ON SOME DIFFERENTIAL SANDWICH THEOREMS USING SĂLĂGEAN OPERATOR AND RUSCHEWEYH OPERATOR

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ABSTRACT

In this work we define a new operator using the Sălăgean operator and Ruscheweyh operator. Denote by $SR^{m,n}$ the Hadamard product of the Sălăgean operator S^m and Ruscheweyh operator R^n, given by $SR^{m,n}:\mathcal{A}\to\mathcal{A}$, $SR^{m,n}f(z)=\left(S^m * R^n\right)f(z)$ and $\mathcal{A}_n=\{f\in\mathcal{H}(U): f(z)=z+a_{n+1}z^{n+1}+\ldots,\quad z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. The purpose of this paper is to introduce sufficient conditions for subordination and superordination involving the operator $SR^{m,n}$ and also to obtain sandwich-type results.

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1. INTRODUCTION

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$.
Let $\mathcal{A}_n=\{f\in\mathcal{H}(U): f(z)=z+a_{n+1}z^{n+1}+\ldots, z\in U\}$ and $\mathcal{A}=\mathcal{A}_1$.

Denote by $K=\left\{f\in\mathcal{A}: Re\,\frac{zf''(z)}{f'(z)}+1>0,\ z\in U\right\}$, the class of normalized convex functions in U.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f\prec g$, if there exists a Schwarz function w, analytic in U, with $w(0)=0$ and $|w(z)|<1$, for all $z\in U$, such that $f(z)=g(w(z))$, for all $z\in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z)\prec h(z),\quad for\ z\in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ and h analytic in U. If p and $\psi(p(z),zp'(z),z^2p''(z);z)$ are univalent and if p satisfies the second order differential superordination

$$h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z),\qquad z\in U, \tag{2}$$

then p is a solution of the differential superordination (2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q\prec p$ for all p satisfying (2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

Miller and Mocanu [8] obtained conditions h, q and ψ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution product) of $f(z)$ and $g(z)$, written as $(f * g)(z)$, is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

Definition 1.1 *(Sălăgean [11]) For* $f \in \mathcal{A}$, *and* $n \in \mathbb{N}$, *the operator* S^n *is defined by* $S^n : \mathcal{A} \to \mathcal{A}$,

$$S^0 f(z) = f(z)$$
$$S^1 f(z) = zf'(z)$$
$$\ldots$$
$$S^{n+1} f(z) = z\left(S^n f(z)\right)', \quad z \in U.$$

Remark 1.1 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.

Definition 1.2 *(Ruscheweyh [10]) For* $f \in \mathcal{A}$ *and* $n \in \mathbb{N}$, *the operator* R^n *is defined by* $R^n : \mathcal{A} \to \mathcal{A}$,

$$R^0 f(z) = f(z)$$
$$R^1 f(z) = zf'(z)$$
$$\ldots$$
$$(n+1) R^{n+1} f(z) = z\left(R^n f(z)\right)' + nR^n f(z), \quad z \in U.$$

Remark 1.2 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ *for* $z \in U$.

Definition 1.3 *([7]) Let* $n, m \in \mathbb{N}$. *Denote by* $SR_{\lambda}^{m,n} : \mathcal{A} \to \mathcal{A}$ *the operator given by the Hadamard product of the generalized Sălăgean operator* D_{λ}^m *and the Ruscheweyh operator* R^n,

$$SR^{m,n} f(z) = \left(S^m * R^n\right) f(z), \tag{3}$$

for any $z \in U$ and each nonnegative integers $m, n.$

Remark 1.3 *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $SR^{m,n} f(z) = z + \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Remark 1.4 *For* $m = n$, *we obtain the Hadamard product* SR^n *[1] of the Sălăgean operator* S^n *and Ruscheweyh derivative* R^n, *which was studied in [2], [3].*

Using simple computation one obtains the next result.

Proposition 1.1 *([7]) For* $m,n\in\mathbb{N}$ *we have*

$$SR^{m+1,n}f(z)=z\left(SR^{m,n}f(z)\right)' \tag{4}$$

and

$$z\left(SR^{m,n}f(z)\right)'=(n+1)SR^{m,n+1}f(z)-nSR^{m,n}f(z). \tag{5}$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of M. Darus, K. Al-Shaqs [6], Shanmugam, Ramachandran, Darus and Sivasubramanian [12].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.4 *[9] Denote by* Q *the set of all functions* f *that are analytic and injective on* $\overline{U}\setminus E(f)$*, where* $E(f)=\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\}$*, and are such that* $f'(\zeta)\neq 0$ *for* $\zeta\in\partial U\setminus E(f)$.

Lemma 1.1 *[9] Let the function* q *be univalent in the unit disc* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w)\neq 0$ *when* $w\in q(U)$*. Set* $Q(z)=zq'(z)\phi(q(z))$ *and* $h(z)=\theta(q(z))+Q(z)$*. Suppose that*

1. Q is starlike univalent in U and
2. $Re\left(\frac{zh'(z)}{Q(z)}\right)>0$ for $z\in U$.

If p is analytic with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z)),$$

then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 1.2 *[5] Let the function* q *be convex univalent in the open unit disc* U *and* ν *and* ϕ *be analytic in a domain* D *containing* $q(U)$*. Suppose that*

1. $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)>0$ for $z\in U$ and
2. $\psi(z)=zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p(z)\in\mathcal{H}[q(0),1]\cap Q$, with $p(U)\subseteq D$ and $\nu(p(z))+zp'(z)\phi(p(z))$ is univalent in U and

$$\nu(q(z))+zq'(z)\phi(q(z))\prec\nu(p(z))+zp'(z)\phi(p(z)),$$

then $q(z)\prec p(z)$ and q is the best subordinant.

2. MAIN RESULTS

Considering $\lambda=1$ in [4] we obtain the following results.

Theorem 2.1 *Let* $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\in\mathcal{H}(U)$, $z\in U$, $f\in\mathcal{A}$, $m,n\in\mathbb{N}$, $\lambda\geq 0$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0)=1$*. Assume that*

$$Re\left(1+\frac{\alpha}{\mu}+\frac{2\beta}{\mu}q(z)+\frac{zq''(z)}{q'(z)}\right)>0,\ \ z\in U, \tag{6}$$

for $\alpha,\beta,\mu,\in\mathbb{C}$, $\mu\ \ \neq 0$, $z\in U,$ and

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z):=(-n\mu+\alpha)\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}+ \tag{7}$$

$$-\mu(n+1)^2\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}+\mu(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n}f(z)}+(\beta-\mu)\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^2.$$

If q satisfies the following subordination

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha q(z)+\beta(q(z))^2+\mu zq'(z), \tag{8}$$

for $,\alpha,\beta,\mu\in\mathbb{C}$, $\mu\ \ \neq 0$ then

$$\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\prec q(z),\ \ z\in U, \tag{9}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z):=\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}$, $z\in U$, $z\neq 0,\quad f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$

Differentiating this function, with respect to $z,$ we get

$$zp'(z)=\frac{z\left(SR^{m+1,n}f(z)\right)'}{SR^{m,n}f(z)}-\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\frac{z\left(SR^{m,n}f(z)\right)'}{SR^{m,n}f(z)}$$

By using the identity (4)and (5), we obtain

$$zp'(z)=-n\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}-(n+1)^2\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}+$$

$$(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n}f(z)}-\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^2+$$

$$(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n}f(z)}-\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^2 \tag{10}$$

By setting $\theta(w):=\alpha w+\beta w^2$ and $\phi(w):=\mu$, $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\ \ \neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0,\ \ w\in\mathbb{C}\setminus\{0\}.$

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\mu zq'(z),$ we find that $Q(z)$ is starlike univalent in $U.$

Let $h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta(q(z))^2+\mu zq'(z),\ \ z\in U$

If we derive the function Q, with respect to z, perform calculations, we have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(1+\frac{\alpha}{\mu}+\frac{2\beta}{\mu}q(z)+\frac{zq''(z)}{q'(z)}\right)>0$.

By using (10), we obtain

$$\alpha p(z)+\beta(p(z))^2+\mu z p'(z)=$$

$$(-n\mu+\alpha)\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}-\mu(n+1)^2\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}+\mu(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n}f(z)}+(\beta-\mu)\left(\frac{SR^{m+1,n}f(z)}{DS^{m,n}f(z)}\right)^2.$$

By using (8), we have $\alpha p(z)+\beta(p(z))^2+\mu z p'(z) \prec \alpha q(z)+\beta(q(z))^2+\mu z q'(z)$.

Therefore, the conditions of Lemma 1.1 are met, so we have $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\prec q(z)$, $z\in U$, and q is the best dominant.

Corollary 2.2 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$, $z\in U$. *Assume that (6) holds. If* $f\in\mathcal{A}$ *and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)\prec\alpha\frac{1+Az}{1+Bz}+\beta\left(\frac{1+Az}{1+Bz}\right)^2+\mu\frac{(A-B)z}{(1+Bz)^2},$$

for $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\ne 0$, $-1\le B<A\le 1$, where $\psi_\lambda^{m,n}$ is defined in (7), then

$$\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\prec\frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, in Theorem 2.1 we get the corollary.

Corollary 2.3 *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^\gamma, m,n\in\mathbb{N}$, $\lambda\ge 0, z\in U$. *Assume that (6) holds. If* $f\in\mathcal{A}$ *and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)\prec\alpha\left(\frac{1+z}{1-z}\right)^\gamma+\beta\left(\frac{1+z}{1-z}\right)^{2\gamma}+\mu\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1},$$

for $\alpha,\mu,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\mu\ne 0$, where $\psi_\lambda^{m,n}$ is defined in (7), then

$$\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^\gamma,$$

and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best dominant.

Proof. Corollary follows by using Theorem 2.1 for $q(z)=\left(\frac{1+z}{1-z}\right)^\gamma$, $0<\gamma\le 1$.

Theorem 2.4 *Let* q *be convex and univalent in* U, *such that* $q(0)=1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. *Assume that*

$$Re\left(\frac{q'(z)}{\mu}(\alpha+2\beta q(z))\right)>0,\ for\, \alpha,\mu,\beta\in\mathbb{C},\mu\neq 0, z\in U. \tag{11}$$

If $f\in\mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$ is univalent in U, where $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$ is as defined in (7), then

$$\alpha q(z)+\beta(q(z))^2+\mu z q'(z)\prec\psi_\lambda^{m,n}(\alpha,\beta,\mu;z),\ z\in U, \tag{12}$$

implies

$$q(z)\prec\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)},\ z\in U, \tag{13}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z):=\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$.

By setting $\nu(w):=\alpha w+\beta w^2$ and $\phi(w):=\mu$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\setminus\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))}=\frac{q'(z)}{\mu}(\alpha+2\beta q(z))$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\frac{q'(z)}{\mu}(\alpha+2\beta q(z))\right)>0$, for $\mu,\xi,\beta\in\mathbb{C},\ \mu\neq 0.$

By using (12) we obtain

$$\alpha q(z)+\beta(q(z))^2+\mu z q'(z)\prec\alpha q(z)+\beta(q(z))^2+\mu z q'(z).$$

Using Lemma 1.2, we have

$$q(z)\prec p(z)=\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)},\ z\in U,$$

and q is the best subordinant.

Corollary 2.5 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1,\ m,n\in\mathbb{N}$, $\lambda\ge 0.$ *Assume that (11) holds. If* $f\in\mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha\frac{1+Az}{1+Bz}+\beta\left(\frac{1+Az}{1+Bz}\right)^2+\mu\frac{(A-B)z}{(1+Bz)^2}\prec\psi_\lambda^{m,n}(\alpha,\beta,\mu;z),$$

for $\alpha,\mu,\beta\in\mathbb{C}$, $\mu\neq 0,\ -1\le B<A\le 1$, where $\psi_\lambda^{m,n}$ is defined in (7), then

$$\frac{1+Az}{1+Bz}\prec\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ in Theorem 2.4 we get the corollary.

Corollary 2.6 *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}, m,n\in\mathbb{N}$, $\lambda\ge 0.$ *Assume that (11) holds. If* $f\in\mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\beta\left(\frac{1+z}{1-z}\right)^{2\gamma}+\mu\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1}\prec\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z),$$

for $\alpha,\mu,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\mu\ne 0,$ where $\psi_{\lambda}^{m,n}$ is defined in (7), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.

Proof. Corollary follows by using Theorem 2.4 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1.$

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z)\ne 0$ *and* $q_2(z)\ne 0$, *for all* $z\in U$, *with* $zq_1'(z)$ *and* $zq_2'(z)$ *being starlike univalent. Suppose that* q_1 *satisfies (6) and* q_2 *satisfies (11). If* $f\in\mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and* $\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (7) univalent in* U, *then*

$$\alpha q_1(z)+\beta(q_1(z))^2+\mu z q_1'(z)\prec\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha q_2(z)+\beta(q_2(z))^2+\mu z q_2'(z),$$

for $\alpha,\mu,\beta\in\mathbb{C},$ $\mu\ne 0,$ implies

$$q_1(z)\prec\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\prec q_2(z),\ \delta\in\mathbb{C},\delta\ne 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\le B_2<B_1<A_1<A_2\le 1$, we have the following corollary.

Corollary 2.8 *Let* $m,n\in\mathbb{N}$, $\lambda\ge 0.$ *Assume that (6) and (11) hold for* $q_1(z)=\frac{1+A_1z}{1+B_1z}$ *and* $q_2(z)=\frac{1+A_2z}{1+B_2z}$, *respectively. If* $f\in\mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha\frac{1+A_1z}{1+B_1z}+\beta\left(\frac{1+A_1z}{1+B_1z}\right)^2+\mu\frac{(A_1-B_1)z}{(1+B_1z)^2}\prec\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z)$$

$$\prec \alpha \frac{1+A_2 z}{1+B_2 z}+\beta\left(\frac{1+A_2 z}{1+B_2 z}\right)^2+\mu \frac{(A_2-B_2)z}{(1+B_2 z)^2},$$

for $\alpha,\mu,\beta \in \mathbb{C}$, $\mu \neq 0$, $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, where $\psi_\lambda^{m,n}$ is defined in (7), then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \prec \frac{1+A_2 z}{1+B_2 z},$$

hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.

Theorem 2.9 *Let* $\left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \in \mathcal{H}(U)$, $f \in \mathcal{A}$, $z \in U$, $\delta \in \mathbb{C}, \delta \neq 0$, $m,n \in \mathbb{N}$, $\lambda \ge 0$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0)=1$, $z \in U$. *Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+\frac{zq''(z)}{q'(z)}\right)>0, \tag{14}$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and

$$\psi_\lambda^{m,n}(\alpha,\beta;z) := \left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \Big[\alpha - n\delta\beta -$$

$$\delta\beta(n+1)^2 \frac{SR^{m,n+1} f(z)}{SR^{m+1,n} f(z)}+\delta\beta(n+1)(n+2)\frac{SR^{m,n+2} f(z)}{SR^{m+1,n} f(z)}-\delta\beta \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\Big] \tag{15}$$

If q satisfies the following subordination

$$\psi_\lambda^{m,n}(\alpha,\beta;z) \prec \alpha q(z)+\beta z q'(z), \tag{16}$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then

$$\left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \prec q(z),\ z \in U, \delta \in \mathbb{C}, \delta \neq 0, \tag{17}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0)=1$

We have $zp'(z) = \delta z \left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \frac{SR^{m,n} f(z)}{SR^{m+1,n} f(z)}\left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)' =$

$$\delta\left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \frac{SR^{m,n} f(z)}{SR^{m+1,n} f(z)}\left(\frac{z\left(SR^{m+1,n} f(z)\right)'}{SR^{m,n} f(z)} - \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\frac{z\left(SR^{m,n} f(z)\right)'}{DS^{m,n} f(z)}\right).$$

By using the identity (4) and (5), we obtain

$$zp'(z) = \delta\left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \frac{SR^{m,n} f(z)}{SR^{m+1,n} f(z)}\Big[-n\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} -$$

$$(n+1)^2\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}+(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n}f(z)}-\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^2\Bigg] \tag{18}$$

so, we obtain

$$zp'(z)=\delta\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\Big[-n-$$

$$(n+1)^2\frac{SR^{m,n+1}f(z)}{SR^{m+1,n}f(z)}+(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n}f(z)}-\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\Bigg] \tag{19}$$

By setting $\theta(w):=\alpha w$ and $\phi(w):=\beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0,\ \ w\in\mathbb{C}\setminus\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta zq'(z),$ we find that $Q(z)$ is starlike univalent in U.

Let $h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta zq'(z)$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+\frac{zq''(z)}{q'(z)}\right)>0$.

By using (19), we obtain $\alpha p(z)+\beta zp'(z)=\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\Big[\alpha-n\delta\beta-$

$$\delta\beta(n+1)^2\frac{SR^{m,n+1}f(z)}{SR^{m+1,n}f(z)}+\delta\beta(n+1)(n+2)\frac{SR^{m,n+2}f(z)}{SR^{m+1,n}f(z)}-\delta\beta\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\Bigg].$$

By using (16), we have $\alpha p(z)+\beta zp'(z)\prec\alpha q(z)+\beta zq'(z).$

From Lemma 1.1, we have $p(z)\prec q(z)$, $z\in U,$ i.e. $\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\prec q(z)$, $z\in U,\delta\in\mathbb{C},\delta\neq 0$

and q is the best dominant.

Corollary 2.10 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $z\in U$, $-1\le B<A\le 1$, $m,n\in\mathbb{N}$, $\lambda\ge 0.$ *Assume that (14) holds. If* $f\in\mathcal{A}$ *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta;z)\prec\alpha\frac{1+Az}{1+Bz}+\beta\frac{(A-B)z}{(1+Bz)^2},$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0,\ \ -1\le B<A\le 1$, where $\psi_{\lambda}^{m,n}$ is defined in (15), then

$$\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\prec\frac{1+Az}{1+Bz},\ \delta\in\mathbb{C},\delta\neq 0,$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. For $q(z)=\dfrac{1+Az}{1+Bz}$, $-1\le B<A\le 1,$ in Theorem 2.9 we get the corollary.

Corollary 2.11 *Let* $q(z)=\left(\dfrac{1+z}{1-z}\right)^{\gamma}, m,n\in\mathbb{N}$, $\lambda\ge 0.$ *Assume that (14) holds. If* $f\in\mathcal{A}$ *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\beta\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1},$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\ne 0,$ where $\psi_{\lambda}^{m,n}$ is defined in (15), then

$$\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},\ \delta\in\mathbb{C},\delta\ne 0,$$

and $\left(\dfrac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.

Proof. Corollary follows by using Theorem 2.9 for $q(z)=\left(\dfrac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1.$

Theorem 2.12 *Let* q *be convex and univalent in* U *such that* $q(0)=1$. *Assume that*

$$Re\left(\frac{\alpha}{\beta}q'(z)\right)>0,\ for\,\alpha,\beta\in\mathbb{C},\beta\ne 0. \tag{20}$$

If $f\in\mathcal{A}$, $\left(\dfrac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_{\lambda}^{m,n}(\alpha,\beta;z)$ is univalent in U, where $\psi_{\lambda}^{m,n}(\alpha,\beta;z)$ is as defined in (15), then

$$\alpha q(z)+\beta zq'(z)\prec\psi_{\lambda}^{m,n}(\alpha,\beta;z) \tag{21}$$

implies

$$q(z)\prec\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta},\ \delta\in\mathbb{C},\delta\ne 0,z\in U, \tag{22}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z):=\left(\dfrac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}$, $z\in U$, $z\ne 0$, $\delta\in\mathbb{C},\delta\ne 0,$ $f\in\mathcal{A}$.

The function p is analytic in U and $p(0)=1.$

By setting $\nu(w):=\alpha w$ and $\phi(w):=\beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\ne 0$, $w\in\mathbb{C}\setminus\{0\}$.

Since $\dfrac{\nu'(q(z))}{\phi(q(z))}=\dfrac{\alpha}{\beta}q'(z)$, it follows that $Re\left(\dfrac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\dfrac{\alpha}{\beta}q'(z)\right)>0$, for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0.$

Now, by using (21) we obtain

$$\alpha q(z)+\beta zq'(z)\prec \alpha q(z)+\beta zq'(z),\ \ z\in U.$$

From Lemma 1.2, we have

$$q(z)\prec p(z)=\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta},\ \ z\in U,\delta\in\mathbb{C},\delta\neq 0,$$

and q is the best subordinant.

Corollary 2.13 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. *Assume that (20) holds. If* $f\in\mathcal{A}$, $\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$, $\delta\in\mathbb{C}$, $\delta\neq 0$ *and*

$$\alpha\frac{1+Az}{1+Bz}+\beta\frac{(A-B)z}{(1+Bz)^2}\prec\psi_{\lambda}^{m,n}(\alpha,\beta;z),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\le B<A\le 1$, where $\psi_{\lambda}^{m,n}$ is defined in (15), then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta},\delta\in\mathbb{C},\delta\neq 0,$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, in Theorem 2.12 we get the corollary.

Corollary 2.14 *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. *Assume that (20) holds. If* $f\in\mathcal{A}$, $\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\beta\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1}\prec\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z),$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\neq 0$, where $\psi_{\lambda}^{m,n}$ is defined in (15), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta},\delta\in\mathbb{C},\delta\neq 0,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.

Proof. Corollary follows by using Theorem 2.12 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1$.

Combining Theorem 2.9 and Theorem 2.12, we state the following sandwich theorem.

Theorem 2.15 *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z)\neq 0$ *and* $q_2(z)\neq 0$, *for all*

$z \in U$. *Suppose that* q_1 *satisfies (14) and* q_2 *satisfies (20). If* $f \in \mathcal{A}$, $\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$, $\delta \in \mathbb{C}, \delta \neq 0$ *and* $\psi_{\lambda}^{m,n}(\alpha, \beta; z)$ *is as defined in (15) univalent in* U, *then*

$$\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda}^{m,n}(\alpha, \beta; z) \prec \alpha q_2(z) + \beta z q_2'(z),$$

for $\alpha, \beta \in \mathbb{C}, \ \beta \neq 0,$ implies

$$q_1(z) \prec \left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta} \prec q_2(z), \ z \in U, \delta \in \mathbb{C}, \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.16 *Let* $m, n \in \mathbb{N}$, $\lambda \geq 0.$ *Assume that (14) and (20) hold for* $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ *and* $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, *respectively. If* $f \in \mathcal{A}$, $\left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$ *and*

$$\alpha \frac{1+A_1 z}{1+B_1 z} + \beta \frac{(A_1 - B_1)z}{(1+B_1 z)^2} \prec \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z)$$

$$\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \beta \frac{(A_2 - B_2)z}{(1+B_2 z)^2}, \ z \in U,$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0,$ $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda}^{m,n}$ is defined in (7), then

$$\frac{1+A_1 z}{1+B_1 z} \prec \left(\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}\right)^{\delta} \prec \frac{1+A_2 z}{1+B_2 z}, \ z \in U, \delta \in \mathbb{C}, \delta \neq 0,$$

hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.

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