FIXED POINT TYPE THEOREM FOR WEAK CONTRACTIONS IN S-METRIC SPACES

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ABSTRACT
In this paper we prove a fixed point theorem for weak contraction self mappings in S -metric spaces. [1]

1 INTRODUCTION
The importance of fixed point theorems can not be overemphasized. The study of fixed point theory has been extensively developed in the past decades [2]. Different researchers have attempted to generalize the notion of metric space to a more general setting. The concept of G -metric spaces as a generalization of metric spaces (X,d) was introduced by [5, 6]. The concept of weak contraction was introduced by [1]. It was proved the existence of fixed points for single-valued maps satisfying weak contraction conditions on Hilbert spaces. Trueness of most results of [1] for any metric space showed by [8]. Not long ago Sedghi. et al [7] introduced the concept of S -metric spaces. Some new properties of this space was explained by [4]. In the peresent paper we prove a fixed point type theorem for a weak contraction self mapping on S -metric spaces. The result is supported by an example.

2 BASIC CONCEPTS
We briefly give some basic definitions of concepts which serve a background to this work.

Definition 2.1 *Let* X *be a nonempty set. We call S-metric on* X *is a function* $S : X^3 \to [0,\infty)$ *which satisfies the following conditions for each* $x, y, z, a \in X$

(i) $S(x, y, z) \geq 0,$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The set X with a S -metric is called *S*-metric space.

Example 2.1 *For any metric space* (X,d) , $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ *is a S-metric on* X .

Example 2.2 *Let* P *be a real line. Then* $S(x, y, z) =| x - z | + | y - z |$ *for all* $x, y, z \in \mathrm{P}$ *is a* S *-metric on* P *. This* S *-metric is called the usual* S *-metric on* P .

Lemma 2.1 *(See[7]) In a* S *-metric space, we have* $S(x, x, y) = S(y, y, x)$.

There exists a natural topology on a S-metric spaces, for more details we refer to [4].

Lemma 2.2 *(See[4]) Any* S *-metric space is a Hausdorff space.*

Lemma 2.3 *Let* (X,S) *be a* S *-metric space. If there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$ *, then* $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

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Lemma 2.4 *(See[3]) Let (X,S) be a S-metric space. Then for all $x,y,z\in X$*

$$S(x,x,z)\le 2S(x,x,y)+S(y,y,z).$$

By the following lemma, every metric space is a S-metric space.

Lemma 2.5 *(See[3]) Let (X,d) be a metric space. Then we have*

(1)- $S_d(x,y,z)=d(x,z)+d(y,z)$ is a S-metric on X.

(2)- $x_n\to x$ in (X,d) if and only if $x_n\to x$ in (X,S_d).

(3)- $\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X,S_d).

(4)- (X,d) is complete if and only if (X,S_d) is complete.

The inverse of Lemma 2.5 does not hold(see[3] Example 1.11).

Definition 2.2 *A self mapping T on X, where (X,d) is a metric space, is said to be a ϕ-weak contraction if*

$$d(Tx,Ty)\le d(x,y)-\phi(d(x,y)) \tag{1}$$

for some $\phi:[0,1)\to[0,1)$, continuous and nondecreasing function such that $\phi(t)=0$ if and only if $t=0$.

3 MAIN RESULT

The following theorem was proved in [8]:

Theorem 3.1 *Let (X,d) be a complete metric space, and let T be a ϕ-weak contraction with $\phi(t)>0$ for $t\in(0,1)$. Then T has a unique fixed point.*

We extend this result by the following way.

Theorem 3.2 *Let (X,S) be a complete S-metric space and let T be a self mapping on X satisfying*

$$S(Tx,Tx,Ty)\le S(x,x,y)-\phi(S(x,x,y)) \tag{2}$$

for all $x,y\in X$ and ϕ being as in Theorem 3.1. Then T has a unique fixed point in X.

Proof. Let $x_0\in X$. We construct the sequence $\{x_n\}$ by $x_n=Tx_{n-1}$, $n\in\mathrm{N}$. If $x_{n+1}=x_n$ for some n, then trivially T has a fixed point. We assume $x_{n+1}\ne x_n$, for all $n\in\mathrm{N}$. From (2), we have

$$S(x_n,x_n,x_{n+1})=S(Tx_{n-1},Tx_{n-1},Tx_n)\le S(x_{n-1},x_{n-1},x_n)-\phi(S(x_{n-1},x_{n-1},x_n)) \tag{3}$$

By the property of ϕ, we have

$$S(x_n,x_n,x_{n+1})\le S(x_{n-1},x_{n-1},x_n).$$

This shows that $S(x_n,x_n,x_{n+1})$ is a non-increasing sequence, and therefore there exists $r\ge 0$ such that

$$\lim_{n\to\infty}S(x_n,x_n,x_{n+1})=r \tag{4}$$

Letting $n\to\infty$ in (3), we have $r\le r-\phi(r)$. Then $r=0$. Hence

$$\lim_{n\to\infty}S(x_n,x_n,x_{n+1})=0 \tag{5}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose it is false. Then there exists $\varepsilon>0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k)>m(k)>k$ such that

$$S(x_{m(k)},x_{m(k)},x_{n(k)})\ge\varepsilon. \tag{6}$$

For any fixed k we can choose $n(k)$, such that it is the smallest integer with $n(k)>m(k)$ and satisfying (6).

Then

$$S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \varepsilon.$$

Hence by (iii) we get

$$\begin{aligned}
&\varepsilon \le S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \le 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) \\
&\le 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) \\
&+ S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) \\
&\le 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) \\
&+ 2S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) + S(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}) \\
&< 4S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ and by (5),

$$\lim_{k\to\infty} S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = \varepsilon \tag{7}$$

Now by (2) and (7)

$$\begin{aligned}
&S(x_{m(k)}, x_{m(k)}, x_{n(k)}) = S(Tx_{m(k)-1}, Tx_{m(k)-1}, Tx_{n(k)-1}) \\
&\le S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) - \phi(S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})).
\end{aligned}$$

Letting $k \to \infty$, we have

$$\varepsilon \le \varepsilon - \phi(\varepsilon)$$

Then we have $\varepsilon = 0$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete S-metric space, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Now we claim that $Tp = p$. We have

$$\begin{aligned}
&S(x_n, x_n, Tp) = S(Tx_{n-1}, Tx_{n-1}, Tp) \\
&\le S(x_{n-1}, x_{n-1}, p) - \phi(S(x_{n-1}, x_{n-1}, p)).
\end{aligned}$$

By taking $n \to \infty$ we obtain $S(x_{n-1}, x_{n-1}, p) \le 0$. Taking into account (i), that means p is a fixed point of T. Suppose T has another fixed point q, then

$$\begin{aligned}
&S(p, p, q) = S(Tp, Tp, Tq) \\
&\le S(p, p, q) - \phi(S(p, p, q)).
\end{aligned}$$

By the properties of ϕ, $S(p, p, q) = 0$. , and then $p = q$.

Theorem 3.3 *Let* (X, S) *be a complete* S *-metric space and let* T *be a self mapping on* X *satisfying*

$$S(Tx, Ty, Tz) \le S(x, y, z) - \phi(S(x, y, z)) \tag{8}$$

for all $x, y, z \in X$ and ϕ being as in Theorem 3.1. Then T has a unique fixed point in X.

Proof. Follows from Theorem 3.2 by taking $x = y$.

Example 3.1 *Let* $X = [0,1]$ *and let S be the usual* S *-metric on X. Then* (X, S) *is a complete S-metric space. Let* $T(x) = x - \frac{x^2}{2}$ *and* $\phi(t) = \frac{t^2}{2}$ *.Without loss of generality, we suppose* $x > y > z$. *Then*

$$\begin{aligned}
&S(Tx, Ty, Tz) = |Tx - Tz| + |Ty - Tz| \\
&= |(x - \frac{x^2}{2}) - (z - \frac{z^2}{2})| + |(y - \frac{y^2}{2}) - (z - \frac{z^2}{2})|
\end{aligned}$$

$$= (x - \frac{x^2}{2}) - (z - \frac{z^2}{2}) + (y - \frac{y^2}{2}) - (z - \frac{z^2}{2})$$
$$= [(x - z) + (y - z)] - \frac{1}{2}[(x^2 - z^2) + (y^2 - z^2)]$$
$$\leq [(x - z) + (y - z)] - \frac{1}{2}[(x - z)^2 + (y - z)^2]$$
$$= S(x, y, z) - \phi(S(x, y, z)).$$

Obviousley T satisfies (2). By Theorem 3.2, T has a unique fixed point which is obviously 0.

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