiliary functions and the auxiliary functions are properly chosen, then the series equation 2.2 converges at $q=1$ and

$$\phi_i(x,t;1) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t),$$

which must be one of solutions of the original nonlinear equations, as proved by Liao [2]. As $h_i = -1$, Eq 2.1 becomes

$$(1-q)L[\phi_i(x,t:q) - z_{i,0}(x,t)] = qN_i[\phi_i(x,t:q)], \tag{2.4}$$

which are frequently used in the homotopy-perturbation method [29].
According to 2.3, the governing equations can be deduced from the *zero-order deformation equations* 2.1. Define the vectors

$$\mathbf{z}_{i,n} = \{z_{i,o}(x,t), z_{i,1}(x,t).\cdots, z_{i,n}(x,t)\}.$$

Differentating 2.1 m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!,$ we have the so-called *mth-order deformation equations*

$$L[z_{i,m}(x,t) - \chi_m z_{i,m}(x,t)] = h_i R_{i,m}(\mathbf{z}_{i,m-1}). \tag{2.5}$$

where

$$R_{i,m}(\mathbf{z}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i[\phi_i(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0} \tag{2.6}$$

and

$$\chi_m = \begin{Bmatrix} 0, & m \le 1, \\ 1, & m > 1. \end{Bmatrix}$$

2. APPLICATIONS

We will apply HAM to nonlinear coupled system of partial differential equation and ordinary differential equation to illustrate the strenght of the method and to establish approximate series solutions for these problems.

2.1 A. An Example with the Solution Known Explicitly

2.1.1 Example 1

Consider the following differential system:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u = v + q_1(x,t), \quad 0 < t < T, 0 < x \le 1 \tag{3.1}$$

$$\frac{\partial v}{\partial t} + v = u + \lambda e^{v} + q_2(x,t), \quad 0 < t < T, 0 < x \le 1$$

$$u(0,t) = 2 - e^{-t}, \quad 0 < t < T$$

$$u(x,0) = 1 - x^2, \quad v(x,0) = 1 - x^2, \quad 0 < x \le 1.$$

where $\lambda > 0$ is a parameter and

$$q_1(x,t) = (1-x^2)e^{-t}, \quad q_2(x,t) = (1+x^2)e^{-t} - (\lambda e^2)e^{-(1+x^2)e^{-t}}. \tag{3.2}$$

To solve system 3.1 by means of homotopy analysis method, let us choose the initial approximations

$u_0(x,t) = 1 - x^2, \qquad v_0(x,t) = 1 - x^2$

and the linear operator

$L[\phi_i(x,t;q)] = \dfrac{\partial \phi_i(x,t;q)}{\partial t}, \qquad i = 1,2,$

with the property

$L[c_i] = 0,$

where $c_i (i = 1,2)$ are integral constants. Using these definitions, the *zeroth-order deformation equations* are

$$(1-q)L[\phi_i(x,t:q)-z_{i,0}(x,t)] = qhN_i[\phi_i(x,t:q)], \quad i=1,2.$$

For $q=0$ and $q=1$, we have

$\phi_1(x,t;0) = z_{1,0}(x,t) = u_0(x,t), \qquad \phi_1(x,t;1) = u(x,t)$

$\phi_2(x,t;0) = z_{2,0}(x,t) = v_0(x,t) \qquad \phi_2(x,t;1) = v(x,t).$

Thus as q increases from 0 to 1, the solution varies from the initial guess to the solutions . Expanding in Taylor series with respect to q , one has

$$\phi_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t)q^m,$$

where

$$z_{i,m} = \frac{i}{m!}\frac{\partial^m \phi_i(x,t;q)}{\partial q^m}|_{q=0}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameters and the auxiliary functions and the auxiliary functions are properly chosen, then the series equation 2.2 converges at $q=1$ and we have

$$u(x,t) = z_{1,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t),$$

$$v(x,t) = z_{2,0}(x,t) + \sum_{m=1}^{+\infty} z_{2,m}(x,t),$$

which must be one of the solutions of the system. Consider a vector which is defined as

$$\mathbf{z}_{i,n} = \{z_{i,o}(x,t), z_{i,1}(x,t).\cdots, z_{i,n}(x,t)\}.$$

So the *mth-order deformation equations* are

$$L[z_{i,m}(x,t) - \chi_m z_{i,m}(x,t)] = h_i R_{i,m}(\mathbf{z}_{i,m-1}). \tag{3.3}$$

with the initial conditions

$$z_{i,m}(x,0) = 0, \tag{3.4}$$

and

$$R_{1,m}(\mathbf{z}_{i,m-1}) = (z_{1,m-1})_t + (z_{1,m-1})_x + (z_{1,m-1}) - (z_{2,m-1}) - q_1(x,t)$$

$$R_{2,m}(\mathbf{z}_{i,m-1}) = (z_{2,m-1})_t + (z_{2,m-1}) - (z_{1,m-1}) - \lambda e^v - q_2(x,t)$$

The solution of the *m*th-order deformation equation 3.3 for $m \geq 1$ is given as

$$z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) + h_{i0}^{t} R_{i,m}(\mathbf{z}_{i,m-1}) d\tau + c_i,$$

where the integration constants $c_i(i=1,2)$ are obtained by the initial conditions 3.4. The series solutions expressions by HAM can be written in the form

$$u(x,t) = z_{1,0}(x,t) + z_{1,1}(x,t) + z_{1,2}(x,t) + z_{1,3}(x,t) + \cdots,$$

$$v(x,t) = z_{2,0}(x,t) + z_{2,1}(x,t) + z_{2,2}(x,t) + z_{2,3}(x,t) + \cdots.$$

For $h=-1$, the approximated series solutions are computed as follows.

$$u(x,t) = 1 - x^2 + h[-2xt + (1-x^2)e^{-t}] - h(1-x^2) + \cdots, 3.5 \tag{3.5}$$

$$v(x,t) = 1 - x^2 + h[-\lambda t e^{(1-x^2)} + (1+x^2)e^{-t}] - 2(1+x^2)e^{-t} + \frac{(1+x^2)}{4}e^{-2t} + \cdots.$$

For $t=1$, the graphs of $u(x,t)$ and $v(x,t)$ are presented in Figures below.

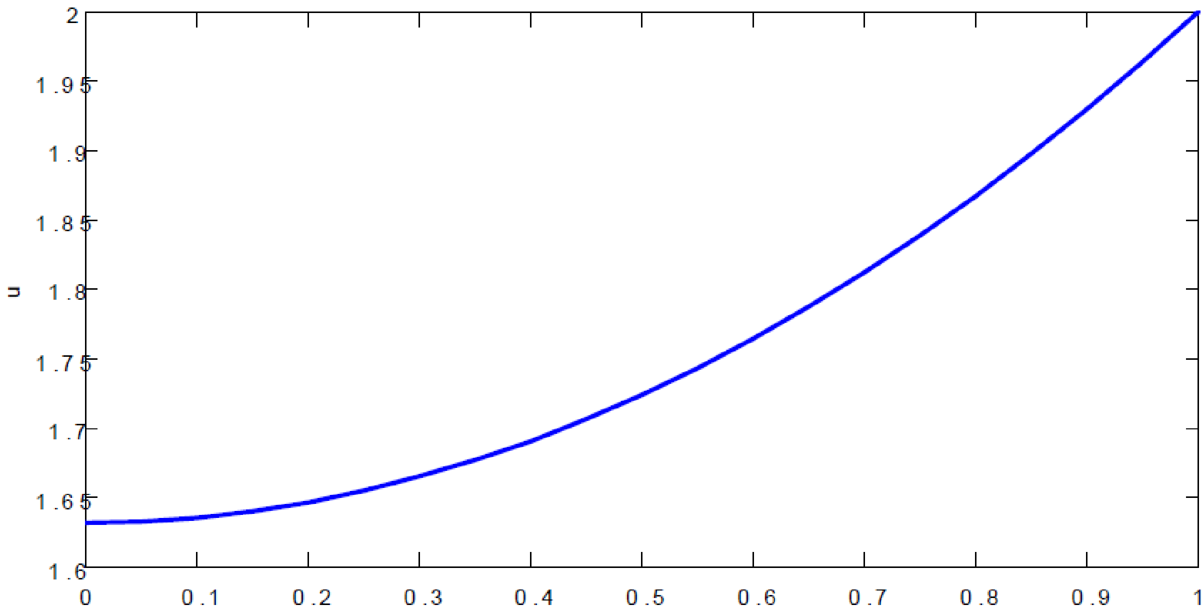

Fig. 1. Figure 1: Graph of u(x,t) at t = 1.0

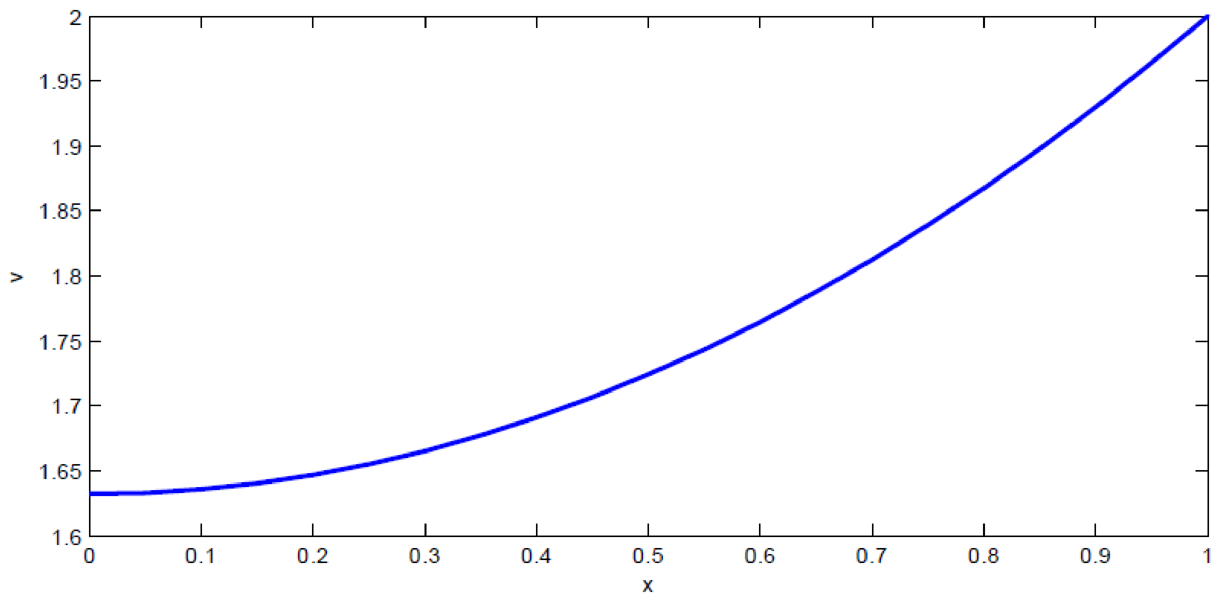

Fig. 2. Figure 2: Graph of v(x,t) for t = 1

2.1.2 Example 2

To emonstrate the application of HAM for a larger class of reaction functions, we consider the following more general system.

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+cu=a_1u^{\alpha_1}+a_2v^{\alpha_2},\quad 0<t<T,0<x\leq 1 \tag{3.6}$$

$$\frac{\partial v}{\partial t}+cv=b_1u^{\beta_1}+b_2e^{\beta_2 v},\quad 0<t<T,0<x\leq 1$$

$$u(0,t),=1 \qquad 0<t<T$$
$$u(x,0)=0, \quad v(x,0)=0, \quad 0<x\le 1.$$

where $a_i, b_i, \alpha_i,$ and $\beta_i (i=1,2)$ are nonnegative constants Series solution for the case $a=2, b=c=10, a_1=1, a_2=2, b_2=2, b_3=3/2, \alpha_1=3/2, \alpha_2=1, \beta_1=1/2$ and $\beta_2=1$ are computed by HAM as follows.

$$u(x,t)=-a_2h^2.b_2.\frac{(hb_2t)^{\alpha_2+1}}{\alpha_2+1}+\cdots,3.7 \tag{3.7}$$
$$v(x,t)=2hb_2t+h[hb_2t+chb_2t^2/2-hb_2[\beta_2ht^2/2+\beta_2^2h^2t^3/6]]+\cdots.$$

The validity of the method is based on the assumption that the series (2) converges at $q=1$. It is the auxiliary parameter h which ensures that this assumption can be satisfied. In general, by means of the so-called h-curve, it is straightforward to choose a proper value of h which ensures that the solution series is convergent.

3. CONCLUSION

In this paper, it is shown that how HAM can be applied to the coupled system of equations related to automobile engineering. The series solutions are obtained by applying HAM. The analytical approximate solutions for more general problems are presented first time. The plotted results agree with those numerical reults already presented in the literature. The advantage of HAM is the auxiliary parameter which provides a convenient way of controlling the convergence region of series solutions which is not possible in other analytical methods.

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