GM(2,1) RECURSIVE MODEL BASED ON PARTICLE SWARM OPTIMIZATION ALGORITHM

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ABSTRACT

Firstly, aiming at the shortcomings of the traditional GM (2, 1) model, to build a new GM (2, 1) model on the basis of the optimize background values. Secondly, from the grey equation of the model to deduce the recursive prediction expressions .Lastly, to introduce the particle swarm optimization algorithm and using GM(2,1) recursive model as objective function to identify the model parameters. Thus the new outcome effectively avoids the jump error. The error caused by the difference equation solving the parameters, and takes it into the time response type of differential equation. Finally, the proposed model is confirmed to have higher accuracy by examples.

Keywords: *GM (2, 1) model, particle swarm optimization, parameter identification*

1. INTRODUCTION

Since the early 1980s, Professor Deng [1] created the grey system theory, and the grey system prediction method has been widespread concern of scholars. GM (2, 1) model is an important class of models in grey forecasting models. GM (2, 1) model has two characteristic roots, so you can reflect monotonous or non-monotonic or swing on the dynamic characteristics and theoretically it should have a wider range of applications. But in real life, the simulation accuracy of GM (2, 1) model was not very deal when it was used to the actual cases. However, at present, the improving methods of GM (2, 1) model are not much. Zeng *et al.* [2] introduced a cumulative method to parameters estimation of GM (2, 1) model, and reduced the pathological on the multiple transformation. Liao *et al.* [3] proposed a new GM (2, 1) model based on calculus, and the model was applied to power load forecasting. Shen *et al.* [4] optimized GM (2, 1) model by using the least squares method. Niu *et al.* [5] combined the first-order grey derivative and background values by using weight p_1 and p_2, so it established GM $(2,1,p_1,p_2)$ model. Firstly Liu *et al.* [6] proposed linear combination of the forward and backward difference model, secondly introduced parameter ρ to multiply transformation, lastly the established GM(2,1,λ,ρ) model has good simulation accuracy by cases. To sum up, it is not difficult to find that the existing researches are identified the parameters of grey equation of GM (2, 1) model by least square method, and then the identification result was substituted into the time equation of whitening equation to simulate and forecast. However, the grey equation and the white equation is not strictly match, so the model must lower simulation prediction accuracy. Although literatures made some improvements on the model, it can not avoid GM (2, 1) model error caused by the difference equation to jump differential equation [2-6].

Firstly, this paper optimize the background value of GM (2, 1) model, and then obtain the new GM (2, 1) model. Secondly, its grey equation is derived to establish recursive prediction expression of the model, and the parameter identification choice the particle swarm algorithm. At last, the paper verify the validity of the model by examples.

2. TRADITIONAL MODEL

The raw series $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots x^{(0)}(\mathrm{n}))$, let $X^{(1)}$ be AGO series of $X^{(0)}$, let $X^{(-1)}$ be IAGO series of $X^{(0)}$, $Z^{(1)}$ is said to be MEAN series of $X^{(1)}$.

Definition 2.1 Grey differential equation of GM (2, 1):

$$x^{(-1)}(k) + ax^{(0)}(k) + bz^{(1)}(k) = c \tag{1}$$

Definition 2.2 White differential equation of GM (2, 1):

$$\frac{d^2x^{(1)}}{dt^2}+a\frac{dx^{(1)}}{dt}+bx^{(1)}=c \quad (2)$$

Using the least square method to calculate parameters, $[a\ b\ c]=(B^TB)^{-1}B^TY$
Among:

$$B=\begin{bmatrix}-x^{(0)}(2) & -z^{(1)}(2) & 1\\ -x^{(0)}(3) & -z^{(1)}(3) & 1\\ \vdots & \vdots & \vdots\\ -x^{(0)}(n) & -z^{(1)}(\mathrm{n}) & 1\end{bmatrix} \quad Y=\begin{bmatrix}x^{(-1)}(2)\\ x^{(-1)}(3)\\ \vdots\\ x^{(-1)}(n)\end{bmatrix}$$

The solution of white equation of GM (2, 1) model has three kinds of cases: the characteristic equation has two unequal real roots, two equal real root, and a pair of conjugate complex roots. It usually have two unequal real roots. This moment, white response of GM (2, 1)

$$x^{(1)}(t)=\mathrm{c}_1\mathrm{e}^{\lambda_1 t}+\mathrm{c}_2\mathrm{e}^{\lambda_2 t}+\frac{c}{b} \quad (3)$$

Among:

$$\lambda_1,\lambda_2=\frac{-a\pm\sqrt{(a^2-4b)}}{2}$$

3. NEW GM (2, 1) MODEL BASED ON PARTICLE SWARM OPTIMIZATION ALGORITHM

3.1. Optimized GM (2, 1) Model

Suppose curve $y=x^{(1)}(t)$ have discrete points that it composed series $X^{(1)}=(x^{(1)}(1),x^{(1)}(2),\cdots x^{(1)}(\mathrm{n}))$, according to the Lagrange theorem, it exits a point ξ in $(k-1,k)$ that $\frac{dx^{(1)}}{dt}|_{t=\xi}=x^{(1)}(k)-x^{(1)}(k-1)$ might be able to. Therefore, grey derivatives $x^{(0)}(\mathrm{k})=x^{(1)}(k)-x^{(1)}(k-1)$ is the derivative of ξ when GM(2,1) is established, then its background value is $x^{(1)}(\xi)=\lambda x^{(1)}(k)+(1-\lambda)x^{(1)}(k-1)$. However, $\lambda=\frac{1}{2}$ only a special case when background value take $\frac{1}{2}[x^{(1)}(k)+x^{(1)}(k-1)]$ to replace $x^{(1)}(\xi)$. It clearly not the optimal situation.

Definition 3.1.1 The raw series $X^{(0)}=(x^{(0)}(1),x^{(0)}(2),\cdots x^{(0)}(\mathrm{n}))$, let $X^{(1)}$ be AGO series of $X^{(0)}$, let $X^{(-1)}$ be IAGO series of $X^{(0)}$, grey differential equation of new GM (2, 1) model:

$$x^{(-1)}(\mathrm{k})+\mathrm{a}\,x^{(0)}(\mathrm{k})+b(\lambda x^{(1)}(k)+(1-\lambda)x^{(1)}(k-1))=c \quad (4)$$

At this time, according to solve the parameters of grey model, it take the least square method, and then take the results into the white equation. If it take the method that it get only approximate solutions, and is not the best solution. So, this paper consider from grey differential equation of GM (2, 1) model which it optimized the background value, at last, the recursive prediction expressions are derived.

3.2. Forecasting Expression of GM (2, 1) Model

Let $X^{(1)}$ be AGO series of $X^{(0)}$, let $X^{(-1)}$ be IAGO series of $X^{(0)}$, namely $x^{(1)}(k)=x^{(0)}(k)+x^{(1)}(k-1)$,

$x^{(-1)}(k) = x^{(0)}(k) - x^{(0)}(k-1)$ are substituted into (4) formula:

$$x^{(0)}(k) - x^{(0)}(k-1) + \mathrm{a}\, x^{(0)}(k) + b[\lambda x^{(0)}(k) + \lambda x^{(1)}(k-1) + (1-\lambda) x^{(1)}(k-1)] = \mathrm{c} \quad (5)$$

Finishing (5) formula:

$$(1 + \mathrm{a} + b\lambda) x^{(0)}(k) - x^{(0)}(k-1) + b x^{(1)}(k-1) = \mathrm{c} \quad (6)$$

Recursive (6) formula:

$$(1 + \mathrm{a} + b\lambda) x^{(0)}(k+1) - x^{(0)}(k) + b x^{(1)}(k) = \mathrm{c} \quad (7)$$

(7) formula minus (6) formula and arrange it:

$$x^{(0)}(k+1) = \frac{2 + \mathrm{a} + b\lambda - b}{1 + \mathrm{a} + b\lambda} x^{(0)}(k) - \frac{1}{1 + \mathrm{a} + b\lambda} x^{(0)}(k-1) \quad (8)$$

Let $A = \frac{2 + \mathrm{a} + b\lambda - b}{1 + \mathrm{a} + b\lambda}$, $\mathrm{B} = -\frac{1}{1 + \mathrm{a} + b\lambda}$, $x^{(0)}(1) = x^{(0)}(1)$, $x^{(0)}(2) = C$

Namely,

$$\begin{cases} x^{(0)}(k+1) = Ax^{(0)}(k) + Bx^{(0)}(\mathrm{k}-1) \\ x^{(0)}(1) = x^{(0)}(1), x^{(0)}(2) = C \end{cases} \quad (9)$$

Definition 3.2.1 Forecasting model of new GM(2,1)

$$\begin{cases} x^{(0)}(k+1) = Ax^{(0)}(k) + Bx^{(0)}(\mathrm{k}-1) \\ x^{(0)}(1) = x^{(0)}(1), x^{(0)}(2) = C \end{cases} \quad (10)$$

Among, $x^{(0)}(k)$ is simulation forecasting value of the k time, A, B, C are parameters to be identified.

3.3. Parameter Identification of Model Based on Particle Swarm Optimization

Particle swarm optimization algorithm is put forward by Kennedy and Eberhart in 1995. It was a kind of evolutionary computation technology, and originated in the simulation of a simplified social model, namely the observation for the flock foraging behavior. And based on the research of the similar biological group behavior, they found that there is a kind of social information sharing mechanism in biological group, which it provides an advantage for the evolution of the group and also the basis for the formation of PSO algorithm. Particle swarm optimization algorithm is a kind of intelligent evolutionary computation technology, its advantage is that the small number of individuals, simple calculation, easy to implement, fast convergence speed, and has been applied in many fields of [7-8]. Particle swarm optimization algorithm had some shortcomings compared with other evolutionary algorithms, so it is easy to fall into local optimal.

Therefore this paper uses the particle swarm optimization algorithm to initialize a set of random solutions for the target function. Each individual can be regarded as a particle. Each particle has a fitness value for the objective function, and a speed determines the direction and distance of particle search. In the process of iteration, according two extreme to update itself on each particle, one extreme is searched the best solution of particle, called individual extremum (pbest). The other extreme is searched the best solution of all particles, called the global extremum (gbest). According to the following formula, each particle update their speed and position:

$$\begin{cases} v_{k+1} = wv_k + c_1(pbest_k - x_k) + c_2(\mathrm{g}best_k - x_k) \\ x_{k+1} = x_k + v_{k+1} \end{cases} \tag{11}$$

Among, v_k is the velocity vector of the particles, x_k is the current position of the particle, $pbest_k$ is the optimal solution of the particle itself, $\mathrm{g}best_k$ is the global extremum of the whole particle swarm. w is the inertia weight coefficient, c_1 and c_2 is called the acceleration coefficient. It take a random number in (0, 2).
In order to optimize parameters of GM (2, 1) model, PSO fitness function is configured that

$$\min F = \sum_{k=1}^{n} \left| \frac{\hat{x}^{(0)}(k)\text{-}x^{(0)}(k)}{x^{(0)}(k)} \right| \tag{12}$$

Among, $\hat{x}^{(0)}(k)$ and $x^{(0)}(k)$ respectively are the simulation forecast value and the original value for the k item of data sequence, n are the number of raw series.
The calculation processes of particle swarm algorithm are as follows:

Ⅰ.To initialize the particle swarm and particle velocity.
Ⅱ.To calculate the fitness for each particle and select the pbest and gbest.
Ⅲ.If the calculation results achieve the precision or the maximum number of iterations, turn into the fifth step, or transferred to the fourth step.
Ⅳ.According the (11) formula to update the speed and position of each particle, and then take into the second step.
Ⅴ.The gbest can be output and the corresponding solution, and then exit the loop.

4. APPLICATION EXAMPLES

Example 1 The actual value of table 1 come from literature [6], respectively establish traditional GM(2,1) model, GM(2,1,λ,ρ) model of literature 6(remark it as literature 6 model) and GM(2,1) model of this paper. To analyze and compare them with the simulation and prediction accuracy.
Three parameters A B C are obtained by the particle swarm algorithm GM (2, 1) model, they respectively are 0.6977, 0.3653, 3.278.Take them into GM (2, 1) recursion model.

$$\begin{cases} \hat{x}^{(0)}(k+1) = 0.6977\hat{x}^{(0)}(k) + 0.3653\hat{x}^{(0)}(\mathrm{k}-1) \\ \hat{x}^{(0)}(1) = 2.874, \hat{x}^{(0)}(2) = 3.278 \end{cases}$$

The results in table 1.

Table 1. The simulation value and relative error of three models to raw series.

Raw value	Traditional GM(2,1)model		Literature[6]model		GM(2,1)model of this paper	
	Simulation value	Relative error (%)	Simulati on value	Relative error (%)	Simulati on value	Relative error (%)
2.874	2.874	0.00	2.874	0.000	2.8740	0.0000
3.278	-6.28	291.67	3.080	6.040	3.2780	0.0000
3.337	-68.5	2152.82	3.433	2.865	3.3369	0.0020
3.39	-539.14	16003.93	3.595	6.041	3.5256	4.0009
3.679	-4093.14	111356.8	3.457	6.041	3.6788	0.0050
Average relative error (%)	25961.01		4.179		0.8016	

Note：The actual value, the simulation value and the relative error of traditional GM (2, 1) model and literature [6] model, which come from table 1 in literature [6].

Example 2 According to the distribution of lifetime data samples of single hydraulic prop, there is no replacement Censored life test in 20 by random selected method. Results are {20,50,640,750,890,970,1110,1160,1560,2140}h. Due to the before 1-2 data of reliability test have an apparent randomness, then remove it and get the forecasting raw series {640,750,890,970， 1110,1160,1560,2140}h. To establish the traditional GM(2,1) model, GM(2,1,λ,ρ) model of literature 6(remark it as literature 6 model) and GM(2,1) model of this paper. To analyze and compare them with the simulation and prediction accuracy.

Three parameters A B C are obtained by the particle swarm algorithm GM (2, 1) model, they respectively are 3.9916, -3.2365, 724.9316.Take them into GM (2, 1) recursion model.

$$\begin{cases} x^{(0)}(k+1) = 3.9916x^{(0)}(k)\text{-}3.2365x^{(0)}(\mathrm{k}-1) \\ x^{(0)}(1) = 640, x^{(0)}(2) = 724.9316 \end{cases}$$

The results in table 2.

Table 2 The simulation value and relative error of three models to raw series.

Raw value	Traditional GM(2,1)model		Literature[6]model		GM(2,1)model of this paper	
	Simulation value	Relative error (%)	Simulation value	Relative error (%)	Simulation value	Relative error (%)
2.874	2.874	0.00	2.874	0.000	2.8740	0.0000
3.278	-6.28	291.67	3.080	6.040	3.2780	0.0000
3.337	-68.5	2152.82	3.433	2.865	3.3369	0.0020
3.39	-539.14	16003.93	3.595	6.041	3.5256	4.0009
3.679	-4093.14	111356.8	3.457	6.041	3.6788	0.0050
Average relative error (%)	25961.01		4.179		0.8016	

Note：The actual value, the simulation value and the relative error of traditional GM (2, 1) model and literature [6] model, which come from table 2 in literature [6].

From table 1, 2, the GM (2, 1) recursion model based on the particle swarm optimization algorithm, not only their simulation values and relative error are far below the traditional GM (2, 1) model, but also they are significantly lower than reference [6], which has greatly improved the prediction accuracy. Therefore, on the whole, the proposed model of this paper has more advantages.

5. CONLUSION

Grey equation and white equation of GM (2,1) model are not strictly matched, so it is not the optimal solution but approximate solution, Which is from the parameters of grey equation and then substituting the white equation; therefore, this paper firstly get a new GM (2,1) model by optimizing GM (2,1) model's background value, then according the grey equation of new GM (2,1) model to establish recursion expression by derive method and it use particle swarm optimization algorithm to efficient optimization of parameters. At last this paper uses an example to verify the validity of the model.

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