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PRECISE ASYMPTOTIC IN THE LAW OF THE ITERATED LOGARITHM AND COMPLETE CONVERGENCE FOR U-STATISTICS

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ABSTRACT

This paper presents the precise asymptotic of U-statistics of i.i.d. absolutely continuous random variables. We argue that this can help us describe the relations among the boundary function, weighted function, and convergence rate and limit value in the study of the complete convergence.

KEYWORDS: *U-statistics; Precise asymptotic.*

1. INTRODUCTION:

Since Hsu and Robbins (1947) introduced the concept of complete convergence, there have been extensions in two directions. Let {X,Xk : k _ 1}be a sequence of i.i.d. random variables, Sn = $\sum^{n}_{i=1}$ Xk, n ≥1, and φ(x) and f(x)be the positive functions defined on [0, ∞). One extension is to discuss the moment conditions, from which it follows that

$$\sum_{n=1}^{\infty} \varphi(n) P(|S_n| \geq \epsilon f(n)) < \infty, \quad \epsilon > 0, \tag{1}$$

Where $\sum_{n=1}^{\infty} \varphi(n) = \infty$. In this direction, one can refer to Hsu and Robbins (1947), Erd¨os (1949,1950) and Baum and Katz (1965), etc. they, respectively, studied the cases in which

$$\varphi(n) \equiv 1,\ f(n) = n \text{ and } \varphi(n) = n^{r/p-2},$$
$$f(n) = n^{1/p}, \text{ where } 0 < p < 2,\ r \geq p.$$

Another extension departs from the observation that the convergence rate and limit value of

$$\sum_{n=1}^{\infty} \varphi(n) P(|S_n| \geq \epsilon n) = EX^2, \tag{2}$$

where EX = 0 and $EX^2 < \infty$. For analogous results in the more general case, see (R. Chen, 1978 and A. Spˇataru, 1999), etc.

Research in this field is called the precise asymptotic. Suppose that $h(X_1, ..., X_m)$ is some real-valued function of m arguments in which $X_1, ..., X_m$ are i.i.d. observations from some CDF, and for a given m ≥ 1 we want to estimate or make inferences about the parameter $\theta = \theta(F) = E_F\, h(X_1, ...,X_m)$.

We assume n ≥m. of course, one unbiased estimator for **θ** is $h(X_1, ...X_m)$ itself. But one should be able to find a better unbiased estimate if n > m because $h(X_1, ...,X_m)$ does not use all the sample data. For example, if the X_i are real valued, then the set of order statistics $X_{(1)}, ...,X_{(n)}$ is always sufficient and the Rao-Blackwellization E[h(X1, ...,Xm)|X(1), ...,X(n)] is a better unbiased estimate than $h(X_1, ...,X_m)$.

Indeed, in this case

$$E[h(X_1, ...X_m)|X_{(1)}, ..., X_{(n)}] = \frac{1}{\binom{n}{m}} \sum_{1\le i_1 < i_2 < \cdots < i_m \le n} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}). \tag{3}$$

Statistics of this form are called U-statistics (U for unbiased), and h is called the kernel and m its order. They were introduced in Hoffding (1948).
Consider an i.i.d. sequence $\{X_i\}$ with a distribution function F. For each sample of size n, $\{X_1, . . . ,X_n\}$, a corresponding sample distribution function Fn is constructed by placing at each observation Xi a mass of 1/n. Thus F_n may be presented as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x), \qquad -\infty < x < \infty. \tag{4}$$

Let F be a distribution function (continuous at the right, as usual). For 0 < p < 1, the pth quintile of F is defined as

$$\xi_p = \inf\{x : F(x) \ge p\} \tag{5}$$

and is alternately denoted by $F^{-1}(p)$. Note that ξ_p satisfies

$$F(\xi_p) \le p \le F(\xi_p). \tag{6}$$

2. REVIEW OF RELATED LEMMAS AND THEOREMS:

First, we reproduce some Lemmas and Theorems.

Lemma 1 (lemmas 5.2.1.A in Serfling) the variance of U_n is given by

$$var_F\{U_n\} = \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{m}{c}\binom{n-m}{m-c}\xi_c$$

and satisfies

$(i)\ \frac{m^2}{n}\xi_1 \leq var_F\{U_n\} \leq \frac{m}{n}\xi_m;$

$(ii)\ (n+1)var_F\{U_{n+1}\} \leq n\ var_F\{U_n\};$

$(iii)\ var_F\{U_n\} = \frac{m^2\xi_1}{n} + O(n^{-2}),\ n \to \infty.$

Theorem 1 (Theorem 5.5.1.A in Serfling)

$if\ E_F^{h^2} < \infty\ and\ \xi_1 > 0,$

$then\ n^{1/2}(U_n - \theta) \xrightarrow{d} N(0, m^2\xi_1)\ that\ is$

$$U_n \qquad is \qquad AN(\theta, \frac{m^2\xi_1}{n}).$$

Theorem 2 (Theorem 5.5.1.B in Serfling)

$$if\ \nu = E|h|^3 < \infty\ and \quad \xi_1 > 0,\ then$$

$$\sup_{-\infty<t<\infty}\left|P(\frac{n^{1/2}(U_n-\theta)}{m\xi_1^{1/2}} \leq t) - \Phi(t)\right| \leq C\nu(m^2\xi_1)^{-3/2}n^{-1/2},$$

Where C is an absolute constant, and

$$\Delta_n = \sup_t \left|P(|U_n - \theta| > t) - \bar{\Phi}(tn^{1/2}/m\xi_1^{1/2})\right|.$$

Theorem 3 (Theorem 5.6.1.A in Serfling)

let h = h($X_1, \ldots, X_m$) be a kernel for

$\theta = \theta(F),\ with\ \alpha \leq h(x_1, \ldots, x_m) \leq b.\ put\ \theta = E\{h(X_1, \ldots, X_m)\}$

$and\ \sigma^2 = var\{h(X_1, \ldots, X_m)\},\ then,\ for\ t > 0\ and\ n \geq m,$

$(1) \qquad P(U_n - \theta \geq t) \leq \exp\{-2[n/m]t^2/(b-a)^2\}.$

And

$$(2) \qquad P(U_n - \theta \geq t) \leq \exp\{-[n/m]t^2/2[\sigma^2 + (1/3)(b-\theta)t]\}.$$

Let $X, X_1, X_2, \ldots$ be i.i.d. absolutely continuous random variables, and .

$$\bar{\Phi}(x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2} dt, \; x \geq 0.$$

before presenting the main results, we first discuss the general form and conditions of precise asymptotic. Assume there exist some n_0 2 Z^+, and the following functions are all defined on $[n_0,\infty)$. Denote

$$f(x) = \frac{m\xi_1^{1/2}}{n^{1/2}} h(x), \qquad x \geq n_0,$$

Where $\frac{m\xi_1^{1/2}}{n^{1/2}}$ is the normalizing function of U_n, and h(x) is differentiable.
Let g(x) be differentiable,

$$\varphi(x) = g'(h(x))h'(x), \qquad x \geq n_0$$

we want to find an appropriate $a \geq 0$, and for any $\epsilon > \alpha$, to find an appropriate $G_0(\epsilon)$ satisfying

$$G_0(\epsilon) < \infty, \quad \epsilon > a \qquad and \qquad \lim_{\epsilon \downarrow a} G_0(\epsilon) = 0. \qquad (7)$$

so that for all $G(\epsilon) \sim (G_0(\epsilon))^{-1}$, $\epsilon \downarrow a$, we have that

$$\lim_{\epsilon \downarrow a} G(\epsilon) \sum_{n \geq n_0} \varphi(n) P(|U_n - \theta| > \epsilon \; m\xi^{1/2} h(n)/n^{1/2}) = 1 \qquad (8)$$

It can be seen that $G_0(\epsilon)$ includes the information of the convergence rate, limit value of the series and limit position of ϵ.
Throughout the following, we assume that g(x), h(x), $x \geq n_0$, be positive, which both strictly increase to ∞, g(h(x)) is defined on $[n_0,\infty)$, and $g^{-1}(x)$, $h^{-1}(x)$ are the inverse functions of f(x) and h(x) respectively.
Choose

$$G_0(\epsilon) = \sqrt{\frac{2}{\pi}} \, \epsilon \int_{h(n_0)}^\infty g(y) e^{-\epsilon^2 y^2/2} dy, \qquad \epsilon > a, \qquad (9)$$

where $a \geq 0$, such that (7) holds.

3. RESULTS

Theorem 3.1
Assume that

$$\varphi(x) = g'(h(x))h'(x)$$

is monotone, and if $\varphi(x)$ is monotone non decreasing, we assume

$\lim_{n\to\infty}(\varphi(n+1)/\varphi(n)) = 1.$

and assume that there exist a ≥ 0 such that, in (9), $G_0(\epsilon)$ satisfies (7). And also assume that g(x), x ≥ n_0, satisfy the following conditions:

$$\sum_{n=n_0}^{\infty} \varphi(n)n^{-1/2} < \infty. \quad (10)$$

$$\lim_{M\to\infty} \lim_{\epsilon\downarrow a} \epsilon\ G(\epsilon) \int_{g^{-1}(G_0(\epsilon)M)}^{\infty} g(y)e^{-\epsilon^2 y^2/2} dy = 0, \quad (11)$$

$$\lim_{M\to\infty} \lim_{\epsilon\downarrow a} G(\epsilon) \int_{h^{-1}og^{-1}(G_0(\epsilon)M)}^{\infty} \exp\Big\{ -\frac{2\epsilon^2 m\xi_1 t^2 x^2 h^2(x)}{(b-a)^2}\Big\} dg(h(x)) = 0, (12)$$

$$\lim_{M\to\infty} \lim_{\epsilon\downarrow a} G(\epsilon) \int_{h^{-1}og^{-1}(G_0(\epsilon)M)}^{\infty} \exp\Big\{ -\frac{m\xi_1 t^2\epsilon^2 x^2 h^2(x)}{2[\sigma^2 + \frac{\epsilon m\xi_1^{1/2} th(x)}{3x^{1/2}}(b-\theta)]}\Big\} dg(h(x)) = 0. (13)$$

Then (8) holds, when a > 0 or a = 0.

Choose $g(x) = x^r l(x)$, r ≥ 0, where $l \in R_0$ is a slowly varying function.

By (9), set $G(\epsilon) = \epsilon^r (l(\epsilon^{-1}))^{-1}(E|N|^r)^{-1}$, $\epsilon > 0 = a$;
then we have

Corollary 3.1 Let h(x) be a positive and differentiable function defined on $[n_0,\infty)$, which is strictly increasing to

∞, $\varphi(x) = r(h(x))^{r-1}h'(x)$

be monotone,
and if φ(x) is monotone nondecreasing, we assume
$\lim_{n\to\infty}(\varphi(n+1)/\varphi(n)) = 1.$

further, let L is bounded away from 0 and ∞ on every compact subset of $[n_0,\infty)$. then

$$\lim_{\epsilon\downarrow 0} \epsilon^r (l(\epsilon^{-1}))^{-1} \sum_{n=n_0}^{\infty} \varphi(n) P(|U_n - \theta| > \epsilon\ tm\xi_1^{1/2} h(n)/n^{1/2}) = r^{-1}E|N|^r. (14)$$

Choose

$g(x) = x^{\frac{2(r-p)}{2-p}} l(x),\ r \geq p,$

where $l \in R_0$ is a slowly varying function.
By (9),

$G(\epsilon) = \epsilon^{\frac{2(r-p)}{2-p}}(l(\epsilon^{-1}))^{-1}(E|N|^{\frac{2(r-p)}{2-p}})^{-1},\ \epsilon > 0 = a;$

then we have following corollary.

Corollary 3.2

Let h(x) be a positive and differentiable function defined on $[n_0,\infty)$, which is strictly increasing to $\infty,\ \varphi(x) = 2(r-p)/(2-p)(h(x))^{\frac{2(r-p)}{2-p}}h'(x)$

be monotone, and if φ(x) is monotone nondecreasing, we assume

$\lim_{n\to\infty}(\varphi(n+\ 1)/\varphi(n)) = 1.$

Further, let L is bounded away from 0 and ∞ on every compact subset of $[n_0,\infty)$. Then

$$\lim_{\epsilon\downarrow 0}\epsilon^{\frac{2(r-p)}{2-p}}(l(\epsilon^{-1}))^{-1}\sum_{n=n_0}^{\infty}\varphi(n)P(|U_n-\theta| > \epsilon\ m\xi_1^{1/2}h(n)/n^{1/2}) = \frac{2-p}{2(r-p)}E|N|^{\frac{2(r-p)}{2-p}}. \qquad (15)$$

Choose $g(x) = e^{rx^2}$, $r > 0$, then by (9), set $G(\epsilon) = (\epsilon^2 - 2r)^{1/2}(2r)^{-1/2}$, $\epsilon > (2r)^{1/2} = a$, hence we have

Corollary 3.3

let h(x) be a positive and differentiable function defined on $[n_0,\infty)$, which is strictly increasing to $\infty,\ \varphi(x) = 2re^{rh^2(x)}h(x)h'(x)$

Be monotone, and if φ(x)is monotone nondecreasing, we assume

$\lim_{n\to\infty}(\varphi(n+\ 1)/\varphi(n)) = 1.$

finally, assume that h(x) satisfies (10).

Then

$$\lim_{\epsilon\downarrow(2r)^{1/2}}(\epsilon^2-2r)^{1/2}\sum_{n=n_0}^{\infty}e^{rh^2(n)}h(n)h'(n)P(|U_n-\theta| > \epsilon\ tm\xi_1^{1/2}h(n)/n^{1/2}) = (2r)^{-1/2} \qquad .(16)$$

Choose

$g(x) = e^{rx},\ r > 0,$ by (9) set $G(\epsilon) = e^{-r^2/2\epsilon^2}/2,\ \epsilon > 0 = a,$

then

it follows that:

Corollary 3.4

Let h(x) be a positive and differentiable function defined on $[n_0,\infty)$, which is strictly increasing to ∞, $\varphi(x) = re^{rh(x)}h'(x)$ be monotone, and if φ(x)is monotone nondecreasing, we assume

$\lim_{n\to\infty}(\varphi(n+1)/\varphi(n)) = 1.$

Finally, assume that h(x) satisfies (10). then

$$\lim_{\epsilon\downarrow 0} e^{-\frac{r^2}{2\epsilon^2}} \sum_{n=n_0}^{\infty} e^{rh(n)} h'(n) P(|U_n - \theta| > \epsilon\ t\ m\xi_1^{1/2} h(n)/n^{1/2}) = 2r^{-1}. \quad (17)$$

4. PROOFS:

Proof of Theorem.3.1
If φ(x) is nondecreasing, then by (7), (9) and integration by parts, we have

$$\int_{n_0+1}^{\infty} \varphi(x)\bar{\Phi}(\epsilon h(x))dx \le \sum_{n=n_0+1}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n)) \le \int_{n_0}^{\infty} \varphi(x)\bar{\Phi}(\epsilon h(x)). \quad (18)$$

If φ(x) is nondecreasing, then by

$\lim_{n\to\infty}(\varphi(n+1)/\varphi(n)) = 1.$

For any

$0 < \delta < 1$, there exist $n_1 = n_1(\delta)$, when

$n \ge n_1,\ \varphi(n+1)/\varphi(n) < 1+\delta$

And

$\varphi(n)/\varphi(n+1) > 1-\delta.$

Thus we have that

$$(1+\delta)^{-1} \int_{n_1+1}^{\infty} \varphi(x)\bar{\Phi}(\epsilon h(x))dx \le \sum_{n=n_1+1}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n))$$
$$\le (1-\delta)^{-1} \int_{n_1}^{\infty} \varphi(x)\bar{\Phi}(\epsilon h(x))dx. \quad (19)$$

Hence by integration by part, (7), (8), (18), (19), we have

$$\liminf_{\epsilon\downarrow a} G(\epsilon) \sum_{n=n_0}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n)) \ge (1+\delta)^{-1} \quad (20)$$

and

$$\limsup_{\epsilon\downarrow a} G(\epsilon) \sum_{n=n_0}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n)) \le (1-\delta)^{-1}. \quad (21)$$

Let $\delta \downarrow 0$, then we conclude

$$\lim_{\epsilon\downarrow a} G(\epsilon) \sum_{n=n_0}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n)) = 1. \quad (22)$$

If φ(x) is nondecreasing, then by lemma 3.1 in Wang Wang (2003),

$$\begin{aligned}4G_0(\epsilon)M &= 4g\Big(h(b(\epsilon))\Big) \geq goh(b(\epsilon)+1) \geq \sum_{n=n_0}^{[b(\epsilon)]} \varphi(n) \\ &= \sum_{n=n_0}^{n_1-1} \varphi(n) + \sum_{n=n_1}^{[b(\epsilon)]} \varphi(n) \geq \sum_{n=n_0}^{n_1-1} \varphi(n) + (1-\delta) \sum_{n=n_1}^{[b(\epsilon)]} \varphi(n+1) \\ &\geq (1-\delta)\Big(\sum_{n=n_0}^{[b(\epsilon)]+1} \varphi(n) - \varphi(n_1)\Big), \end{aligned} \tag{23}$$

where $b(\epsilon) = h^{-1}og^{-1}(G_0(\epsilon)M)$.

If φ(x) is non-increasing, similarly we have

$$4G_0(\epsilon)M \geq \sum_{n=n_0}^{[b(\epsilon)]+1} \varphi(n) - \varphi(n_0). \tag{24}$$

By Theorems (23), (24) and Toepliz lemma, we get

$$\lim_{\epsilon \downarrow a} G(\epsilon) \sum_{n=n_0}^{[b(\epsilon)]+1} \varphi(n)\Delta_n = 0. \tag{25}$$

By integration by parts and (7),

$$(1-\delta)^{-1} \int_{b(\epsilon)}^{\infty} \varphi(x)\bar{\Phi}(\epsilon h(x))dx \geq \sum_{n=[b(\epsilon)]+1}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n)) \tag{26}$$

$$\begin{aligned}(1-\delta)^{-1} \int_{b(\epsilon)}^{\infty} \varphi(x)\bar{\Phi}(\epsilon h(x))dx &= (1-\delta)^{-1} \int_{h(b(\epsilon))}^{\infty} \bar{\Phi}(\epsilon y)dg(y) \\ &\leq (1-\delta)^{-1} \int_{g^{-1}(G_0(\epsilon)M)}^{\infty} \epsilon\sqrt{\frac{2}{\pi}} g(y) e^{-\epsilon^2 y^2} dy \end{aligned} \tag{27}$$

By (11), (26), (27) and (18), we deduce

$$\lim_{\epsilon \downarrow a} G(\epsilon) \sum_{n=[b(\epsilon)]+1}^{\infty} \varphi(n)\bar{\Phi}(\epsilon h(n)) = 0, \qquad as \quad M \to \infty. \tag{28}$$

In the following, we prove that

$$\lim_{M\to\infty} \lim_{\epsilon \downarrow a} G(\epsilon) \sum_{n=[b(\epsilon)]+1}^{\infty} \varphi(n) P\Big(|U_n - \theta| > \epsilon t m \xi_1^{1/2} h(n)/n^{1/2}\Big) = 0. \tag{29}$$

By Theorem 3, we have

$$G(\epsilon)\sum_{n=[b(\epsilon)]+1}^{\infty}\varphi(n)P\Big(|U_n-\theta|>\epsilon t m\xi_1^{1/2}h(n)/n^{1/2}\Big)$$
$$\leq CG(\epsilon)\sum_{n=[b(\epsilon)]+1}^{\infty}\varphi(n)\exp\{-\frac{2m\xi_1 t^2 n^2 h^2(n)\epsilon^2}{(b-a)^2}\}$$
$$\leq CG(\epsilon)\int_{b(\epsilon)}^{\infty}\varphi(x)\exp\{-\frac{2\epsilon^2 m\xi_1 t^2 h^2(x)}{(b-a)^2}x^2\}dx$$
$$= CG(\epsilon)\int_{b(\epsilon)}^{\infty}\varphi(x)\exp\{-\frac{2\epsilon^2 m\xi_1 t^2 h^2(x)}{(b-a)^2}x^2\}dg(h(x))$$

Or by Theorem 3, we have

$$G(\epsilon)\sum_{n=[b(\epsilon)]+1}^{\infty}\varphi(n)P\Big(|U_n-\theta|>\epsilon t m\xi_1^{1/2}h(n)/n^{1/2}\Big)$$
$$\leq CG(\epsilon)\sum_{n=[b(\epsilon)]+1}^{\infty}\varphi(n)\exp\{-\frac{m\xi_1 t^2 n^2 h^2(n)\epsilon^2}{2[\sigma^2+\frac{\epsilon m\xi_1^{1/2}th(n)}{3n^{1/2}}(b-\theta)]}$$
$$\leq CG(\epsilon)\int_{b(\epsilon)}^{\infty}\varphi(x)\exp\{-\frac{m\xi_1 t^2 x^2 h^2(x)\epsilon^2}{2[\sigma^2+\frac{\epsilon m\xi_1^{1/2}th(x)}{3x^{1/2}}(b-\theta)]}\}dx$$
$$= CG(\epsilon)\int_{b(\epsilon)}^{\infty}\exp\{-\frac{m\xi_1 t^2 x^2 h^2(x)\epsilon^2}{2[\sigma^2+\frac{\epsilon m\xi_1^{1/2}th(x)}{3x^{1/2}}(b-\theta)]}\}dg(h(x))$$

Together with (12) or (13), we get (29).

Proof of corollary 3.1:

By properties of slowly varying functions and dominant convergence theorem and Potter's theorem and Theorem 1.5.6 and 1.5.12 in Bingham, we have

$$\lim_{\epsilon\downarrow 0}\sqrt{\frac{2}{\pi}}\,\epsilon\int_{h(n_0)}^{\infty}\frac{y^r l(y)e^{-\epsilon^2y^2/2}}{\epsilon^{-r}l(\epsilon^{-1})}dy$$
$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\lim_{\epsilon\downarrow 0}\frac{z^r l(\epsilon^{-1}z)e^{-z^2/2}}{l(\epsilon^{-1})}I(z>\epsilon h(n_0))dz$$
$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}z^r e^{-z^2/2}dz=E|N|^r.$$

When ∈ is small enough,

$$g^{-1}\Big(g(\epsilon^{-1})ME|N|^r\Big) \leq \epsilon^{-1}B(ME|N|^r)^\delta \qquad (30)$$

we know that when ∈ is small enough,

$$\begin{aligned}\epsilon G(\epsilon)\int_{g^{-1}(G_0(\epsilon)M)}^{\infty} g(y)e^{-\epsilon^2y^2/2}dy &= G(\epsilon)\int_{\epsilon g^{-1}(G_0(\epsilon)M)}^{\infty} g(\epsilon^{-1}z)e^{-z^2/2}dz \\ &\leq (E|N|^r)^{-1}\int_{2\epsilon g^{-1}(g(\epsilon^{-1})ME|N|^r)}^{\infty}\frac{z^r l(\epsilon^{-1}z)}{l(\epsilon^{-1})}e^{-z^2/2}dz \\ &\leq A(E|N|^r)^{-1}\int_{2B(ME|N|^r)^\delta}^{\infty} z^{r+\delta}e^{-z^2/2}dz \to 0 \quad as \quad M\to\infty\end{aligned}$$

i.e., (11) is satisfied.

By Karamata's theorem and Potter's theorem and (30), when ∈ is small
enough, we have

$$\lim_{M\to\infty}\lim_{\epsilon\downarrow a} G(\epsilon)\int_{h^{-1}og^{-1}(G_0(\epsilon)M)}^{\infty} e^{-\frac{2\epsilon^2 m\xi_1 t^2 x^2 h(x)}{(b-a)^2}}dg(h(x)) = 0$$

Hence by theorem.3.1, we have

$$\lim_{\epsilon\downarrow 0}\epsilon^r(l(\epsilon^{-1}))^{-1}\sum_{n=n_0}^{\infty}\varphi(n)P\Big(|U_n-\theta| > \epsilon t m\xi_1^{1/2}h(n)/n^{1/2}\Big) = r^{-1}E|N|^r.$$

The proof of Corollaries 3.2 and 3.3 and 3.4 are just to verify the conditions
of theorem.3.1 straightly, we omit them.

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